SOLUTIONS:

Question 1

Qn		Key Points
1	Consider auxiliary equation $m^2+m+1=0$ $m=\frac{1}{2}(-1\pm\sqrt{3}i)$ $y_n=Ae^{i2n\pi/3}+Be^{-i2n\pi/3}$ $y_n=Ae^{i2n\pi/3}+Be^{-i2n\pi/3}$ $y_0=\sqrt{3}\Rightarrow y_n=Ae^{i0}+Be^{i0}=A+B=\sqrt{3}$ $y_1=0=Ae^{i2\pi/3}+Be^{-i2\pi/3}=(A+B)\cos(2\pi/3)+(A-B)\sin(2\pi/3)$ $-\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}(A-B)=0$ $A-B=1$ $A=\frac{\sqrt{3}+1}{2},B=\frac{\sqrt{3}-1}{2}$ $y_n=\frac{\sqrt{3}+1}{2}(\cos(2n\pi/3)+i\sin(2n\pi/3))+\frac{\sqrt{3}-1}{2}(\cos(2n\pi/3)-i\sin(2n\pi/3))$ $=\sqrt{3}\cos(2n\pi/3)+\sin(2n\pi/3)$ $=2\sin(2n\pi/3+\tan^{-1}\frac{\sqrt{3}}{1})$ $=2\sin(2n\pi/3+\pi/3)$	Auxiliary equation Solve for P.S

Question 2

		Key Points
2(a)	$\begin{pmatrix}1&a&4\\a&4&7\\-2&2&0\end{pmatrix}\xrightarrow[2R_1+R_3\to R_3]{-aR_1+R_2\to R_2}\begin{pmatrix}1&a&4\\0&-a^2+4&-4a+7\\0&2a+2&8\end{pmatrix}\xrightarrow[\frac{1}{2a+2}R_3\to R_3]{R_3\leftrightarrow R_2}\begin{pmatrix}1&a&4\\0&1&\frac{4}{a+1}\\0&-a^2+4&-4a+7\end{pmatrix}$ $\xrightarrow{(a^2-4)R_2+R_3}\begin{pmatrix}1&a&4\\0&1&\frac{4}{a+1}\\0&0&(a^2-4)\frac{4}{a+1}-4a+7\end{pmatrix}$ Since dim(ker(T))=1, $(a^2-4)\frac{4}{a+1}-4a+7=0$ $4a^2-16-4a^2+3a+7=0$ $a=3$ Note that $\begin{pmatrix}1&3&4\\0&1&1\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ Clearly, $\begin{pmatrix}1\\1\\-1\end{pmatrix}$ is the basis for the kernel of T, thus $\mathbf{r}=\lambda\begin{pmatrix}1\\1\\-1\end{pmatrix},\lambda\in\mathbb{R}$,	**Perform correct EROs** **Eqn in *a***
(b)	Since $\left\{\begin{pmatrix}1\\3\\-2\end{pmatrix},\begin{pmatrix}3\\4\\2\end{pmatrix}\right\}$ is the basis for $R(T)$, then $\mathrm{r}=s\begin{pmatrix}1\\3\\-2\end{pmatrix}+t\begin{pmatrix}3\\4\\2\end{pmatrix}$, Normal $=\begin{pmatrix}1\\3\\-2\end{pmatrix}\times\begin{pmatrix}3\\4\\2\end{pmatrix}=\begin{pmatrix}14\\-8\\-5\end{pmatrix}$ Equation of plane is: $14x-8y-5z=0$	**State basis or linear combination of basis**

