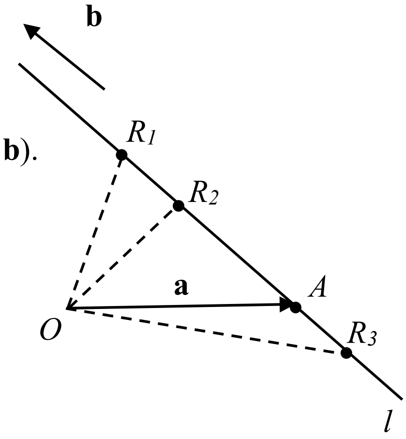
Chapter 1(b): Lines and Vector Product

1. Vector Equation of a Line

Consider the line l passing through the point A with position vector **a** and parallel to the vector **b**.

Let R_1, R_2 and R_3 be points on l.



Then we have $\overrightarrow{OR_1} = \overrightarrow{OA} + \overrightarrow{AR_1} = \mathbf{a} + \lambda_1\mathbf{b}$, for some λ_1. (since $\overrightarrow{AR_1}$ // **b**).

Similarly, we have,

$\overrightarrow{OR_2} =$, for some λ_2.

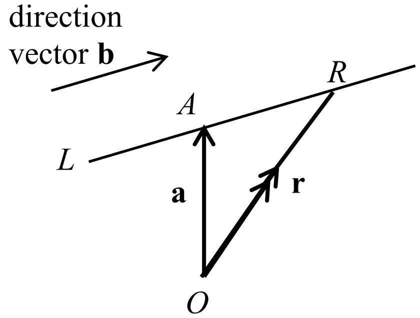
$\overrightarrow{OR_3} =$, for some λ_3.

Hence, the position vector of any point on a line can be obtained by identifying a point that the line passes through and a vector that is parallel to the line.

Therefore, the **vector equation of a line** is as follows:

Vector equation of a line:

$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, $\lambda \in \mathbb{R}$



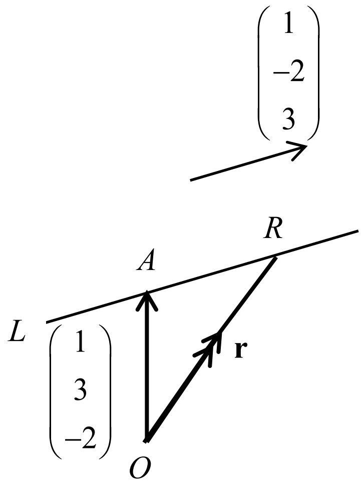
Note:

1. λ is a real value that gives a particular point on the line.
2. **r** is the position vector of a point on the line corresponding to a λ value.
3. **a** is the position vector of **any** known point on the line.
4. **b** is a vector (called direction or displacement vector) parallel to the line.
5. Vector equation of a line is <u>not unique</u> because of the different choices of **a** and **b**.

<u>Example 1</u>

Write down a vector equation of the line passing through the point (1, 3, –2) and parallel to the vector $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$.

Solution



The position vector of a point on the line is

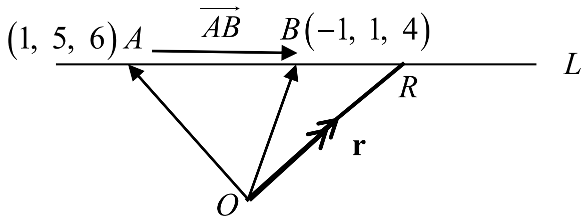
Direction vector of the line is

$\therefore$ Vector eqn of the line is

<u>Example 2</u>

Find a vector equation of the line passing through the points A(1, 5, 6) and B(–1, 1, 4).

Solution



$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix} =$

Direction vector of the line is :

The position vector of a point on the line is :

Vector eqn of the line is $\qquad\qquad\qquad\qquad \lambda \in \mathbb{R}$

<u>Note:</u> (1) The direction vector can also be found using $\overrightarrow{BA}$.
(2) Only directional vector can be reduced.

<u>Example 3</u>

The equation of a straight line is given by $\mathbf{r}=\begin{pmatrix}1\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\3\end{pmatrix}, \lambda\in\mathbb{R}$. Find the value of λ that will give the position vectors below:

(a) $\mathbf{r}=\begin{pmatrix}1\\1\\1\end{pmatrix}$, therefore $\Rightarrow$

(b) $\mathbf{r}=\begin{pmatrix}3\\5\\7\end{pmatrix}$, therefore $\begin{pmatrix}3\\5\\7\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\3\end{pmatrix}$ $\Rightarrow$

(c) $\mathbf{r}=\begin{pmatrix}0\\-1\\-2\end{pmatrix}$, therefore $\begin{pmatrix}0\\-1\\-2\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\3\end{pmatrix}$ $\Rightarrow$

<u>Note</u>: If there is a **unique** solution for λ, the point P lies on the line, otherwise the point is not on the line.

<u>Example 4</u>

Determine whether the points $P(2, 3, 4)$ and $Q(-1, -3, -6)$ lie on the line $\mathbf{r}=(\mathbf{i}+\mathbf{j}+\mathbf{k})+\lambda(\mathbf{i}+2\mathbf{j}+3\mathbf{k})$, $\lambda\in\mathbb{R}$.

Solution

Vector equation of line: $\lambda\in\mathbb{R}$

Consider point $P(2, 3, 4)$

$$\mathbf{r}=\begin{pmatrix}2\\3\\4\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\3\end{pmatrix}$$

Equate components and solve for λ in each equation:

$1+\lambda=2 \Rightarrow \lambda=1$ - (1)
$1+2\lambda=3 \Rightarrow \lambda=1$ - (2)
$1+3\lambda=4 \Rightarrow \lambda=1$ - (3)

<u>Conclusion</u>: Since an unique solution of λ is found, the point $P(2, 3, 4)$ _______on the line.

