3

EQUATIONS AND INEQUALITIES

SYLLABUS

- Formulate an equation, a system of linear equations, or inequalities from a problem situation
- Solve an equation exactly or approximately using a graphing calculator or a graphing software
- Solve a system of linear equations using a graphing calculator or a graphing software
- Solve inequalities of the form $\frac{\mathrm{f}(x)}{\mathrm{g}(x)} > 0$ where $\mathrm{f}(x)$ and $\mathrm{g}(x)$ are linear expressions or quadratic expressions
- Concept of $|x|$, and use of relations $|x-a| < b \Leftrightarrow a-b < x < a+b$ and $|x-a| > b \Leftrightarrow x < a-b$ or $x > a+b$
- Solve inequalities by graphical methods

CONTENTS

LECTURE 1

Lesson Outline

- Solving an equation exactly or approximately using a graphing calculator
- Solving a system of linear equations using a graphing calculator
- Formulation and solution of equations or a system of linear equations from problem situations

---

1 EQUATIONS

1.1 Exact or Approximate Solutions of Equations

Example 1

Solve the following equations **exactly**:

(a) $2x^3 - 9x^2 + 10x - 3 = 0$

(b) $\cot\theta = 2\tan\theta + 1 \left(0 \le \theta \le 2\pi\right)$

(c) $2e^x = 3 - e^{x+1}$

Solution

(a)

$$2x^3 - 9x^2 + 10x - 3 = 0$$
$$\Rightarrow (x-1)\left(2x^2 - 7x + 3\right) = 0$$
$$\Rightarrow (x-1)(2x-1)(x-3) = 0$$
$$\therefore x = 1 \text{ or } \frac{1}{2} \text{ or } 3$$

The solutions should show the factorisation because exact solutions are required

(b)

$$\cot\theta = 2\tan\theta + 1$$
$$\frac{1}{\tan\theta} = 2\tan\theta + 1$$
$$1 = 2\tan^2\theta + \tan\theta$$
$$2\tan^2\theta + \tan\theta - 1 = 0$$
$$(2\tan\theta - 1)(\tan\theta + 1) = 0$$
$$\tan\theta = \frac{1}{2} \text{ or } \tan\theta = -1$$
$$\theta = \tan^{-1}\frac{1}{2}, \pi + \tan^{-1}\frac{1}{2},$$
$$\frac{3\pi}{4} \text{ or } \frac{7\pi}{4}$$

(c)

$$2e^x = 3 - e^{x+1}$$
$$2e^x + e^{x+1} = 3$$
$$e^x(2 + e) = 3$$
$$e^x = \frac{3}{2+e}$$
$$x = \ln\left(\frac{3}{2+e}\right)$$

■

Example 2

Solve the following equations:

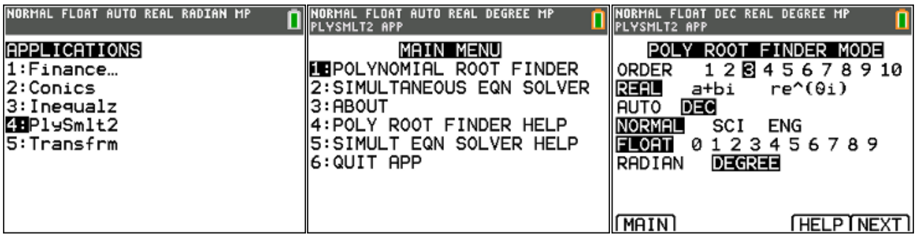
(a) $x^3+200x-4000=0$ **(b)** $\mathrm{e}^{\frac{x}{2}}=x^2$

(c) $x-\ln\left(x^7+1\right)=0$

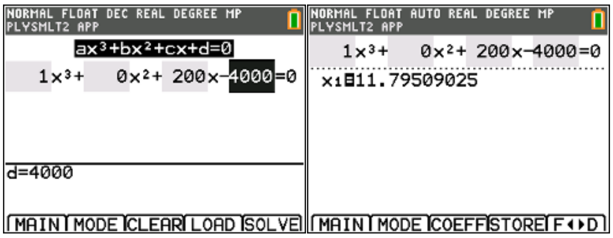
Solution

(a) Here we shall use an APP to solve the polynomial equation $x^3+200x-4000=0$.