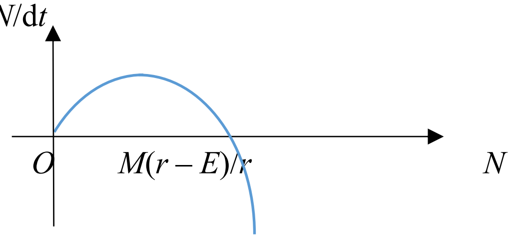
Question 3

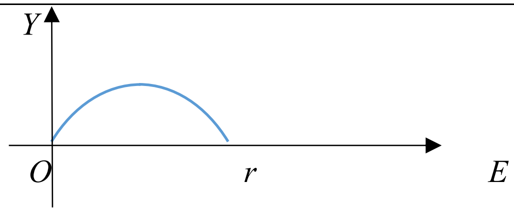
		Key Points
	$1+e^{2i\alpha}=e^{i\alpha}\left(e^{-i\alpha}+e^{i\alpha}\right)=(2\cos\alpha)e^{i\alpha}$ $\left\lvert 1+e^{2i\alpha}\right\rvert=2\cos\alpha>0$ since $\cos\alpha>0$ for $-\frac{1}{2}\pi<\alpha<\frac{1}{2}\pi$. $\arg\left(1+e^{2i\alpha}\right)=\arg\left[(2\cos\alpha)e^{i\alpha}\right]$ $=\underbrace{\arg(2\cos\alpha)}_{0}+\arg\left(e^{i\alpha}\right)$ $=\alpha$ $\sum_{r=0}^{n}\frac{n!}{r!(n-r)!}\left[\cos(2r+1)\alpha+i\sin(2r+1)\alpha\right]$ $=\sum_{r=0}^{n}\frac{n!}{r!(n-r)!}e^{i(2r+1)\alpha}$ $=e^{i\alpha}\sum_{r=0}^{n}\binom{n}{r}\left(e^{i2\alpha}\right)^{r}$ $=e^{i\alpha}\left(1+e^{i2\alpha}\right)^{n}$ using $\sum_{r=0}^{n}\binom{n}{r}x^{r}=(1+x)^{n}$ $=e^{i\alpha}\left[(2\cos\alpha)e^{i\alpha}\right]^{n}$ $=(2\cos\alpha)^{n}e^{i(n+1)\alpha}$ $=(2\cos\alpha)^{n}\left[\cos(n+1)\alpha+i\sin(n+1)\alpha\right]$ by de Moivre's Equating imaginary parts, $\sum_{r=0}^{n}\frac{n!}{r!(n-r)!}\sin(2r+1)\alpha=(2\cos\alpha)^{n}\sin(n+1)\alpha$.	Exponential form Justify modulus Justify argument Consider sum Write as $\sum_{r=0}^{n}\binom{n}{r}x^{r}$ Binomial expansion Use previous result Use de Moivre's theorem

Question 4

		Key Points
	Define $\mathrm{f}(x)=\tan^{-1}x-\dfrac{x}{1+x^2}$.	
	$\mathrm{f}(0)=0$.	State $\mathrm{f}(0)=0$
	$\mathrm{f}'(x)=\dfrac{1}{1+x^2}-\dfrac{1+x^2-x(2x)}{\left(1+x^2\right)^2}$	
	$=\dfrac{2x^2}{\left(1+x^2\right)^2}\geq 0$ for all $x\geq 0$	Conclude $\mathrm{f}'(x)\geq 0$
	Since $\mathrm{f}(0)=0$ and $\mathrm{f}'(x)\geq 0$ for all $x\geq 0$,	
	$\tan^{-1}x\geq\dfrac{x}{1+x^2}$ for all $x\geq 0$ with equality at $x=0$.	State $x=0$
	Required volume	
	$=2\pi\displaystyle\int_0^1 x\left(\tan^{-1}x-\frac{x}{1+x^2}\right)\mathrm{d}x$	
	$=2\pi\left\{\displaystyle\int_0^1 x\tan^{-1}x\ \mathrm{d}x-\int_0^1\frac{x^2}{1+x^2}\ \mathrm{d}x\right\}$	
	$=2\pi\left\{\dfrac{1}{2}\left[x^2\tan^{-1}x\right]_0^1-\dfrac{1}{2}\displaystyle\int_0^1\frac{x^2}{1+x^2}\ \mathrm{d}x-\int_0^1\frac{x^2}{1+x^2}\ \mathrm{d}x\right\}$	Integration by parts
	$=2\pi\left\{\dfrac{1}{2}\left(\dfrac{\pi}{4}\right)-\dfrac{3}{2}\displaystyle\int_0^1\frac{x^2}{1+x^2}\ \mathrm{d}x\right\}$	
	$=2\pi\left\{\dfrac{\pi}{8}-\dfrac{3}{2}\displaystyle\int_0^1\left(1-\frac{1}{1+x^2}\right)\mathrm{d}x\right\}$	Writing in correct form to attempt integration
	$=2\pi\left\{\dfrac{\pi}{8}-\dfrac{3}{2}\left[x-\tan^{-1}x\right]_0^1\right\}$	
	$=2\pi\left\{\dfrac{\pi}{8}-\dfrac{3}{2}\left(1-\dfrac{\pi}{4}\right)\right\}$	
	$=2\pi\left(\dfrac{\pi}{2}-\dfrac{3}{2}\right)$	
	$=\pi(\pi-3)$	