Consider Point $Q(-1, -3, -6)$ $\qquad \mathbf{r} = \begin{pmatrix} -1 \\ -3 \\ -6 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Equate components and solve for λ in each equation:

$$1 + \lambda = -1 \quad \Rightarrow$$

$$1 + 2\lambda = -3 \quad \Rightarrow$$

$$1 + 3\lambda = -6 \quad \Rightarrow$$

Conclusion: Since the solution of λ ___________, the point $Q(-1,-3,-6)$__________ on the line.

Note: All 3 equations must be used to check the uniqueness / non- uniqueness of λ. If there is a unique value of λ , the point lies on the line. Otherwise, the point is not on the line.

2. Cartesian Equation of a Line

The Cartesian equation of a line can be derived from its vector equation. Generally, from the vector equation, we have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \qquad \text{so} \qquad \begin{aligned} x &= a_1 + \lambda b_1 \\ y &= a_2 + \lambda b_2 \\ z &= a_3 + \lambda b_3 \end{aligned}$$

Making λ the subject, we have

$$\lambda = \frac{x - a_1}{b_1}, \qquad \lambda = \frac{y - a_2}{b_2}, \qquad \lambda = \frac{z - a_3}{b_3}$$

The Cartesian equation of the line is:

$$\boxed{\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}}$$

Example 5

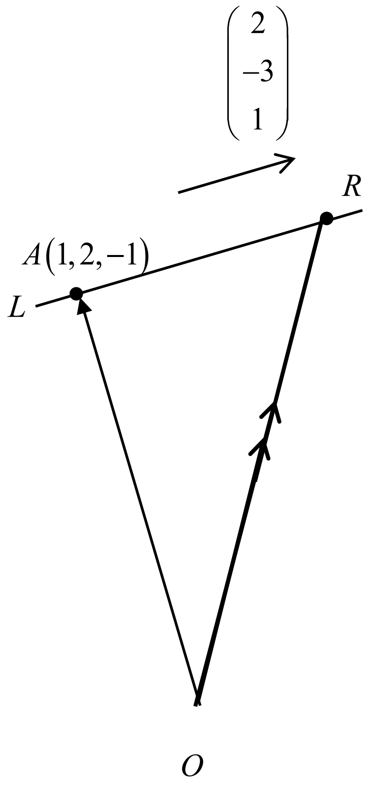
Find the Cartesian equation of the line passing through the point A with position vector $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and parallel to the vector $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$. Hence check whether the point B with position vector $\overrightarrow{OB} = \mathbf{i} + 2\mathbf{j}$ lies on the line.

Solution

Vector equation of line is:

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \qquad\qquad , \quad \lambda \in \mathbb{R} \qquad \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

Equate components:



Making λ the subject:

Cartesian equation is

Check if point $B(1, 2, 0)$ is on the line:

Now, and ,

and .

Since , the point $B(1, 2, 0)$ ____________ on the line.

<u>Example 6</u>

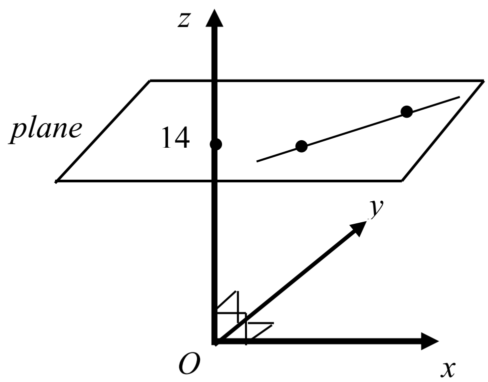
Express the vector equation $\mathbf{r} = \begin{pmatrix} 0 \\ -7 \\ 14 \end{pmatrix} + \lambda \begin{pmatrix} 12 \\ 5 \\ 0 \end{pmatrix}$, $\lambda \in \mathbb{R}$ in Cartesian form.

Solution

$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$

$\therefore$ The Cartesian equation is

Geometrically, what does the Cartesian equation in **Example 6** mean?



Example 7

Express the Cartesian equation $\frac{x+3}{4}=\frac{y-3}{2}=\frac{z}{3}$ in vector equation form.

Solution

Let

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} =$

Vector equation is

Example 8

Express $\frac{13-x}{7}=z, \ \ y=-1$ in vector equation form.

Solution

Let

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} =$

Vector equation is

Geometrical meaning of the equation of the line:

3. Relationships between Two Lines

In a 2-Dimensional situation, e.g. on the whiteboard, any two straight lines drawn will either

(i) intersect at 1 point (the two lines are not parallel).

(ii) do not intersect (the two lines are parallel but not the same line).

There is a case where the two lines intersect infinitely many times. This is when the two lines are the same.

Give a geometrical interpretation when two lines are parallel and intersect.

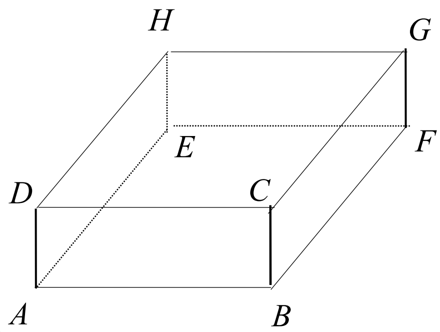
However, in a **3-Dimensional** situation, any two straight lines may

(a) be parallel and do not intersect;

(b) not be parallel and intersect; or

(c) not be parallel and do not intersect. Such lines are called *skew lines*.

