

11

DEFINITE INTEGRALS

SYLLABUS

- Concept of definite integral as a limit of sum
 - Understand the idea that the area under a curve is a limit of a sum of the areas of rectangles, and use a graphing calculator to illustrate the limiting process (simple cases only)
- Definite integral as the area under a curve
 - Define the definite integral as a limit of sums and interpret it as the area under a curve
 - Students are expected to know the different perspectives of the concept of definite integral

$$\int_a^b f(x) dx = F(b) - F(a), \text{ where } \frac{d}{dx} F(x) = f(x) \text{ (anti-derivative)}$$

$$\int_a^b f(x) dx = \lim_{\delta x \rightarrow 0} \sum y \cdot \delta x, \text{ where } y = f(x) \text{ (limit of sums)}$$

$$\int_a^b f(x) dx \text{ is the area under the curve } y = f(x) \text{ between } x = a \text{ and } x = b$$

- Use anti-derivatives to evaluate definite integrals
- Finding the area of a region bounded by a curve and lines parallel to the coordinate axes, between a curve and a line, or between 2 curves
- Finding area below the x -axis
- Finding the volume of revolution about the x - or y -axis
- Finding the approximate value of a definite integral using a graphing calculator or a graphing software

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LECTURE 1

Lesson Outline

- Evaluating Definite Integrals
 - Definite Integrals Involving Modulus
 - Definite Integral as a Limiting Sum
-

1 DEFINITE INTEGRALS

1.1 Evaluating Definite Integrals

A function $F(x)$ is an *anti-derivative* of a function $f(x)$ if $F'(x) = f(x)$ for all x in the domain of f .

If a function f has an anti-derivative F on $a \leq x \leq b$, then

$$\int_a^b f(x) \, dx = [F(x)]_a^b = F(b) - F(a).$$

Example 1

Evaluate the following definite integrals exactly.

$$(a) \int_1^2 3x^2 - \frac{2}{x} \, dx \qquad (b) \int_0^\pi x \cos x \, dx$$

Solution

$$\begin{aligned} (a) \quad \int_1^2 3x^2 - \frac{2}{x} \, dx &= \left[x^3 - 2 \ln |x| \right]_1^2 \\ &= \left[x^3 - 2 \ln x \right]_1^2 \quad (\because 1 < x < 2 \Rightarrow |x| = x) \\ &= (2^3 - 2 \ln 2) - (1 - 2 \ln 1) = 7 - 2 \ln 2 \end{aligned}$$

$$\begin{aligned} (b) \quad \int_0^\pi x \cos x \, dx &= [x \sin x]_0^\pi - \int_0^\pi \sin x \, dx \\ &= [\cos x]_0^\pi = \cos \pi - \cos 0 = -2 \end{aligned}$$

■

Example 2

Find $\int x e^{x^2} dx$. Hence find $\int_0^1 x^3 e^{x^2} dx$ exactly.

Solution

$$\int x e^{x^2} dx = \frac{1}{2} \int 2x e^{x^2} dx = \frac{1}{2} e^{x^2} + c \quad \left(\text{using } \int f'(x) e^{f(x)} dx = e^{f(x)} + c \right)$$

$$\int_0^1 x^3 e^{x^2} dx = \int_0^1 x^2 \left(\quad \right) dx$$

=

$$= \frac{1}{2} e - \left[\frac{1}{2} e^{x^2} \right]_0^1$$

$$= \frac{1}{2} e - \frac{1}{2} (e - 1) = \frac{1}{2}$$

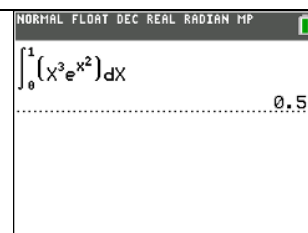
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Calculator Activity – Using GC to verify answers

- 1) Press **[ALPHA]** **[WINDOW]** and select **[4: fnInt(]**.



- 2) Input $\int_0^1 x^3 e^{x^2} dx$ by filling in the appropriate blanks and press **[ENTER]**.



If exact answers are not required, then all definite integrals should be calculated using the GC.

Example 3

By means of the substitution $x = \cos u$, find the exact value of

$$\int_0^1 \sqrt{1-x^2} dx.$$

Solution

$$\int_0^1 \sqrt{1-x^2} dx$$

$$\begin{aligned} x &= \cos u \\ \frac{dx}{du} &= \\ x = 0 &\Rightarrow \\ x = 1 &\Rightarrow \end{aligned}$$

=

$$= \int_{\frac{\pi}{2}}^0 \sin u (-\sin u) du$$

$$= \int_0^{\frac{\pi}{2}} \sin^2 u du$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} 1 - \cos 2u du$$

$$= \frac{1}{2} \left[u - \frac{\sin 2u}{2} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2} \left(\left(\frac{\pi}{2} - 0 \right) - (0 - 0) \right) = \frac{\pi}{4}$$

■ Can we derive the answer without integration?

1.2 Definite Integrals Involving Modulus

To evaluate integrals of functions involving modulus, we often will need to split the integrals into parts where the function is positive or negative. This is because

$$|f(x)| = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x) < 0 \end{cases}$$

This can be done by considering graph of $y = f(x)$ or by solving the inequality $f(x) \geq 0$ and then using the definition of $|f(x)|$.

Example 4

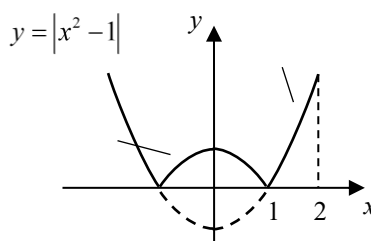
(a) Find $\int_0^2 |x^2 - 1| dx$, showing your working clearly.

(b) Find the exact value of $\int_0^1 x|x-a| dx$, where $0 < a < 1$.

Solution

(a) $\int_0^2 |x^2 - 1| dx$

$$\begin{aligned} &= \left[x - \frac{x^3}{3} \right]_0^1 + \left[\frac{x^3}{3} - x \right]_1^2 \\ &= \left(1 - \frac{1}{3} \right) + \left(\left(\frac{8}{3} - 2 \right) - \left(\frac{1}{3} - 1 \right) \right) \\ &= 2 \end{aligned}$$



(b) $\int_0^1 x|x-a| dx, 0 < a < 1$

$$|x-a| = \begin{cases} x-a, & x \geq a \\ a-x, & x < a \end{cases}$$

$$\begin{aligned} &= \left[\frac{ax^2}{2} - \frac{x^3}{3} \right]_0^a + \left[\frac{x^3}{3} - \frac{ax^2}{2} \right]_a^1 \\ &= \left(\frac{a^3}{2} - \frac{a^3}{3} \right) + \left(\left(\frac{1}{3} - \frac{a}{2} \right) - \left(\frac{a^3}{3} - \frac{a^3}{2} \right) \right) \\ &= \frac{a^3}{3} - \frac{a}{2} + \frac{1}{3} \end{aligned}$$

■

Self-Practice 1

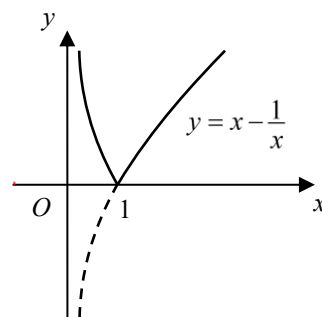
Find the exact value of $\int_{\frac{1}{2}}^2 \left| x - \frac{1}{x} \right| dx$.