Question 5

		Key Points
(a)	EN is the rate at which the species of fish are caught when the population of the fish is N.	State meaning
(b)	For equilibrium populations, $\frac{\mathrm{d}N}{\mathrm{d}t}=0 \Rightarrow r\left(1-\frac{N}{M}\right)N-EN=0$ $\Rightarrow N\left(r-\frac{rN}{M}-E\right)=0$ $\Rightarrow N=0$ or $r-\frac{rN}{M}-E=0$ $\Rightarrow N=0$ or $N=\frac{M}{r}(r-E)>0$ since $r>E$ Thus $N_1=0, N_2=\frac{M}{r}(r-E)$.	Obtain equation and attempt to solve Obtain N_1 and N_2
(c)	 For $0<N<\frac{M}{r}(r-E)$, $\frac{\mathrm{d}N}{\mathrm{d}t}>0 \Rightarrow N\uparrow \frac{M}{r}(r-E)$. For $N>\frac{M}{r}(r-E)$, $\frac{\mathrm{d}N}{\mathrm{d}t}<0 \Rightarrow N\downarrow \frac{M}{r}(r-E)$. Therefore $N=N_1$ is unstable and $N=N_2$ is stable.	Sketch $\frac{\mathrm{d}N}{\mathrm{d}t}$ vs N Argue $N\uparrow \frac{M}{r}(r-E)$ Argue $N\downarrow \frac{M}{r}(r-E)$
(d)	$Y=EN_2=\frac{EM}{r}(r-E)$.	Write Y in terms of E

		Sketch Y vs E
(e)	From the quadratic graph of Y vs E, Y attains a maximum when $E = \frac{r}{2}$ and $Y_m = \frac{\left(\frac{r}{2}\right)M}{r}\left(r - \frac{r}{2}\right) = \frac{Mr}{4}$.	

Question 6 (Source: HCI Promo 2025 Qn 7)

(a)	$V=\int_0^{\frac{\pi}{2}} 2\pi x\sin x\,\mathrm{d}x$ $=2\pi\left(\left[-x\cos x\right]_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\cos x\,\mathrm{d}x\right)$ $=2\pi\left[\sin x\right]_0^{\frac{\pi}{2}}$ $=2\pi$
(bi)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\cos x$ $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\cos^2 x$ $L=\int_0^{\pi}\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$ $L=\int_0^{\pi}\sqrt{1+\cos^2(x)}\,\mathrm{d}x=3.8202$
(bii)	$A=2\pi\int_0^{\pi} y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$ $A=2\pi\int_0^{\pi}\sin x\sqrt{1+\cos^2 x}\,\mathrm{d}x$ Let $u=-\cos x\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=\sin x$ $A=2\pi\int_{-1}^{1}\sqrt{1+u^2}\,\mathrm{d}u$ $A=4\pi\int_0^{1}\sqrt{1+u^2}\,\mathrm{d}u$, where $k=4$ and $u=-\cos x$ **Alternatively:** Let $u=\cos x\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=-\sin x$ $A=2\pi\int_{1}^{-1}-\sqrt{1+u^2}\,\mathrm{d}u$ $A=2\pi\int_{-1}^{1}\sqrt{1+u^2}\,\mathrm{d}u$