Let see the case as in a box:
AD and BC are parallel.
AD and DC intersect at D.
AD and BF are ***skew*** lines since they are not parallel and do not intersect.
Extend the line segments into straight lines, and the results still hold.



Consider 2 lines L_1 and L_2 where L_1 : $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1,\ \lambda \in \mathbb{R}$ and L_2 : $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}_2,\ \mu \in \mathbb{R}$.

(a) If the two lines are parallel, then $\mathbf{b}_1 // \mathbf{b}_2$.

(b) If the two lines are not parallel and intersect, there are unique values of λ and μ such that

$$\mathbf{a}_1 + \lambda\mathbf{b}_1 = \mathbf{a}_2 + \mu\mathbf{b}_2$$

(c) If the two lines are not parallel and no unique values of λ and μ can be found, then the lines are a pair of skew lines.

Note :
If two lines are parallel and do not intersect or two lines are not parallel and intersect at a point, then the two lines lie on the same plane. The lines are said to be coplanar. Conversely, **two lines that are not coplanar are skew lines.**

Example 9

Determine whether the following pair of lines are parallel, intersecting or skew. If the lines intersect, find the position vector of their point of intersection.

$l_1 : \mathbf{r} = \mathbf{i} - \mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k}),\ \lambda \in \mathbb{R}$, $l_2 : \mathbf{r} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} + 3\mathbf{k}),\ \mu \in \mathbb{R}$.

Solution

Check if direction vectors are parallel

Since , the lines are _______________.

Suppose the 2 lines intersect at a point.

Write LHS & RHS in column vectors

Equate components: (3 equations in λ & μ)

Solve any two equations to obtain values of λ and μ

(1) + (2), $0 = 6 + 3\mu \quad \Rightarrow \mu = -2$

Subst into (1), $1 + \lambda = 2 + 2(-2) \Rightarrow \lambda = -3$

Test that the values of λ and μ satisfy remaining equation

Substituting to (3):

$\therefore \lambda = -3$ and $\mu = -2$ satisfy all the 3 equations.

Conclusion

The lines _____________.

The position vector of the point of intersection is

Example 10

Determine whether the following pair of lines are parallel, intersecting or skew. If the lines intersect, find the position vector of their point of intersection.

$$l_1 : \frac{x-1}{2} = \frac{y+5}{-3} = \frac{z+3}{4}, \quad l_2 : \mathbf{r} = \begin{pmatrix} -1 \\ 5 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}, \quad \mu \in \mathbb{R}$$

Solution

l_1 :

$$l_2 : \mathbf{r} = \begin{pmatrix} -1 \\ 5 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}, \quad \mu \in \mathbb{R}$$

Since   , the lines are not parallel.

Suppose the 2 lines intersect at a point.

$$\begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$$

$1 + 2\lambda = -1 + 2\mu \Rightarrow \lambda - \mu = -1$ - (1)

$-5 - 3\lambda = 5 - 3\mu \Rightarrow -3\lambda + 3\mu = 10$ - (2)

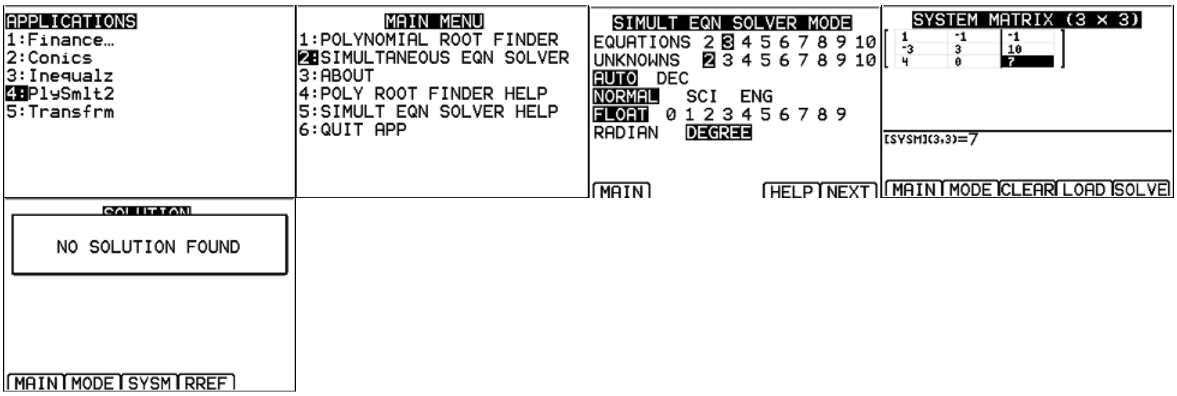
$-3 + 4\lambda = 4 \Rightarrow 4\lambda = 7$ - (3)

Using GC, ________________.

Since there are ____.

Hence, the lines are _________.

Note: All 3 equations must be used to check the uniqueness / non-uniqueness of λ and μ.



4. Acute Angle Between Two Lines

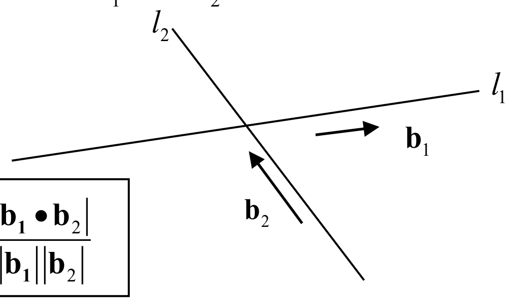
Consider the vector equations of two lines:

$$l_1 : \mathbf{r} = \mathbf{a_1} + \lambda \mathbf{b_1},\ \lambda \in \mathbb{R} \quad \text{and} \quad l_2 : \mathbf{r} = \mathbf{a_2} + \mu \mathbf{b_2},\ \mu \in \mathbb{R}.$$