Solution

$$\int_{\frac{1}{2}}^2 \left| x - \frac{1}{x} \right| dx$$

$$\begin{aligned} &= \left[\ln|x| - \frac{x^2}{2} \right]_{\frac{1}{2}}^1 + \left[\frac{x^2}{2} - \ln|x| \right]_1^2 \\ &= \left(\left(-\frac{1}{2} \right) - \left(\ln \frac{1}{2} - \frac{1}{8} \right) \right) + \left((2 - \ln 2) - \left(\frac{1}{2} \right) \right) \\ &= \frac{17}{8} \end{aligned}$$

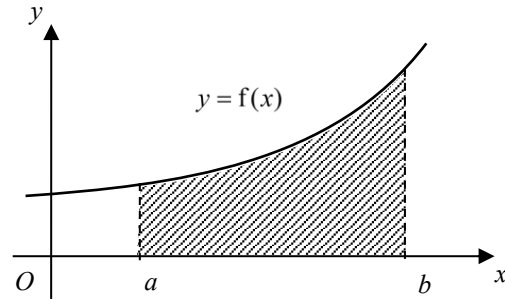
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2 PLANE AREA

2.1 Definite Integral as a Limiting Sum

Consider the curve with equation $y=f(x)$, where $f(x) \geq 0$ for all $x \in [a, b]$.

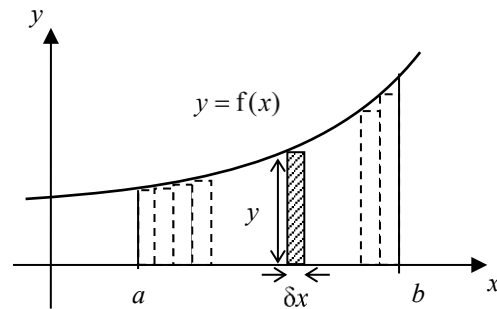


The area bounded by the curve, the x -axis and the lines $x=a$ and $x=b$ is

$$\int_a^b y \, dx = \int_a^b f(x) \, dx.$$

Derivation

Suppose we divide the required area into vertical strips of equal thickness, say δx .



Area of a typical strip = $y \delta x$ where $y = f(x)$

Sum of area of all strips = $\sum_{x=a}^{x=b} y \delta x$

Area bounded by the curve and the x -axis from $x=a$ to $x=b \approx \sum_{x=a}^{x=b} y \delta x$

As δx (thickness of the strips) gets smaller i.e. $\delta x \rightarrow 0$, the approximation gets better. It can be shown that total area tends to the actual area of the region bounded by the curve $y=f(x)$, the x -axis and the lines $x=a$ and $x=b$, i.e.

$$\lim_{\delta x \rightarrow 0} \sum_{x=a}^{x=b} y \delta x = \int_a^b y \, dx.$$



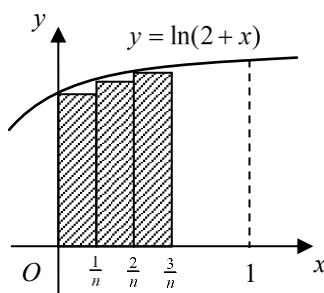
EXPLORATION / Area under graph as a limit of sums of rectangles

Use this GeoGebra applet to visualise the approximation of area under graphs with area of rectangular strips. In the applet, you can change the function, the number of strips and the region.

<https://www.geogebra.org/m/nKU3jkYd>

Example 5

The region R is bounded by the curve with equation $y = \ln(2+x)$, the x -axis and the lines $x=0$ and $x=1$. By considering rectangles of widths $\frac{1}{n}$ as shown in the diagram, show that the sum of areas of the n rectangles is given by $\frac{1}{n} \sum_{r=1}^n \ln\left(2 + \frac{r-1}{n}\right)$.



Using the graphing calculator, estimate the limit of the area of R up to two decimal places.

Solution

Area of first strip =

Area of second strip =

Area of third strip =

Therefore, it can be seen that area of r -th strip =

Hence, area of all n rectangles = $\frac{1}{n} \sum_{r=1}^n \ln\left(2 + \frac{r-1}{n}\right)$. ■

Calculator Activity – Estimating Limits of Sums

- (1) Press **[Y=]**. Type in $\frac{1}{X} \sum_{r=1}^X \ln\left(2 + \frac{r-1}{X}\right)$ for **Y1**. Press **[ENTER]**.

Plot1	Plot2	Plot3
Y1: $\frac{1}{X} \sum_{r=1}^X \left(\ln\left(2 + \frac{r-1}{X}\right) \right)$		
Y2=		
Y3=		
Y4=		
Y5=		
Y6=		
Y7=		
Y8=		

- (2) Press **[2ND] [WINDOW]** to adjust table setup

TABLE SETUP	
TblStart=	30
ΔTbl=	5
Indent:	Auto Ask
Depend:	Auto Ask

- (3) Press **[2ND] [GRAPH]**. (It will take a while. Why?)

X	Y1
30	0.9028
35	0.9037
40	0.9045
45	0.905
50	0.9055
55	0.9059
60	0.9062
65	0.9064
70	0.9066
75	0.9068
80	0.907

Observe that as **X** (which is n , the number of rectangles) increases, the column **Y1** approaches 0.91, which is roughly the area under the curve.

The convergence of the sum of rectangles is generally quite slow and good estimates are reached when n gets large. (> 100).

Example 6

The diagram shows part of the graph of $y = x^2 + 1$, with rectangles approximating the area under the curve between $x = 0$ and $x = 1$.

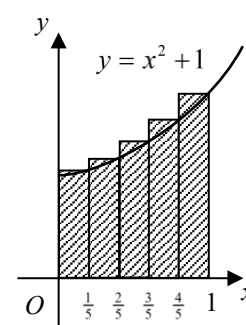
- (i) Prove that A_5 , the total area of the five rectangles shown, may be expressed by $A_5 = \frac{1}{5^3} \sum_{r=1}^5 r^2 + 1$.

In general, when the x -axis between $x = 0$ and $x = 1$ is divided into n equal parts, the area under the curve may be approximated by the total area, A_n , of n rectangles of width $\frac{1}{n}$.

- (ii) Given that $\sum_{r=1}^n r^2 = \frac{1}{6} n(n+1)(2n+1)$, show that

$$A_n = 1 + \frac{1}{6} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right).$$

- (iii) Hence deduce the exact value of $\int_0^1 x^2 + 1 \, dx$. Verify your answer using integration. [J90/I/18b modified]



Solution

(i) $A_5 =$

$$\begin{aligned}
 &= \frac{1}{5} \left(\left(\frac{1}{5} \right)^2 + \left(\frac{2}{5} \right)^2 + \left(\frac{3}{5} \right)^2 + \left(\frac{4}{5} \right)^2 + \left(\frac{5}{5} \right)^2 \right) + \frac{1}{5}(5) \\
 &= \frac{1}{5^3} \sum_{r=1}^5 r^2 + 1
 \end{aligned}$$

(ii) Area of the r -th strip $= \frac{1}{n} \left(\left(\frac{r}{n} \right)^2 + 1 \right)$

$$\begin{aligned}
 \text{Area of } n \text{ rectangles, } A_n &= \frac{1}{n} \sum_{r=1}^n \left(\left(\frac{r}{n} \right)^2 + 1 \right) \\
 &= 1 + \frac{1}{n^3} \sum_{r=1}^n r^2 \\
 &= 1 + \frac{1}{n^3} \left(\frac{1}{6} (n)(n+1)(2n+1) \right) \\
 &= 1 + \frac{1}{6} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) \text{ (proven)}
 \end{aligned}$$

(iii) $\int_0^1 x^2 + 1 \, dx = \lim_{n \rightarrow \infty} A_n$

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{6} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) \right) \\
 &= 1 + \frac{1}{6} (1)(2) = \frac{4}{3}
 \end{aligned}$$

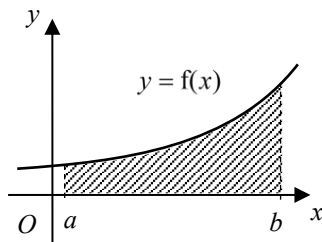
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LECTURE 2

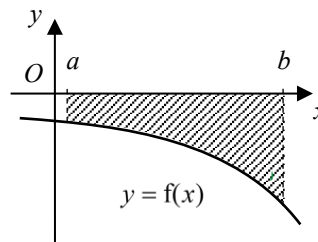
Lesson Outline

- Area Bounded by a Curve and x -axis
- Area Bounded by a Curve and y -axis
- Area Between Two Curves

2.2 Area Bounded by a Curve and the x -axis



When the curve is above the x -axis, $\int_a^b f(x) dx$ is positive as $f(x) \geq 0$ for all $x \in [a, b]$. Hence the area is $\int_a^b f(x) dx$.