	$A=4\pi\int_0^1 \sqrt{1+u^2}\ \mathrm{d}u$, where $k=4$ and $u=\cos x$
(biii)	Let $u=\tan\theta \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}\theta}=\sec^2\theta$ $A=4\pi\int_0^{\frac{\pi}{4}} \sqrt{1+\tan^2\theta}\sec^2\theta\ \mathrm{d}\theta$ $=4\pi\int_0^{\frac{\pi}{4}} \sec\theta\sec^2\theta\ \mathrm{d}\theta$ $=4\pi\left[\left[\sec\theta\tan\theta\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\sec\theta\tan^2\theta\ \mathrm{d}\theta\right]$ $=4\pi\left[\left[\sec\theta\tan\theta\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\sec^3\theta-\sec\theta\ \mathrm{d}\theta\right]$ $8\pi\int_0^{\frac{\pi}{4}}\sec^3\theta\ \mathrm{d}\theta=4\pi\left[\left[\sec\theta\tan\theta\right]_0^{\frac{\pi}{4}}+\int_0^{\frac{\pi}{4}}\sec\theta\ \mathrm{d}\theta\right]$ $A=4\pi\int_0^{\frac{\pi}{4}}\sec^3\theta\ \mathrm{d}\theta=2\pi\left[\sec\theta\tan\theta+\ln\left\|\sec\theta+\tan\theta\right\|\right]_0^{\frac{\pi}{4}}$ $A=2\pi\left[\sec\theta\tan\theta+\ln\left\|\sec\theta+\tan\theta\right\|\right]_0^{\frac{\pi}{4}}$ $A=2\pi\left[\sqrt{2}+\ln\left(\sqrt{2}+1\right)\right]$

Question 7

		Key Points
(a)	Required probability = P(3 draws) + P(4 draws) + P(5 draws) + P(6 draws) = P(111) + P(2111) + P(*2111)+ P(**2111) $= p_1^3 + (1-p_1)p_1^3 + (1-p_1)p_1^3 + (1-p_1)p_1^3$ $= p_1^3(1+3-3p_1) = p_1^3(4-3p_1)$	
(b)	Let X denote the time that a **randomly chosen** battery will last. $P(X>s+t \mid X>t) = \dfrac{P(X>s+t)}{P(X>t)}$ $= \dfrac{P(X>s+t \mid \text{battery is type 1})p_1 + P(X>s+t \mid \text{battery is type 2})p_2}{P(X>t \mid \text{battery is type 1})p_1 + P(X>t \mid \text{battery is type 2})p_2}$ $= \dfrac{e^{-\lambda_1(s+t)}p_1 + e^{-\lambda_2(s+t)}p_2}{e^{-\lambda_1 t}p_1 + e^{-\lambda_2 t}p_2}$	Passing remark: If $\lambda_1 = \lambda_2 = \lambda$, the fraction simplifies to $P(X>s+t \mid X>t) = e^{-\lambda s}$ demonstrating memoryless property. Otherwise, the memoryless property does not hold in such a mixture problem.

Question 8

		Key Points
(a)	$\mathrm{P}(X < Y) = 0.5$	$\mathrm{P}(X < Y) + \mathrm{P}(Y < X) + \mathrm{P}(Y = X) = 1$ Thus $\mathrm{P}(X < Y) + \mathrm{P}(Y < X) = 1$ since for CRVs, $\mathrm{P}(Y = X) = 0$. Since X and Y are i.i.d., $\mathrm{P}(X < Y) = \mathrm{P}(Y < X)$
(b)	Let $M = \max(X, Y)$ $\mathrm{P}(\max(X,Y) \le m) = \mathrm{P}(X \le m)\mathrm{P}(Y \le m)$ $= \left(\dfrac{m-a}{b-a}\right)\dfrac{m-a}{b-a}$ $= \left(\dfrac{m-a}{b-a}\right)^2$ $\mathrm{f}(m) = \dfrac{2(m-a)}{(b-a)^2}$ $\mathrm{E}(M) = \int_a^b m\mathrm{f}(m)\,\mathrm{d}m$ $= \dfrac{2}{(b-a)^2}\int_a^b m(m-a)\,\mathrm{d}m$ $= \dfrac{2}{(b-a)^2}\left[\dfrac{m^3}{3} - \dfrac{am^2}{2}\right]_a^b$ $= \dfrac{2}{(b-a)^2}\left[\dfrac{b^3}{3} - \dfrac{ab^2}{2} - \dfrac{a^3}{3} + \dfrac{a^3}{2}\right]$ $= \dfrac{2}{(b-a)^2}\dfrac{2b^3 - 3ab^2 + a^3}{6}$ $= \dfrac{2}{(b-a)^2}\dfrac{(2b+a)(b-a)^2}{6}$ $= \dfrac{1}{3}(2b+a)$	**Find f(m) by differentiating F(m)** **Apply Expectation formula** **Integration** **Factorization of cubic expression**