Since $\mathbf{b}_1$ and $\mathbf{b}_2$ are the direction vectors of the 2 lines,

the angle between l_1 and l_2 is the same as the angle between $\mathbf{b}_1$ and $\mathbf{b}_2$

$$\Rightarrow \cos\theta = \frac{\mathbf{b_1} \bullet \mathbf{b}_2}{|\mathbf{b_1}||\mathbf{b}_2|}$$



Hence, to find the acute angle between l_1 and l_2 $\Rightarrow$ $\boxed{\cos\theta = \dfrac{|\mathbf{b_1} \bullet \mathbf{b}_2|}{|\mathbf{b_1}||\mathbf{b}_2|}}$

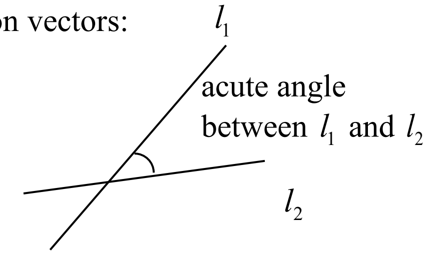
<u>Note</u>: By convention, we normally find the acute angle between 2 lines.

<u>Example 11</u>

Find the acute angle between the lines $\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\lambda \in \mathbb{R}$ and $\mathbf{r} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$ $\mu \in \mathbb{R}$.

Solution

Use the scalar product to find the acute angle between the two direction vectors:



The acute angle = $\theta = 78.5°$ (to 1 d.p.)

Example 12

Prove that the lines $\mathbf{r} = (1+2\lambda)\mathbf{i} + (1-\lambda)\mathbf{j} + (1-\lambda)\mathbf{k}$, $\lambda \in \mathbb{R}$ and $\mathbf{r} = (3\mathbf{i} - \mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})$, $\mu \in \mathbb{R}$ are perpendicular.

Solution
[Write the vector equation of lines:]

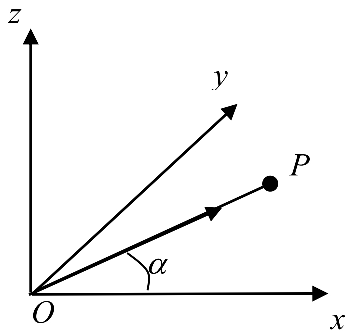
l_1: , $\lambda \in \mathbb{R}$ l_2: , $\mu \in \mathbb{R}$

[Prove that the scalar product of the direction vectors is zero:]

Since the scalar product of the direction vectors is zero, the 2 lines are perpendicular.

5. Angle between a vector and the positive *x*-, *y*-, *z*- axes

Consider a vector $\overrightarrow{OP} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$. The angle between $\overrightarrow{OP}$ and the positive *x*-axis can be found using the scalar product of $\overrightarrow{OP}$ and $\mathbf{i}$:



$$\overrightarrow{OP} \bullet \mathbf{i} = \left|\overrightarrow{OP}\right|\left|\mathbf{i}\right|\cos\alpha$$

$$\begin{pmatrix}2\\3\\1\end{pmatrix} \bullet \begin{pmatrix}1\\0\\0\end{pmatrix} = \sqrt{14}(1)\cos\alpha$$

$$\cos\alpha = \frac{2}{\sqrt{14}} \quad \Rightarrow \quad \alpha = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) = 57.7°$$

Similarly, the angle between $\overrightarrow{OP}$ and the positive *y*-axis, β, can be found using the scalar product $\overrightarrow{OP} \bullet \mathbf{j} = \left|\overrightarrow{OP}\right|\left|\mathbf{j}\right|\cos\beta$ and the angle between $\overrightarrow{OP}$ and the positive *z*-axis, γ, can be found using the scalar product $\overrightarrow{OP} \bullet \mathbf{k} = \left|\overrightarrow{OP}\right|\left|\mathbf{k}\right|\cos\gamma$, which are left as exercises.

Example 13

Find the angles between **a** = 2**i** + 4**j** – 2**k** and the positive x, y, z- axis.

Solution

Let α, β and γ be the angles between **a** and positive x, y, z- axis be respectively.

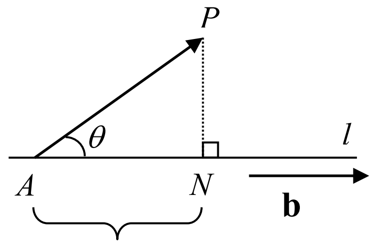
$\alpha =$ $= 65.9°$

$\beta =$ $= 35.3°$

$\gamma =$ $= 114.1°$

6. Length of Projection of a Vector Onto Another Line

> Length of projection of $\overrightarrow{AP}$ onto the line $l = \left|\overrightarrow{AP} \bullet \mathbf{b}\right|$,
> where $\mathbf{b}$ is the direction vector of l.



Projection of $\overrightarrow{AP}$ onto $\mathbf{b}$

Proof:

Length of projection of $\overrightarrow{AP}$ onto the line l

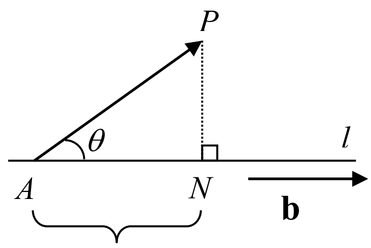
$$\begin{aligned} &= AN \\ &= \left|\left|\overrightarrow{AP}\right|\cos\theta\right| \\ &= \left|\left|\overrightarrow{AP}\right|\frac{\overrightarrow{AP} \bullet \mathbf{b}}{\left|\overrightarrow{AP}\right|\left|\mathbf{b}\right|}\right| \\ &= \left|\overrightarrow{AP} \bullet \mathbf{b}\right| \end{aligned}$$

(by scalar product $\overrightarrow{AP} \bullet \mathbf{b} = \left|\overrightarrow{AP}\right|\left|\mathbf{b}\right|\cos\theta$)

Example 14

Find the length of projection of AP onto the line L: $\mathbf{r} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\2\end{pmatrix}$, $\lambda \in \mathbb{R}$ where the position vector of A and P are $\mathbf{i} + \mathbf{j}$ and $3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ respectively. Hence find the shortest distance from point P to line L.