When the curve is below the x -axis, $\int_a^b f(x) dx$ is negative as $f(x) \leq 0$ for all $x \in [a, b]$. Hence the area is $-\int_a^b f(x) dx$.

In general, we need to sketch the graphs to check if the curve is above or below the x -axis, and find these areas separately.

Example 7

- Find $\int_0^3 x(x-1) dx$ exactly.
- Sketch the graph of $y = x(x-1)$. Hence find the exact area of the region bounded by the x -axis and the part of the curve between $x = 0$ and $x = 3$.

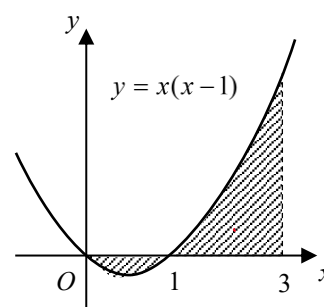
Solution

$$(i) \quad \int_0^3 x(x-1) dx = \int_0^3 x^2 - x dx = \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_0^3 = \left(\frac{27}{3} - \frac{9}{2} \right) = \frac{9}{2}$$

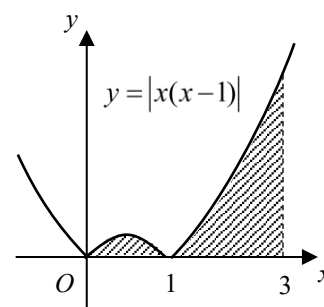
- First we need to correctly identify the region described in the question.

Required area

$$\begin{aligned}
 &= \\
 &= -\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^3 \\
 &= \frac{1}{6} + \left(\left(9 - \frac{9}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)\right) \\
 &= 4\frac{5}{6} \text{ unit}^2
 \end{aligned}$$



The required area in this example can be found by considering $|f(x)|$ and is equivalent to $\int_0^3 |x(x-1)| \, dx$ as shown.

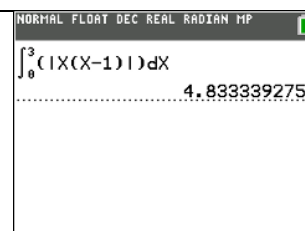


Therefore, you may use the GC to calculate this definite integral to verify your answer **or** obtain the answer directly if the question does not require an exact answer or workings to be shown.

Calculator Activity

Find the area of the region bounded by the x -axis and the part of the curve $y = x(x-1)$ between $x = 0$ and $x = 3$.

- 1) Type $\int_0^3 (|x(x-1)|) \, dx$.
 For integral, press **[ALPHA]**
[WINDOW] and select **[4:fnInt()]**.
 For modulus, press **[ALPHA]**
[WINDOW] and select **[1:abs()]**.



Example 8

Find the area enclosed by the curve $y^2 = \sin x$ and the line $x = \frac{1}{2}\pi$.

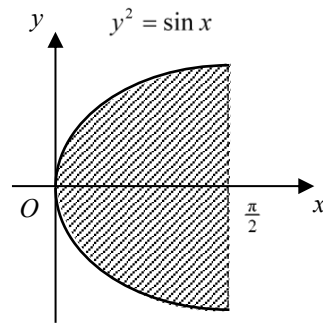
Solution

$$y^2 = \sin x$$

$$\Rightarrow y = \sqrt{\sin x} \quad \text{or} \quad y = -\sqrt{\sin x}$$

By symmetry

$$\text{Area} = 2 \int_0^{\frac{\pi}{2}} \sqrt{\sin x} \, dx = 2.40 \text{ unit}^2.$$

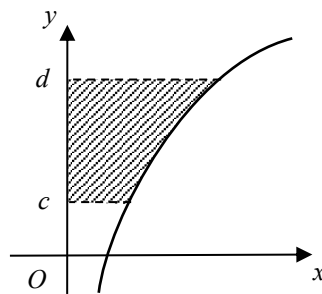


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As exact answers and working are not required by the question, the integral can simply be evaluated using the GC.

Extra Practice

Annex B: Practice Questions 1 – 2

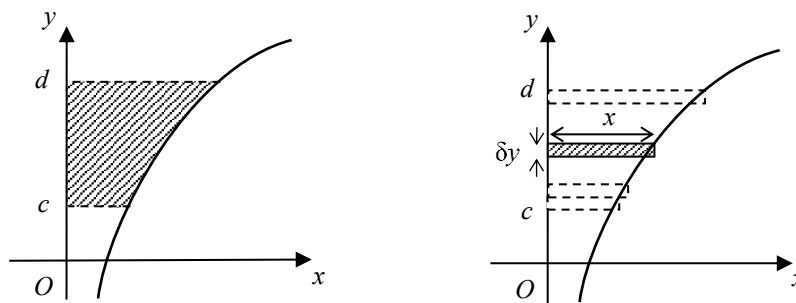
2.3 Area Bounded by a Curve and the y-axis

The area bounded by the curve, the y-axis and the lines $y = c$ and $y = d$ is

$$\int_c^d x \, dy = \int_c^d g(y) \, dy.$$

Derivation

Suppose we divide the required area into horizontal strips of equal thickness, say δy .



Area of a typical horizontal strip = $x \delta y$, where $x = g(y)$

Sum of areas of all horizontal strips = $\sum_{y=c}^{y=d} x \delta y$

Area of region bounded by the curve, y -axis from $y = c$ to $y = d$

$$\approx \sum_{y=c}^{y=d} x \delta y$$

As δy (thickness of the strips) gets smaller i.e. $\delta y \rightarrow 0$, the approximation gets better. It can be shown that total area tends to the area of region bounded by the curve $x = g(y)$, the y -axis and the lines $y = c$ and $y = d$, i.e.

$$\lim_{\delta y \rightarrow 0} \sum_{y=c}^{y=d} x \delta y = \int_c^d x \, dy.$$

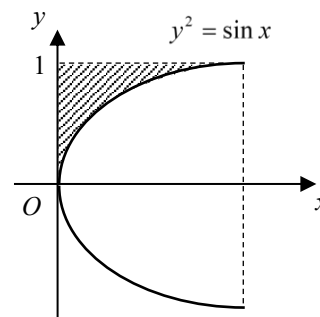
Example 9

Find the area enclosed by the curve $y^2 = \sin x$, the y -axis and the line $y = 1$.

Solution

$$y^2 = \sin x \Rightarrow x = \sin^{-1} y^2$$

Required area



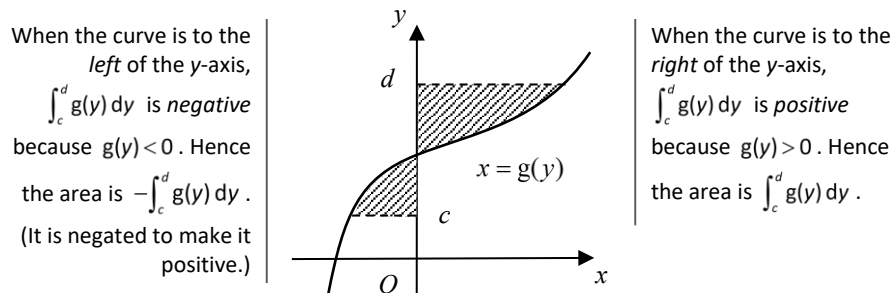
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Question

What is the value of $\int_0^{\frac{\pi}{2}} \sqrt{\sin x} \, dx + \int_0^1 \sin^{-1} y^2 \, dy$?