Question 9

		Key Points
	$\int_{-k}^{k} \frac{2}{\pi(1+x^2)} = 1$ $\left[\tan^{-1} x\right]_{-k}^{k} = \frac{\pi}{2}$ $\tan^{-1} k - \tan^{-1}(-k) = \frac{\pi}{2}$ $2\tan^{-1} k = \frac{\pi}{2}$ $k = 1$ or $\int_{-k}^{k} \frac{2}{\pi(1+x^2)} = 1$ $\frac{2}{\pi}\left[\tan^{-1} x\right]_{0}^{k} = \frac{1}{2}$ (use of symmetry) $\tan^{-1} k = \frac{\pi}{4}$ $k = 1$ $\mathrm{P}(\lvert X \rvert \le \tan\frac{\pi}{8})$ $= \mathrm{P}(-\tan\frac{\pi}{8} \le X \le \tan\frac{\pi}{8})$ $= \int_{-\tan\frac{\pi}{8}}^{\tan\frac{\pi}{8}} \frac{2}{\pi(1+x^2)} dx$ $= \frac{2}{\pi}[2\tan^{-1}(\tan\frac{\pi}{8})]$ $= \frac{4}{\pi}(\frac{\pi}{8}) = \frac{1}{2}$ or $\mathrm{P}(\lvert X \rvert \le \tan\frac{\pi}{8})$ $= 1 - 2\mathrm{P}(X \ge \tan\frac{\pi}{8})$ $= 1 - 2\int_{\tan\frac{\pi}{8}}^{1} \frac{2}{\pi(1+x^2)} dx$ $= 1 - \frac{4}{\pi}[\tan^{-1} x]_{\tan\pi/8}^{1}$ $= 1 - \frac{4}{\pi}(\frac{\pi}{4} - \frac{\pi}{8}) = \frac{1}{2}$	**Area under pdf=1**
(b)	$\mathrm{E}(X) = 0$ by symmetry $\mathrm{Var}(X) = \mathrm{E}(X^2) - \left(\mathrm{E}(X)\right)^2$ $= \int_{-1}^{1} \frac{2x^2}{\pi(1+x^2)} dx - 0$ $= \frac{2}{\pi}\int_{-1}^{1} 1 - \frac{1}{1+x^2} dx$ $= \frac{2}{\pi}[x - \tan^{-1} x]_{-1}^{1}$ $= \frac{2}{\pi}[1 - \tan^{-1} 1 - (-1 - \tan^{-1}(-1))]$ $= \frac{2}{\pi}(2 - \frac{\pi}{2}) = \frac{4}{\pi} - 1$	