Solution



Projection of $\overrightarrow{AP}$ onto $\mathbf{b}$

$$\overrightarrow{AP} = \begin{pmatrix}3\\2\\6\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\6\end{pmatrix}$$

Length of projection of $\overrightarrow{AP}$ onto the line $L = \left|\overrightarrow{AP} \bullet \hat{\mathbf{b}}\right|$

$=$

$\left|\overrightarrow{AP}\right| = \sqrt{2^2 + 1^2 + 6^2} = \sqrt{41}$

By Pythagoras' Theorem, shortest distance from point P to line $L =$

7. Foot of Perpendicular and Perpendicular Distance from a Point to a Straight Line

Example 15

Find the position vector of the foot of perpendicular, N, from the point P (3, 2, 6) to the line

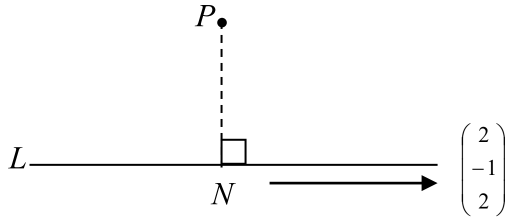
L: $\mathbf{r}=\begin{pmatrix}1\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\2\end{pmatrix}$, $\lambda\in\mathbb{R}$. Deduce the perpendicular distance PN.

Solution

Idea: Let N be the foot of the perpendicular from P to L.

$\overrightarrow{ON}$ satisfies equation of L since N lies on the line L.

Since $\overrightarrow{PN}\perp\begin{pmatrix}2\\-1\\2\end{pmatrix}$, use the fact that $\overrightarrow{PN}\bullet\begin{pmatrix}2\\-1\\2\end{pmatrix}=0$



to form an equation for λ and solve for it.

With the value of λ, obtain the position vector of N.

Since the foot of the perpendicular N is a point on line L,

N lies on the line:

for a value of λ.

Find $\overrightarrow{PN}$ in terms of λ:

Therefore, $\overrightarrow{PN}=\overrightarrow{ON}-\overrightarrow{OP}=\begin{pmatrix}1\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\2\end{pmatrix}-\begin{pmatrix}3\\2\\6\end{pmatrix}$

$\overrightarrow{PN}\perp$ line Use scalar product to solve for λ :

Since $\overrightarrow{PN}\bullet\begin{pmatrix}2\\-1\\2\end{pmatrix}=0$, we have

The position vector of the foot of perpendicular N is

$$\overrightarrow{PN} = \frac{1}{3}\begin{pmatrix} 13 \\ -2 \\ 10 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 13-9 \\ -2-6 \\ 10-18 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 4 \\ -8 \\ -8 \end{pmatrix}.$$

The perpendicular distance, $\left|\overrightarrow{PN}\right| =$

Note: (1) The perpendicular distance is also the shortest distance from a point to a line.

(2) Compare Examples 14 and 15. If the question did not ask for the foot of the perpendicular, then the method used in Example 14 is the shorter one.

Example 16

The position vector of point A is $\begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}$ and the equation of line l is $\mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ $\lambda \in \mathbb{R}$.

Find

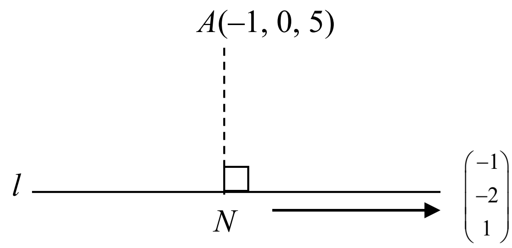
(i) the position vector of N, the foot of perpendicular from A to the line l;

(ii) the perpendicular distance from A to the line l;

(iii) the position vector of the reflection of A in the line l.

Solution

(i) Since N lies on l,



for a value of λ

$$\overrightarrow{AN} = \overrightarrow{ON} - \overrightarrow{OA} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}$$

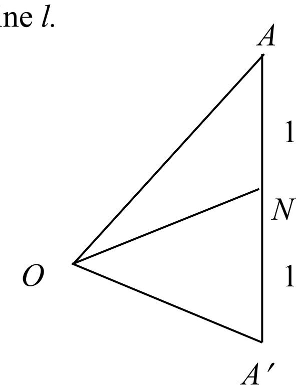
Since $\overrightarrow{AN} \perp l$,

(ii) $\overrightarrow{AN} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}$

Perpendicular distance from A to the line l =

(iii) Let A' be the reflection of A in the line l.

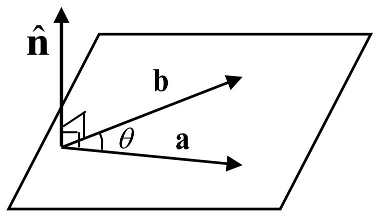
Since N is the midpoint of AA',



8. Vector Product (Cross Product)

The **vector product (cross product)** of two vectors **a** and **b** whose directions are inclined at an angle θ to each other is defined as:

$$\boxed{\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}}$$



where $0^\circ \le \theta \le 180^\circ$ and

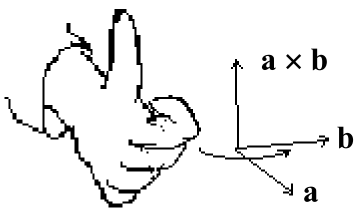
$\hat{\mathbf{n}}$ is the unit vector perpendicular to the plane parallel to both **a** and **b**.