1) Press **[APPS]** and select **[4:PlySmlt2]**. Press **[ENTER]** to display the main menu.
2) Select **[1: POLYNOMIAL ROOT FINDER]**.
3) Select the order of the polynomial, in this example, the order is 3, as it is a cubic equation.
4) Leave the other options as default. Press **[NEXT]**.



5) Key in the coefficients of the equation accordingly, and press **[SOLVE]**.

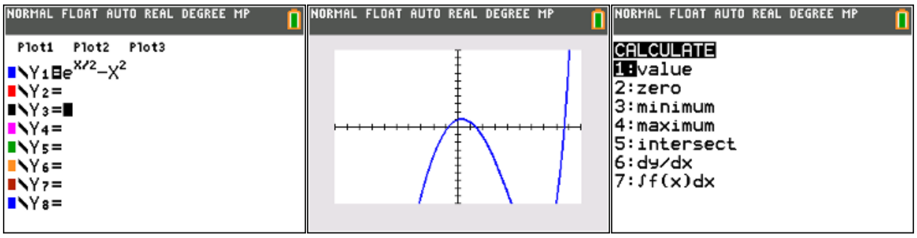


The only real root is 11.8 (to 3 s.f.).

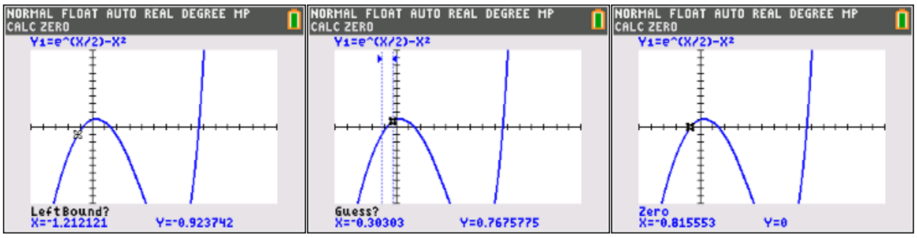
(b) For the non-polynomial equation $e^{\frac{x}{2}} = x^2$, we will use graph to solve it.

1) Press **[Y=]**, type in the equation. Press **[GRAPH]** to display. Press **[2ND][TRACE]**. Select **[2:zero]**.

We are interested in the zeroes, or the x-intercepts, as they are the solutions of the equation.



2) Move the cursor to define the “Left” and “Right” bounds. Do the same to find all x-intercepts.

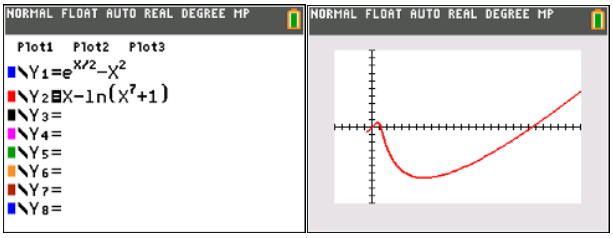


From the GC, $x = -0.816$ or $x = 1.43$ or $x = 8.61$.

(c) For $x - \ln\left(x^7 + 1\right) = 0$, we shall also use graph to solve this equation.

1) Press **[Y=]**, type in the equation. Press **[GRAPH]** to display. Press **[2ND][TRACE]**. Select **[2:zero]**.

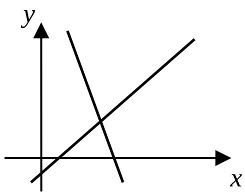
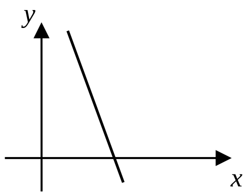
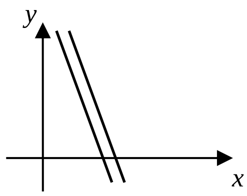
2) Move the cursor to define the “Left” and “Right” bounds. Do the same to find all x-intercepts.