Question 10

		Key Points
(a)	• Arrival times occur independently. (i.e. the arrival of a customer is independent of any other customers) • Assume that no 2 customers arrive at the same time. Note: "Average rate at which customers arrive is constant" cannot be accepted as it is implied by question.	
(b)	*X:* time between arrivals of customers in minutes. $X \sim \text{Exp}(0.5)$ $P(X<1)=\int_0^1 \frac{1}{2}\mathrm{e}^{-\frac{1}{2}x}\,\mathrm{d}x=0.393$	
(c)	*A:* no of customers in a 40 minute period $A \sim \text{Po}\left(\frac{40}{60}(30)\right) \Rightarrow A \sim \text{Po}(20)$ $\text{P}(\lvert A-\mu \rvert < 3\sigma)$ $=\text{P}(\mu-3\sigma < A < \mu+3\sigma)$ $=\text{P}(6.58 < A < 33.4)$ $=\text{P}(A \le 33)-P(A \le 6)$ $=0.997$	
(d)	$N(t) \sim \text{Po}(0.5t)$ $\text{P}(X_1 \le x \mid N(t)=1)=\frac{\text{P}(X_1 \le x \text{ and } N(t)=1)}{\text{P}(N(t)=1)}$ $=\frac{\text{P(1 arrival in } [0,x] \text{ and no arrivals in } (x,t])}{\text{P}(N(t)=1)}$ $=\frac{\text{P}(N(x)=1)\text{P}(N(t-x)=0)}{\text{P}(N(t)=1)}$ $=\frac{\left(\frac{\mathrm{e}^{-0.5x}(0.5x)^1}{1!}\right)\left(\frac{\mathrm{e}^{-0.5(t-x)}(0.5(t-x))^0}{0!}\right)}{\left(\frac{\mathrm{e}^{-0.5t}(0.5t)^1}{1!}\right)}=\frac{x}{t}$ $\text{f}(x)=\frac{\mathrm{d}}{\mathrm{d}x}\frac{x}{t}=\frac{1}{t}$ This $X_1 \sim \text{U}[0,t]$	One should split the interval into two and consider the number of arrivals in each interval instead.

Question 11

		Key Points
(a)	Let X be the random variable denoting the score of appetite for a red background themed restaurant. Let S_X^2 be the unbiased estimate of population variance of the score of appetite for a red background themed restaurant. $\bar{x} = \dfrac{23.29 + 22.28}{2} = 22.785$ $22.785 + t_{0.025}(20-1)\dfrac{s_x}{\sqrt{20}} = 23.29$ $22.785 + 2.0930\dfrac{s_x}{\sqrt{20}} = 23.29$ $s_x = 1.0790$ $\therefore$ unbiased estimate for population variance of the score of appetite for a red background themed restaurant $= 1.0790^2 = 1.1642 = 1.16$ (3 s.f.) An assumption needed is that the scores of appetite for a red background themed restaurant follow a normal distribution.	
(b)(i)	Paired sample test is more appropriate as the "before" and "after" samples measure the same individuals and are not independent. Pairing eliminates the factor of variability between the individuals.	
(b)(ii)	Let X and Y be the percentage of platelets that came together before and after smoking respectively. Let $D = Y - X$, and μ_D be the population mean for D. $H_0 : \mu_D = 0$ $H_1 : \mu_D \neq 0$ at 1% level of significance Test statistic $T = \dfrac{\bar{D} - \mu_D}{S_{\bar{D}}} \sim t(10)$ Reject H_0 if $\lvert t_{\text{cal}} \rvert \geq 3.169$ From sample, $\bar{d} = 10.273$, $s = 7.9761$ and $n = 11$ $t_{\text{cal}} = \dfrac{10.273 - 0}{7.9761/\sqrt{11}} = 4.2716$	

	Since $t_{cal} > 3.169,$ we reject H_0 and conclude that there is sufficient evidence, at 1% significance level, that smoking affects platelet function.	
(b)(iii)	Let μ_Y be the population mean for Y. $H_0 : \mu_Y = 50$ $H_1 : \mu_Y > 50$ Test statistic $T = \dfrac{\overline{Y} - \mu_Y}{S_Y / \sqrt{11}} \sim t(10)$ From GC, $\overline{y} = 52.4545,\ s_y = 18.300$ $p\text{-value} = 0.33295 \quad (\text{or } t_{\text{cal}} = 0.44486)$ • The p-value is much greater than 0.1 large and we cannot reject H_0 even at 10% level of significance. There is insufficient evidence for Andy's claim. • The larger than 50% sample mean can be attributed to variation that is expected to occur with any distribution and is not large enough for us to conclusively say that the actual population mean could not be ≤ 50 .	A point estimate above a threshold is not enough evidence without checking if it's statistically significantly above.