They sum up to $\frac{\pi}{2}$ units². (The two areas form a rectangle with breadth $\frac{\pi}{2}$ and length 1.) We can alternatively obtain the answer to Example 9 by using $\int_0^1 \sin^{-1} y^2 \, dy = \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \sqrt{\sin x} \, dx$.

When finding areas bounded by the y -axis, we need to consider regions to the right or left of the y -axis separately.



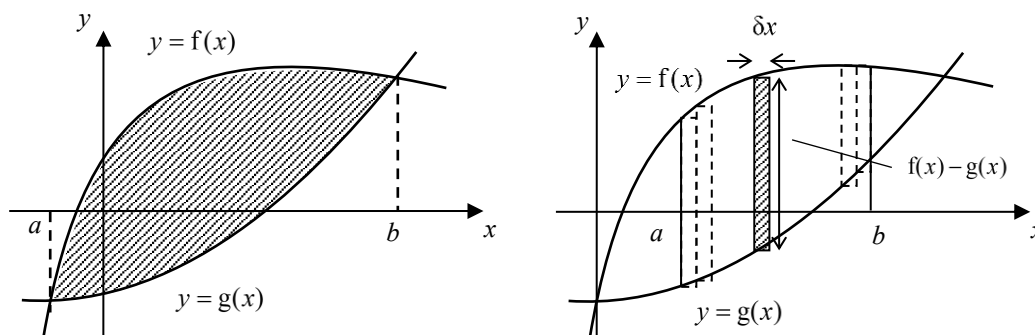
In conclusion, when finding area bounded by curves, we need to sketch the curves clearly to help us determine regions that are above/below x -axis or to the left/right of y -axis.

2.4 Area between Two Curves

If $f(x) \geq g(x)$ over the interval $[a, b]$, then the area bounded by two curves $y = f(x)$, $y = g(x)$, the lines $x = a$ and $x = b$ is given by

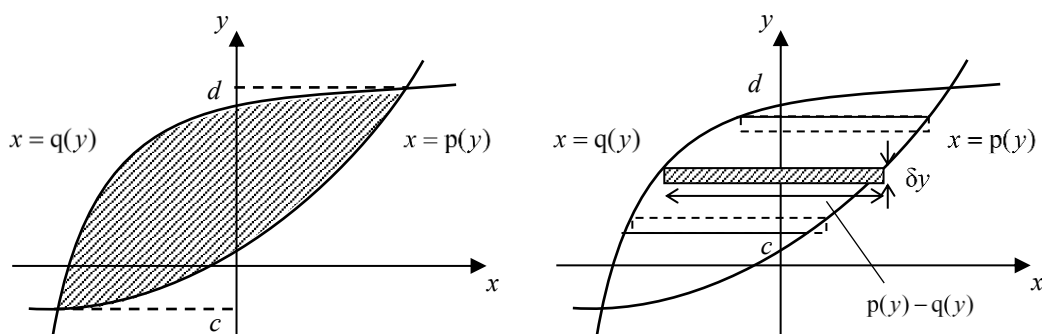
$$\int_a^b f(x) - g(x) \, dx.$$

This result is true even if the functions, or part of, are negative.



Similarly, the area of the region bounded by two curves defined as $x = p(y)$ and $x = q(y)$, where $p(y) \geq q(y)$, the lines $y = c$ and $y = d$ is given by

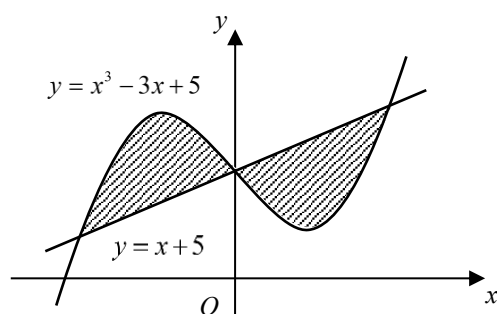
$$\int_c^d p(y) - q(y) \, dy.$$



Example 10

Calculate the area bounded by the curves $y = x^3 - 3x + 5$ and $y = x + 5$.

Solution



By GC, the curves intersect at $x = -2, 0, 2$.

Area =

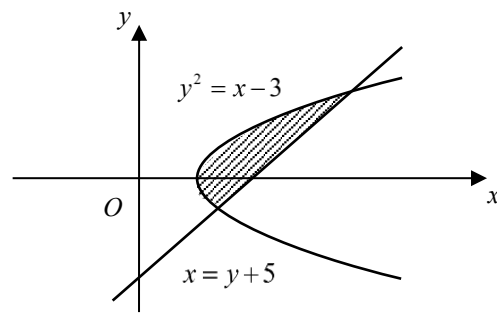
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Example 11

The diagram shows the graphs of $y^2 = x - 3$ and $x = y + 5$. Find the area of the region bounded by the two curves.

Solution



Using GC, the curves intersect at $(4, -1)$ and $(7, 2)$.

Required area =

$$\begin{aligned}
 &= \left[\frac{y^2}{2} - \frac{y^3}{3} + 2y \right]_{-1}^2 \\
 &= \left(2 - \frac{8}{3} + 4 \right) - \left(\frac{1}{2} + \frac{1}{3} - 2 \right) \\
 &= \frac{9}{2} \text{ unit}^2
 \end{aligned}$$

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Example 12

Find the exact area enclosed by the curve $x = (y-2)^2$, the lines $x=1$ and $x=4$.

Solution

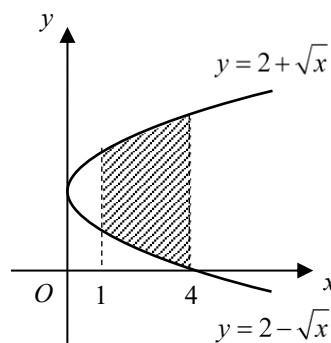
$$x = (y-2)^2 \Rightarrow$$

Required area

=

=

$$= \frac{28}{3} \text{ unit}^2$$



■

Question

This curve has a symmetry. Can we use this symmetry to help us find the area?

As the region is symmetrical about $y = 2$,

$$\text{Required area} = 2 \int_1^4 (2 + \sqrt{x}) - 2 \, dx = 2 \int_1^4 \sqrt{x} \, dx = \frac{28}{3} \text{ unit}^2.$$

LECTURE 3

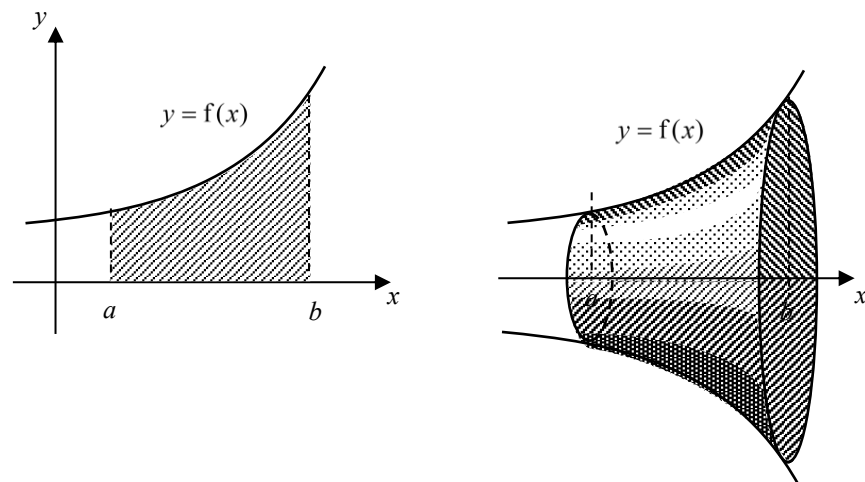
Lesson Outline

- Volume of Solid of Revolution about x -axis
- Volume of Solid of Revolution about y -axis
- Volume of Solid of Revolution Generated by Region Between Two Curves