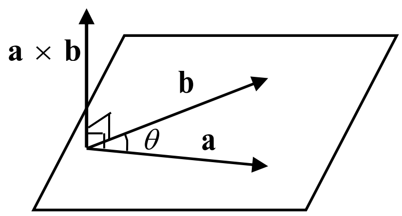
($\hat{\mathbf{n}}$ is a normal to the plane parallel to both **a** and **b**.)

Note:

(1) The resultant of a vector product is a *vector*.

(2) The **direction of a vector product is perpendicular to the directions of both a and b** in the sense of using a *right- hand screw*.

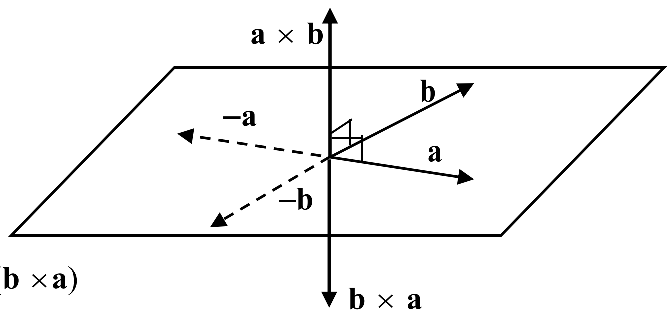




Basic Properties of a Vector Product

(i) $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$

In fact, $\mathbf{a} \times \mathbf{b} = (-\mathbf{b}) \times \mathbf{a}$
$= \mathbf{b} \times (-\mathbf{a})$
$= -(\mathbf{b} \times \mathbf{a})$



Note : $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$ but $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$

(ii) $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$ since $|\hat{\mathbf{n}}| = 1$

(iii) $\mathbf{a} \times (\mathbf{b} \pm \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \pm (\mathbf{a} \times \mathbf{c})$

(iv) $\lambda(\mathbf{a}\times\mathbf{b}) = \lambda\mathbf{a}\times\mathbf{b}$ or $\lambda(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times\lambda\mathbf{b}$

***Important Results**

(i) Parallel Vectors

For any two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$,

if vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel, then θ =___________, $\sin\theta$ =_____ $\Rightarrow$ $\mathbf{a} \times \mathbf{b}$ = _____

Conversely, if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, and $\mathbf{a} \neq \mathbf{0}$, $\mathbf{b} \neq \mathbf{0}$, then $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors.

$$\boxed{\text{i.e. } \mathbf{a} // \mathbf{b} \iff \mathbf{a} \times \mathbf{b} = \mathbf{0}}$$

Recall the alternative way to check that two vectors are parallel:

$$\boxed{\mathbf{a} // \mathbf{b} \iff \mathbf{a} = \lambda\mathbf{b} \text{ for some } \lambda \in \mathbb{R},\ \lambda \neq 0.}$$

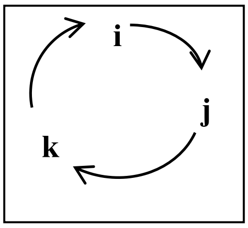
Note: $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$

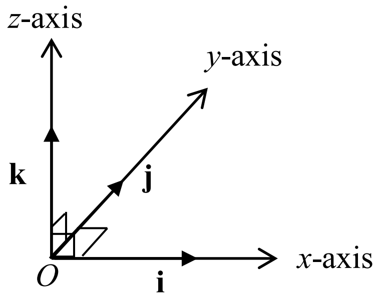
(ii) Mutually Perpendicular Vectors

As the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ form a mutually exclusive perpendicular right-handed system, it follows that:

$\mathbf{i} \times \mathbf{j} = \mathbf{k}$ $\qquad$ $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ $\qquad$ $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

$\mathbf{i} \times \mathbf{k} = -\mathbf{j}$ $\qquad$ $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$ $\qquad$ $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$





Comparison of Key Results for Scalar and Vector Product

For non-zero vectors $\mathbf{a}$ and $\mathbf{b}$, and $\lambda \in \mathbb{R}$

Scalar (Dot) Product	Vector (Cross) Product
$\mathbf{a} \bullet \mathbf{b} = 0 \iff \mathbf{a} \perp \mathbf{b}$	$\mathbf{a} \times \mathbf{b} = \mathbf{0} \iff \mathbf{a} // \mathbf{b}$
$\mathbf{a} \bullet \mathbf{a} = \lvert\mathbf{a}\rvert^2$	$\mathbf{a} \times \mathbf{a} = \mathbf{0}$
$\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$	$\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$
$\lambda(\mathbf{a} \bullet \mathbf{b}) = \lambda\mathbf{a} \bullet \mathbf{b} = \mathbf{a} \bullet \lambda\mathbf{b}$	$\lambda(\mathbf{a} \times \mathbf{b}) = \lambda\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \lambda\mathbf{b}$

What is the difference between $\mathbf{a} \bullet \mathbf{0}$ and $\mathbf{a} \times \mathbf{0}$?

Example 17

Find $\left|\mathbf{a}\times 3(\mathbf{b}+\mathbf{c})\right|$, given that that **a** is parallel to **b** and perpendicular to **c**, $|\mathbf{a}|=4$ and $|\mathbf{c}|=5$.