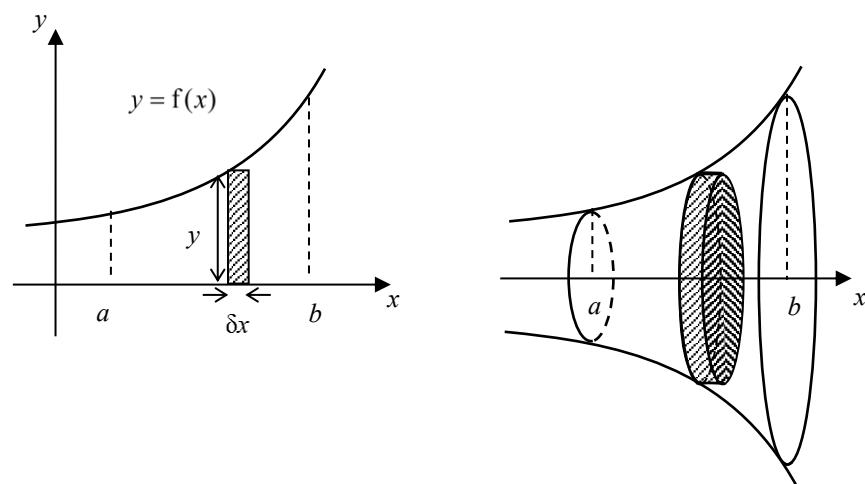
3 VOLUME OF REVOLUTION (DISC METHOD)

A solid of revolution is generated by revolving a region in the xy -plane about a line called the axis of rotation.

3.1 Volume of Solid of Revolution about the x -axis



The region enclosed by $y = f(x)$, the x -axis and the vertical lines $x = a$ and $x = b$ is rotated about the x -axis through 2π radians (or 360°) to form a solid, called the solid of revolution.



Consider the approximation of the plane area with vertical strips of width δx . When each of these strips is rotated through 2π radians about the x -axis, they will form circular discs of volume $\pi y^2 \delta x$.

Adding the volume of all discs from $x=a$ to $x=b$ will give an approximation to the volume of the solid of revolution, i.e. $\sum_{x=a}^{x=b} \pi y^2 \delta x$.

As the thickness of the strips δx gets smaller ($\delta x \rightarrow 0$), the approximation gets better.

Thus, the volume of revolution = $\lim_{\delta x \rightarrow 0} \sum_{x=a}^{x=b} \pi y^2 \delta x$ which can be shown to be

$\pi \int_a^b y^2 dx$. Therefore

Volume of revolution about the x -axis = $\pi \int_a^b y^2 dx$.



VISUALISATION / Solid of revolution about x -axis

A YouTube video to help you visualise the generation of a solid of revolution.

<https://youtu.be/3oAjcLD34kc>



EXPLORATION / Solid of revolution about x -axis and approximations with discs

An interactive applet that helps with the visualisation of the solid of revolution. You can change the function of the curve, the angle of revolution and also the number of discs used.

<https://www.geogebra.org/m/phecwkbc>

Example 13

Find the volume of the solid formed when the area between the curve $y = x^2 + 2$ and the x -axis from $x = 1$ to $x = 3$ is rotated through 2π radians about the x -axis.

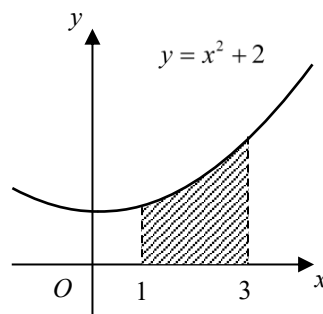
Solution

$$\text{Required volume} = \pi \int_1^3 y^2 dx$$

$$= \pi \int_1^3 x^4 + 4x^2 + 4 dx$$

$$= \pi \left[\frac{x^5}{5} + \frac{4x^3}{3} + 4x \right]_1^3$$

$$= \frac{1366\pi}{15} \text{ unit}^3$$



Example 14

Prove, using integration, that volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

Solution

A sphere of radius r is generated by the circle with equation $x^2 + y^2 = r^2$.

Required volume

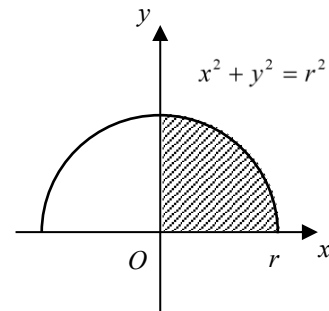
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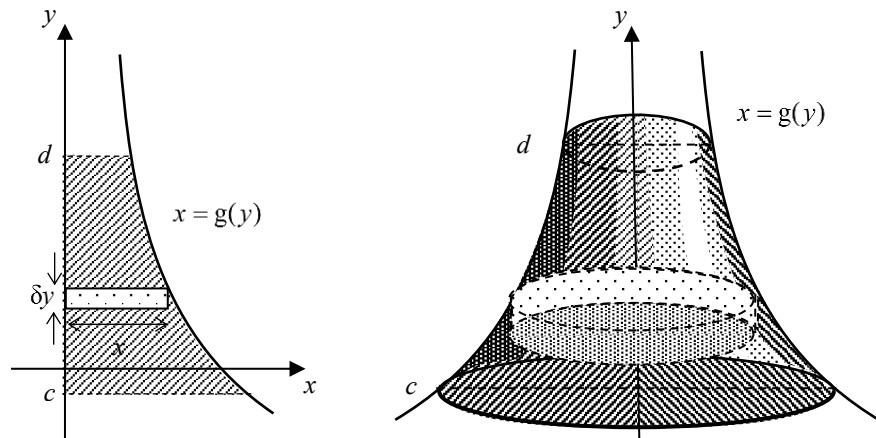
$$= \frac{4}{3}\pi r^3$$



■

3.2 Volume of Solid of Revolution about the y-axis

A region can be rotated about the y-axis through 2π radians to generate a solid of revolution.



The volume of solid formed by rotating the region enclosed by $x = g(y)$, the y-axis and the horizontal lines $y = c$ and $y = d$ through 2π radians about the y-axis, can be approximated by the many horizontal circular discs with radius x and thickness δy , i.e. $\sum_{y=c}^{y=d} \pi x^2 \delta y$.

As the thickness of the strips δy gets smaller ($\delta y \rightarrow 0$), the approximation gets better.

Thus, the volume of revolution = $\lim_{\delta y \rightarrow 0} \sum_{y=c}^{y=d} \pi x^2 \delta y$ which can be shown to be

$\pi \int_c^d x^2 dy$. Therefore

Volume of revolution about the y -axis = $\pi \int_c^d x^2 dy$.

Example 15

Find the exact volume of the solid formed when the area between the curve $y = x^2 + 2$ and the line $y = 3$ is rotated through π radians about the y -axis.

Solution

$$y = x^2 + 2 \Rightarrow x^2 =$$

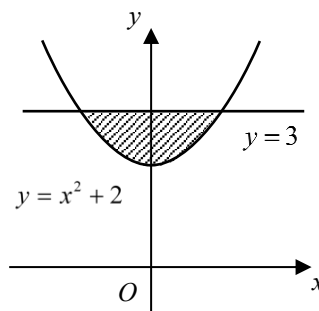
Volume of solid formed

=

=

$$= \pi \left[\frac{y^2}{2} - 2y \right]_2^3$$

$$= \frac{\pi}{2} \text{ unit}^3$$



As the region is symmetric about the y -axis, a rotation of π radians is enough to generate the

■

Example 16

A curve C has equation $y^2 = \frac{1}{4-x^2}$ for $-1 \leq x \leq 1$. The region R is

bounded by the curve C and the line $y = \frac{1}{\sqrt{3}}$.