Solution

$\left|\mathbf{a}\times 3(\mathbf{b}+\mathbf{c})\right|$

since

$=3|\mathbf{a}\times\mathbf{c}|$

$=$ since

$=$ since

$=3(4)(5)$

$=60$

To calculate Vector Product

Consider the vectors $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$.

This is available in MF27.

Note!

$$\mathbf{a}\times\mathbf{b}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=\begin{pmatrix}a_2b_3-a_3b_2\\-\left(a_1b_3-a_3b_1\right)\\a_1b_2-a_2b_1\end{pmatrix}=\begin{pmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{pmatrix}$$

It is easier to calculate the cross product by writing it as follows:

$$\mathbf{a}\times\mathbf{b}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$$

This step is done mentally by "covering" one row each time

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}\qquad\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}\qquad\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$$

$$=\left(a_2b_3-a_3b_2\right)\mathbf{i}\ -\ \left(a_1b_3-a_3b_1\right)\mathbf{j}\ +\ \left(a_1b_2-a_2b_1\right)\mathbf{k}$$

<u>Proof (Read it yourself)</u>:

$$\mathbf{a}\times\mathbf{b}=\left(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}\right)\times\left(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}\right)$$

$$\begin{aligned}&=a_1b_1(\mathbf{i}\times\mathbf{i})+a_1b_2(\mathbf{i}\times\mathbf{j})+a_1b_3(\mathbf{i}\times\mathbf{k})\\&+a_2b_1(\mathbf{j}\times\mathbf{i})+a_2b_2(\mathbf{j}\times\mathbf{j})+a_2b_3(\mathbf{j}\times\mathbf{k})\\&+a_3b_1(\mathbf{k}\times\mathbf{i})+a_3b_2(\mathbf{k}\times\mathbf{j})+a_3b_3(\mathbf{k}\times\mathbf{k})\end{aligned}$$

$$= a_1b_2\,\mathbf{k}+a_1b_3\,(-\mathbf{j})+a_2b_1\,(-\mathbf{k})+a_2b_3\,\mathbf{i}+a_3b_1\,\mathbf{j}+a_3b_2\,(-\mathbf{i})$$

$$=\left(a_2b_3-a_3b_2\right)\mathbf{i}-\left(a_1b_3-a_3b_1\right)\mathbf{j}+\left(a_1b_2-a_2b_1\right)\mathbf{k}$$

(since $\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=\mathbf{0}$)

<u>Example 18</u>

If $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, find $\mathbf{a}\times\mathbf{b}$.

Solution

$$\mathbf{a}\times\mathbf{b}=\begin{pmatrix}2\\3\\1\end{pmatrix}\times\begin{pmatrix}3\\-1\\2\end{pmatrix}=$$

<u>Note</u>: Checking correctness of vector product using scalar product i.e.

Check that and

<u>Example 19</u>

Find the unit vectors which are perpendicular to both the vectors $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + \mathbf{j} - \mathbf{k}$.

Solution

$$\mathbf{a}\times\mathbf{b}=\begin{pmatrix}2\\2\\-1\end{pmatrix}\times\begin{pmatrix}4\\1\\-1\end{pmatrix}=$$

$$|\mathbf{a}\times\mathbf{b}|=|\mathbf{b}\times\mathbf{a}|=\sqrt{41}$$

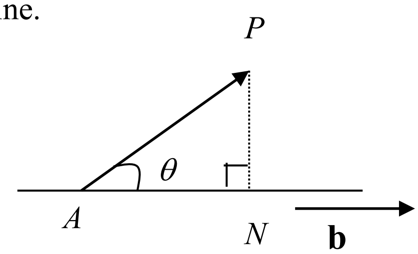
Unit vectors are

9. Applications of Vector Product

9.1 Perpendicular Distance from a Point to a Straight Line/Vector (Shortest Method)

Consider a point P and the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ (i.e. line passing through A and parallel to the vector $\mathbf{b}$).

Let N be the foot of the perpendicular from P to the line.



Perpendicular distance of P to the line

$= PN = \left|\overrightarrow{AP}\right|\sin\theta$

From the definition of vector product,

$\overrightarrow{AP}\times\mathbf{b} = \left|\overrightarrow{AP}\right|\left|\mathbf{b}\right|\sin\theta\,\mathbf{n}$

$\left|\overrightarrow{AP}\times\mathbf{b}\right| = \left|\overrightarrow{AP}\right|\left|\mathbf{b}\right|\sin\theta$ since $\left|\mathbf{n}\right| = 1$

$\left|\overrightarrow{AP}\right|\sin\theta = \dfrac{\left|\overrightarrow{AP}\times\mathbf{b}\right|}{\left|\mathbf{b}\right|}$

Recall that
$\sin\theta = \dfrac{opposite}{hypotenus}$

$\therefore$ Perpendicular distance of P to the line with direction vector $\mathbf{b}$ $= \dfrac{\left|\overrightarrow{AP}\times\mathbf{b}\right|}{\left|\mathbf{b}\right|} = \left|\overrightarrow{AP}\times\hat{\mathbf{b}}\right|$

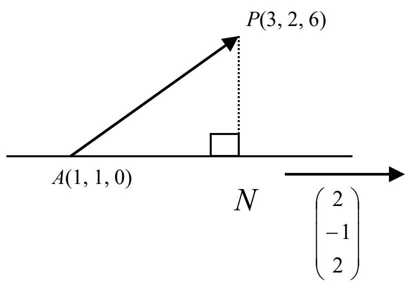
Example 20

Find the perpendicular distance from the point $P(3, 2, 6)$ to the line $\mathbf{r} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\2\end{pmatrix}$, $\lambda \in \mathbb{R}$.

Solution

$\overrightarrow{AP} =$

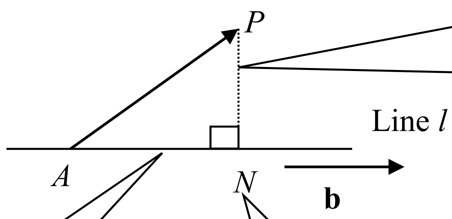
$\hat{\mathbf{b}} =$



Perpendicular distance of P to line
$= \left|\overrightarrow{PN}\right|$

Comparing Examples 14, 15 and 20, what are the methods to find the perpendicular distance from a point to a line?

<u>Summary (3 Methods to find perpendicular distance from a point to a line)</u>



Method 1 (Example 14)

- Find AN, the length of projection of $\overrightarrow{AP}$ onto line l using dot product.
- Then use Pythagoras' Theorem to find PN.

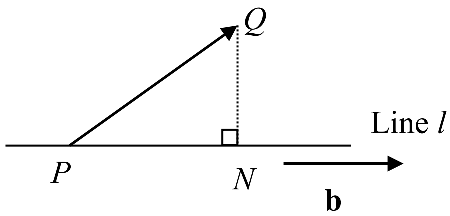
Method 2 (Example 15)

- Find $\overrightarrow{ON}$, the position vector of foot of perpendicular from P to line l.
- Then find $\left|\overrightarrow{PN}\right|$.

Method 3 (Example 20)

- Find $\left|\overrightarrow{PN}\right|$ using cross product.

<u>Important Note</u>:



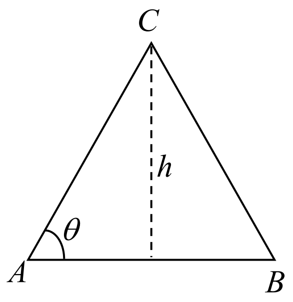
(I) Length of PN $= \left|\overrightarrow{PQ} \bullet \mathbf{b}\right|$ (which is also the length of projection of $\overrightarrow{PQ}$ onto line l.)

(II) Length of QN $= \left|\overrightarrow{PQ} \times \mathbf{b}\right|$ (which is the perpendicular distance from Q to line l.)

9.2 Area of Triangle

(Approach 1)

Area of triangle $ABC = \frac{1}{2}\left|\overrightarrow{AB}\right| h$



h = perpendicular distance from point C to AB $= \left|\overrightarrow{AC} \times \overrightarrow{AB}\right|$

Area of triangle $ABC = \frac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC} \times \overrightarrow{AB}\right|$

$= \frac{1}{2}\left|\overrightarrow{AB}\right| \frac{\left|\overrightarrow{AC} \times \overrightarrow{AB}\right|}{\left|\overrightarrow{AB}\right|}$

$$\boxed{\text{Area of triangle } ABC = \frac{1}{2}\left|\overrightarrow{AC} \times \overrightarrow{AB}\right|}$$

(Approach 2)

Area of triangle $ABC = \frac{1}{2}\left|\overrightarrow{AC}\right|\left|\overrightarrow{AB}\right| \sin\theta$ ----- (*)

From the definition of vector product,

$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \hat{\mathbf{n}}$

$\overrightarrow{AC} \times \overrightarrow{AB} = \left|\overrightarrow{AC}\right|\left|\overrightarrow{AB}\right| \sin\theta \, \mathbf{n}$

$\left|\overrightarrow{AC} \times \overrightarrow{AB}\right| = \left|\overrightarrow{AC}\right|\left|\overrightarrow{AB}\right| \sin\theta$ since $|\mathbf{n}| = 1$

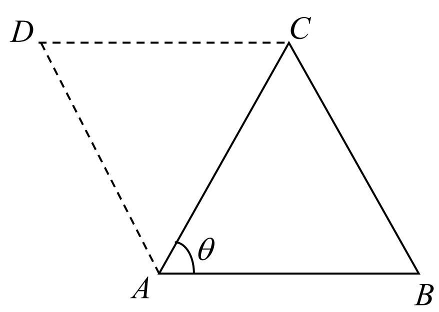
Substitute $\left|\overrightarrow{AC}\right|\left|\overrightarrow{AB}\right| \sin\theta = \left|\overrightarrow{AC} \times \overrightarrow{AB}\right|$ into (*)

Area of triangle $ABC = \frac{1}{2}\left|\overrightarrow{AC} \times \overrightarrow{AB}\right|$

9.3 Area of Parallelogram

Area of parallelogram $ABCD$ = 2 times of area of triangle ABC

$= 2\left(\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|\right)$



$$\boxed{\text{Area of parallelogram } ABCD = \left|\overrightarrow{AB} \times \overrightarrow{AC}\right|}$$

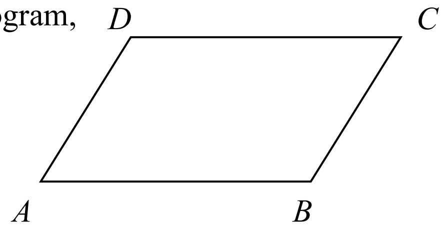
<u>Example 21</u>

The position vectors of A, B and C are given by $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{c} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ respectively. Find

(i) the position vector of the point D such that $ABCD$ is a parallelogram,

(ii) the area of parallelogram $ABCD$, and

(iii) the perpendicular distance from D to AB.



Solution

(i) Since $ABCD$ is a parallelogram,

$$\therefore \ \overrightarrow{OD} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 0 \end{pmatrix}$$

(ii) Area of $ABCD$ =

$\overrightarrow{AD} \times \overrightarrow{AB} =$

Area of $ABCD = \left|\overrightarrow{AD} \times \overrightarrow{AB}\right| = 9$ units^2

(iii) Perpendicular distance from D to AB