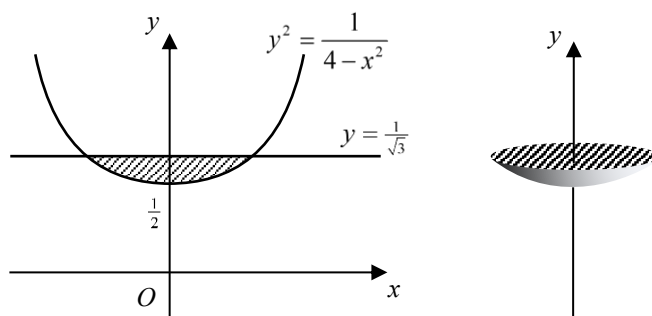
- (i) Find the volume of the solid when R is rotated π radians about the y -axis.

The region S is enclosed by the curve C , the x -axis and the lines $x = -1$ and $x = 1$.

- (ii) Find the volume generated when S is rotated through π radians about the y -axis. [N95/I/17 modified]

Solution

(i) $y^2 = \frac{1}{4-x^2} \Rightarrow x^2 = 4 - \frac{1}{y^2}$



When $x = 0$, $y = \pm \frac{1}{2}$, therefore

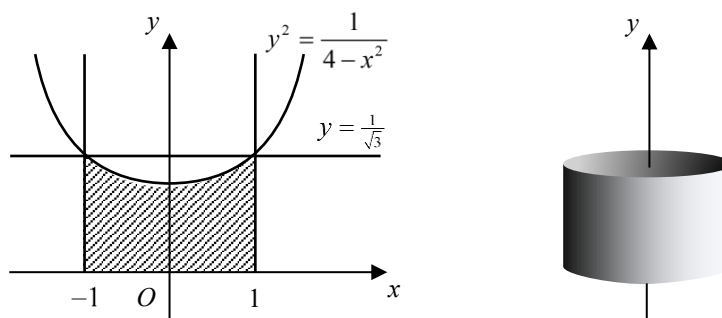
volume generated by R

=

=

= 0.130 (3 s.f.)

(ii)



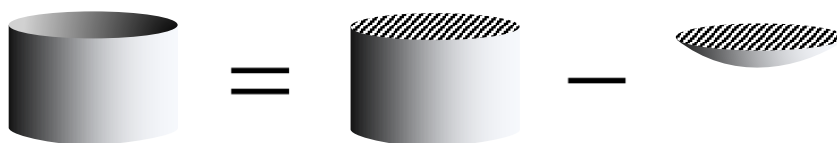
When $x = \pm 1$, $y = \pm \frac{1}{\sqrt{3}}$, therefore

Volume generated by S

= Volume of cylinder – volume generated by R

$$= \pi(1)^2 \frac{1}{\sqrt{3}} - \pi \int_{\frac{1}{2}}^{\frac{1}{\sqrt{3}}} 4 - \frac{1}{y^2} dy$$

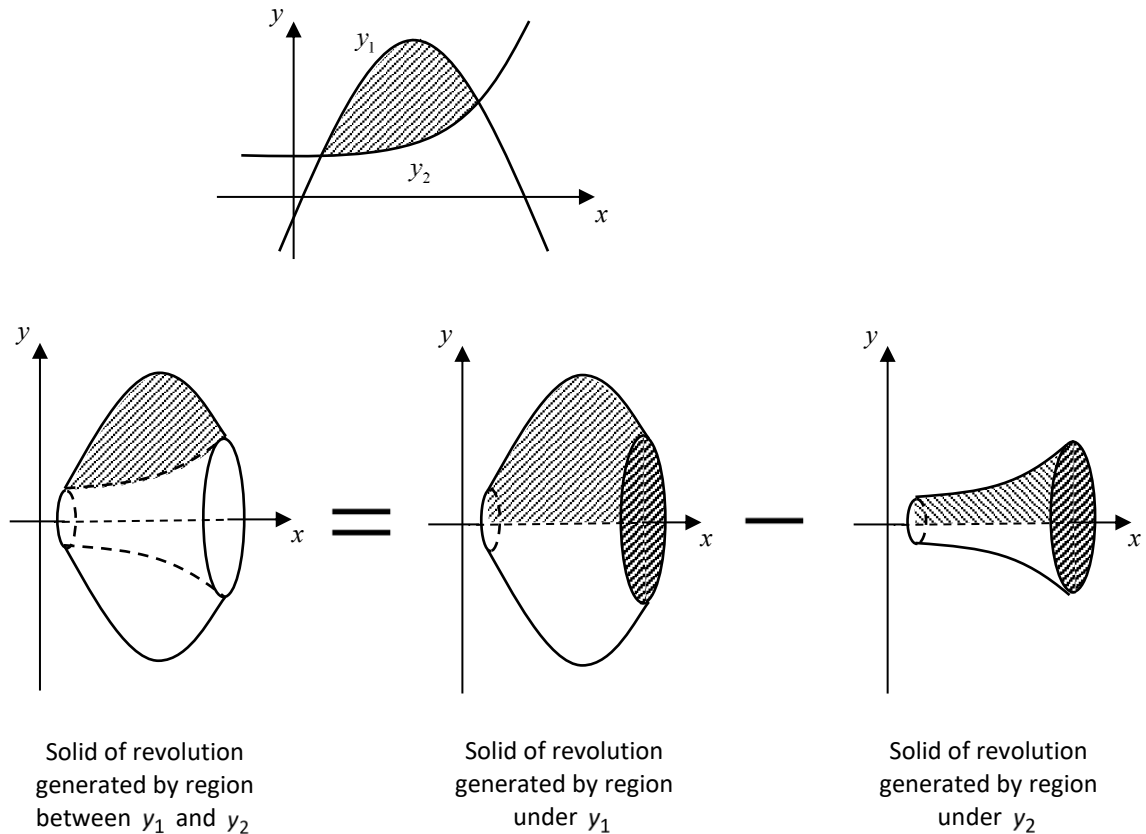
$$= 1.68 \text{ unit}^3$$



■

3.3 Volume of Solid of Revolution Generated by Region between Two Curves

When a region bounded by two curves is rotated through an axis completely, the volume of the solid formed will be that of the smaller solid formed subtracted from that of the larger solid formed. For example,



Volume of solid generated when region between two curves y_1 and y_2 is rotated about the x -axis is

$$\pi \int_a^b y_1^2 \, dx - \pi \int_a^b y_2^2 \, dx$$

which can be evaluated as $\pi \int_a^b y_1^2 - y_2^2 \, dx$.

Note

$$\pi \int_a^b (y_1^2 - y_2^2) \, dx \neq \pi \int_a^b (y_1 - y_2)^2 \, dx$$

Similarly, volume of solid generated when region between two curves is rotated about the y -axis is

$$\pi \int_c^d x_1^2 dy - \pi \int_c^d x_2^2 dy$$

which can be evaluated as $\pi \int_c^d (x_1^2 - x_2^2) dy$.

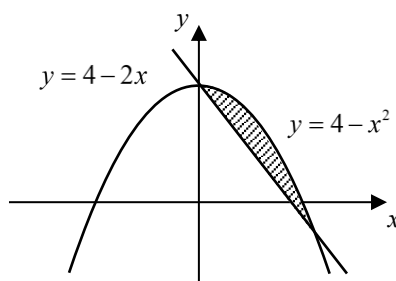


VISUALISATION / Solid of Revolution about x -axis (between two curves)

Use this applet to visualise the volume generated by region bounded by two curves.

<https://www.geogebra.org/m/CxRcJaGu>

Example 17



Find the exact volume of the solid generated when the region enclosed between the curve $y = 4 - x^2$ and the line $y = 4 - 2x$ is rotated through 2π radians about the

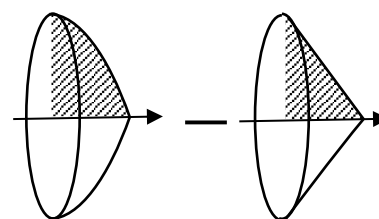
- (i) x -axis,
- (ii) y -axis.

Solution

- (i) Required volume

=

$$\begin{aligned} &= \frac{256}{15}\pi - \frac{32}{3}\pi \\ &= \frac{32}{5}\pi \text{ unit}^3 \end{aligned}$$

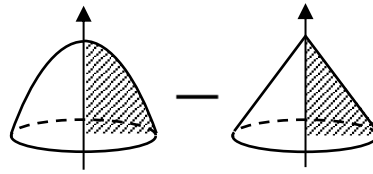


(ii) Required volume

=

$$= 8\pi - \frac{16}{3}\pi$$

$$= \frac{8}{3}\pi \text{ unit}^3$$

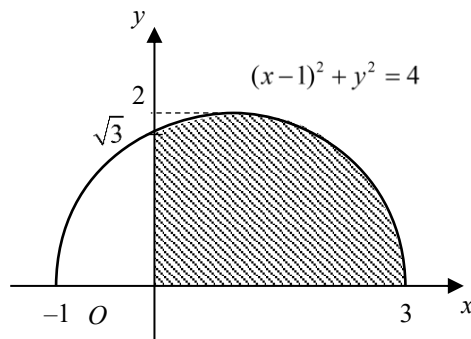


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Example 18

Find the volume of the solid generated when the region shown in the diagram is rotated 2π about the y -axis.

The region bounded by the curve $(x-1)^2 + y^2 = 4$ and the axes in the first quadrant is rotated through 2π radians about the y -axis. Find the volume of the solid generated.

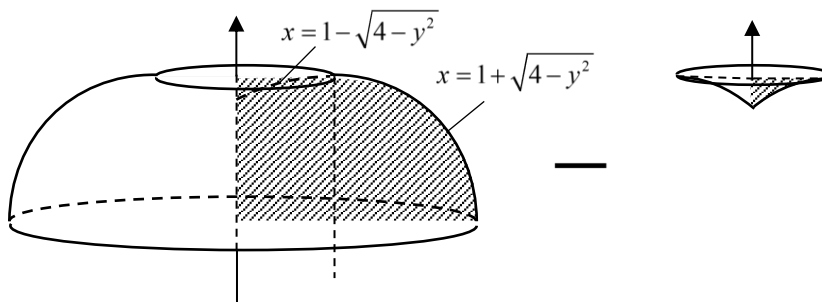
*Solution*

$$(x-1)^2 + y^2 = 4 \Rightarrow x = 1 \pm \sqrt{4-y^2}$$

Required volume

$$= \pi \int_0^2 \left(1 + \sqrt{4-y^2}\right)^2 dy - \pi \int_{\sqrt{3}}^2 \left(1 - \sqrt{4-y^2}\right)^2 dy$$

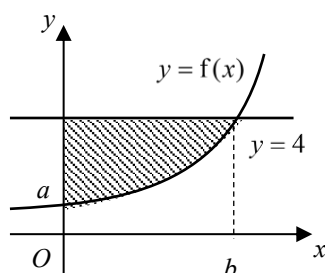
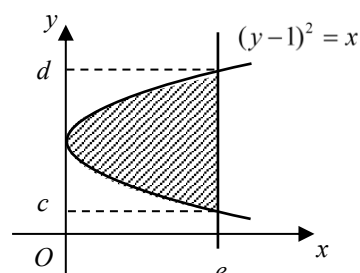
$$= 42.6 \text{ unit}^3$$



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Self-Practice 2

Set up the integral to find the volume of the solid generated when the shaded regions are rotated completely about the (i) x -axis, (ii) y -axis.

(a)**(b)****Example 19 (Revolution about Other Lines)**

The region R is bounded by the part of the curve $y = x^2 + 1$ and the line $y = 4$. Write down the equation of the curve when the graph is translated 4 units in the negative y -direction. Hence, find the volume of solid obtained when R is rotated about the line $y = 4$ through 2π radians.

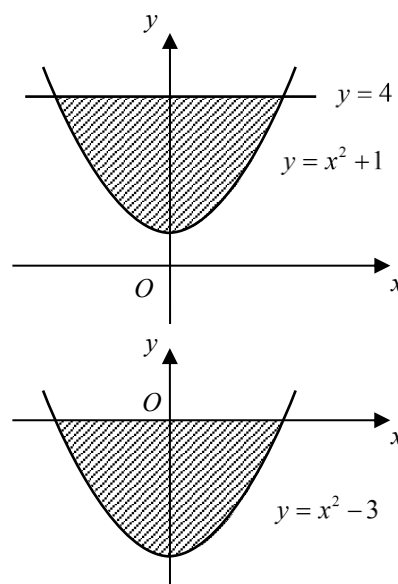
Solution

Translate graph of $y = x^2 + 1$ by 4 units in negative y -direction gives

$$y = (x^2 + 1) - 4 \Rightarrow y = x^2 - 3$$

Required volume

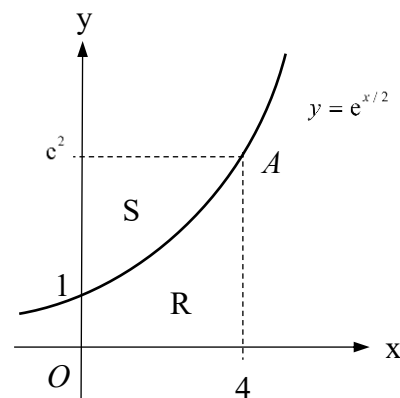
$$\begin{aligned} &= 2\pi \int_0^{\sqrt{3}} y^2 \, dx \\ &= 2\pi \int_0^{\sqrt{3}} (x^2 - 3)^2 \, dx \\ &= 52.2 \text{ unit}^3 \text{ (3 s.f.)} \end{aligned}$$

**Note**

Questions involving solids of revolution about lines other than the coordinate axes will be guided.

Example 20

The point A is on the curve $y = e^{\frac{x}{2}}$ where $x = 4$. The region R is enclosed by the curve, the line $x = 4$, the x -axis and the y -axis. The region S is enclosed by the curve $y = e^{\frac{x}{2}}$, the line $y = e^2$ and the y -axis.



- (i) Find the exact area of region R . Hence find $\int_1^{e^2} \ln y^2 \, dy$ in exact form.
- (ii) Find the volume of the solid of revolution formed when S is rotated through 2π radians about the x -axis, giving your answer in an exact form.

Solution

(i) Area $R = \int_0^4 e^{\frac{x}{2}} \, dx = \left[2e^{\frac{x}{2}} \right]_0^4 = 2(e^2 - 1) \text{ units}^2$

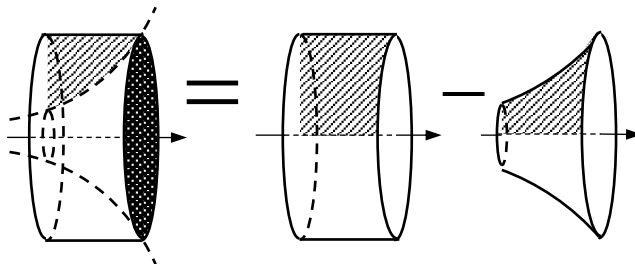
$$y = e^{\frac{x}{2}} \Rightarrow x = 2 \ln y = \ln y^2$$

Hence $\int_1^{e^2} \ln y^2 \, dy$ gives the area of S .

$$\int_1^{e^2} \ln y^2 \, dy = 4(e^2) - \text{area } R = 2(e^2 + 1)$$

- (ii) Volume of solid formed

$$\begin{aligned} &= \underbrace{\pi(e^2)^2(4)}_{\text{cylinder}} - \pi \int_0^4 \left(e^{\frac{x}{2}}\right)^2 \, dx \\ &= 4\pi e^4 - \pi \int_0^4 e^x \, dx \\ &= 4\pi e^4 - \pi \left[e^x \right]_0^4 \\ &= \pi(3e^4 + 1) \text{ units}^3 \end{aligned}$$



■

Extra Practice

Annex B: Practice Question 3

ANNEX A

Riemann and Riemann Sums

In mathematics, a Riemann sum is an approximation that takes the form $\sum f(x) \delta x$. It is named after German mathematician Bernhard Riemann. One very common application is approximating the area of functions or lines on a graph, but also the length of curves and other approximations.



Georg Friedrich Bernhard Riemann

Georg Friedrich Bernhard Riemann (German, 17 September 1826 – 20 July 1866) was a German mathematician who made contributions to analysis, number theory, and differential geometry. In the field of real analysis, he is mostly known for the first rigorous formulation of the integral, the Riemann integral, and his work on Fourier series. His contributions to complex analysis include most notably the introduction of Riemann surfaces, breaking new ground in a natural, geometric treatment of complex analysis. His famous 1859 paper on the prime-counting function, containing the original statement of the Riemann hypothesis, is regarded, although it is his only paper in the field, as one of the most influential papers in analytic number theory. Through his pioneering contributions to differential geometry, Riemann laid the foundations of the mathematics of general relativity.

[Wikipedia, The Free Encyclopedia. (2017, February 1). Bernhard Riemann. Retrieved from https://en.wikipedia.org/wiki/Bernhard_Riemann on 2017, February 6.]

ANNEX B

Practice Questions on Definite Integrals

1 Find the exact values of the integrals.

(a) $\int_{\frac{7}{2}}^4 2x + e^{2x-8} dx$

(b) $\int_0^2 \frac{20}{(5x+2)^3} dx$

(c) $\int_0^3 \frac{6}{9-2x} dx$

(d) $\int_0^2 xe^{-x} dx$

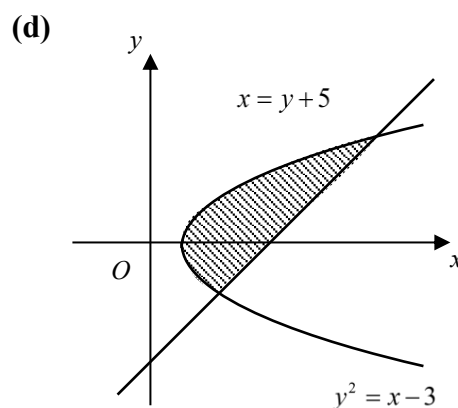
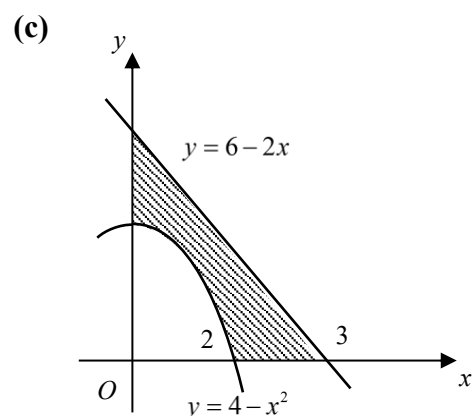
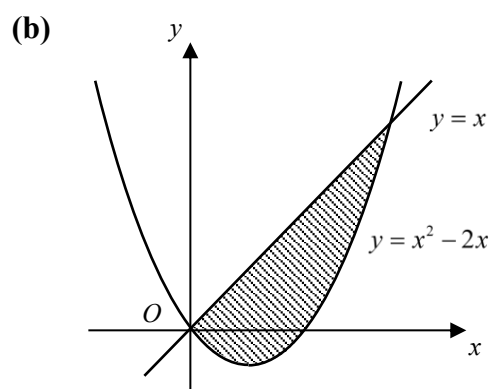
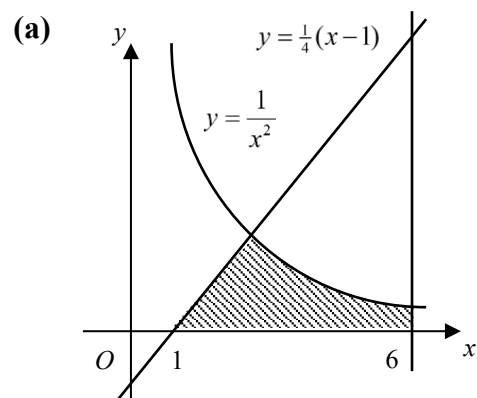
(e) $\int_1^3 \ln(x+1)^2 dx$

(f) $\int_{-1}^1 x \tan^{-1} x dx$

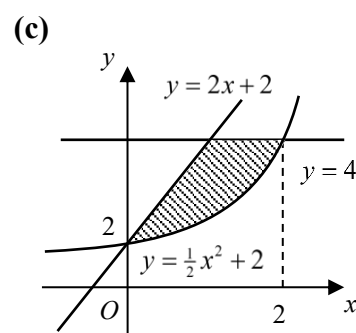
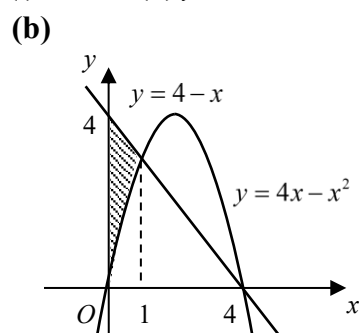
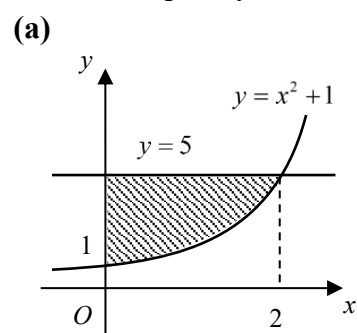
(g) $\int_{\frac{\pi}{2}}^{\pi} \sin^3 x dx$ (let $u = \cos x$)

(h) $\int_{\ln 2}^{\ln 5} \frac{2e^x}{e^{2x}-1} dx$ (let $x = \ln u$)

2 Find the exact area of the shaded regions.



3 Set up the integral to find the volume of the solid generated when the shaded regions are rotated completely about the (i) x -axis, (ii) y -axis.



Answers

1 (a) $\frac{17}{4} - \frac{1}{2e}$ (b) $\frac{35}{72}$ (c) $3\ln 3$ (d) $1 - \frac{3}{e^2}$ (e) $4(3\ln 2 - 1)$ (f) $\frac{\pi}{2} - 1$ (g) $\frac{2}{3}$ (h) $\ln 2$

2 (a) $\frac{11}{24}$ (b) 4.5 (c) $\frac{11}{3}$ (d) 4.5

3 (a) (i) $\underbrace{\pi(5)^2(2)}_{\text{cylinder}} - \pi \int_0^2 (x^2 + 1)^2 dx$ (ii) $\pi \int_1^5 (y - 1) dy$

(b) (i) $\pi \int_0^1 (4 - x)^2 - (4x - x^2)^2 dx$ (ii) $\frac{1}{3} \underbrace{\pi(1)^2(1)}_{\text{cone}} + \pi \int_0^3 (2 - \sqrt{4 - y})^2 dy$

(c) (i) $\left(\underbrace{\pi(4)^2(1)}_{\text{cylinder}} + \pi \int_0^1 4(x + 1)^2 dx \right) - \pi \int_0^2 (0.5x^2 + 2)^2 dx$

(ii) $\pi \int_0^3 4(y - 2)^2 dy - \frac{1}{3} \underbrace{\pi(1)^2(2)}_{\text{cone}}$