From the GC, $x = -0.931$ or $x = 0$ or $x = 1.11$ or $x = 21.5$. ■

1.2 Solutions of Systems of Linear Equations Using Graphing Calculator

Consider the following:

Case 1	**Case 2**	**Case 3**
$2x+y=8$ $2x-3y=4$	$2x+y=8$ $4x+2y=16$	$2x+y=8$ $2x+y=7$
Solving, $4y=4 \Rightarrow y=1,\ x=\dfrac{7}{2}$ A unique solution exists. 	Simplifying, $4x+2y=16$ $\Leftrightarrow 2x+y=8$ Since the 2 lines coincide, infinitely many solutions exist. 	Since the 2 lines are parallel, no solutions exist. 

Note

There are 3 possibilities for the solution of the above system of linear equations.

- There is only **one unique solution** (when the two lines intersect at one point).
- There are **infinitely many solutions** (when the two lines coincide).
- There are **no solutions** (when the two lines are parallel).

Graphically, the solution may be obtained by drawing the graphs of the equations and finding the point(s) of intersection.

In general, any system of linear equations will have one of the above three types of solutions.

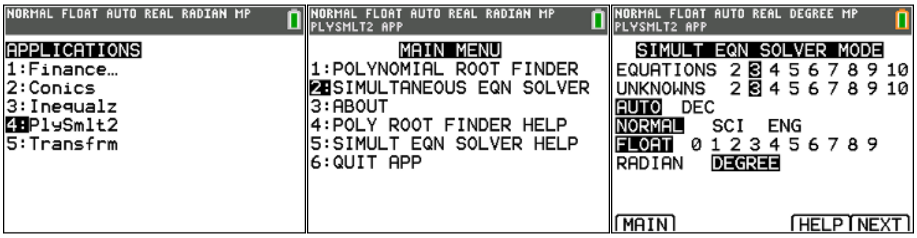
Example 3

Solve the system of linear equations

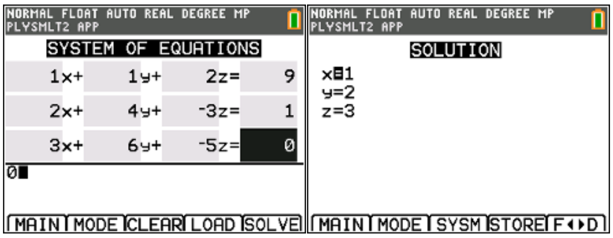
$$\begin{aligned} x+y+2z &= 9 \\ 2x+4y-3z &= 1 \\ 3x+6y-5z &= 0. \end{aligned}$$

Solution

1) Press **[APPS]** and select **[4:PlySmlt2]**. Press **[ENTER]** to display the main menu.
2) Select **[2: SIMULTANEOUS EQN SOLVER]**.
3) Adjust the settings accordingly as depicted on the screen. Press **[NEXT]**.



4) Enter the values of the coefficients accordingly. Press **[SOLVE]**.

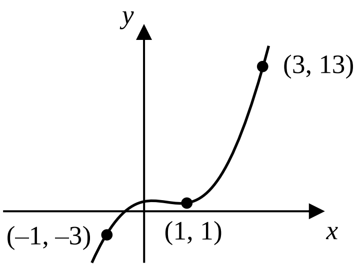


The solution is $x=1, y=2, z=3$. ■

1.3 Formulation of equations or a system of linear equations

Example 4

The diagram shows the graph of a polynomial of degree *n*. Explain why *n* cannot be an even number. Given that $n=3$, find the equation of the graph if it passes through the points $(-1,-3)$ and $(3,13)$, and has a minimum point at $(1,1)$.



Solution

From the given diagram, it is observed that as $x \to \infty, y \to \infty$, and as $x \to -\infty, y \to -\infty$. Hence *n* cannot be an even number.

Let $y = ax^3 + bx^2 + cx + d$.

At $(-1,-3)$: $\quad -a+b-c+d=-3 \quad ---(1)$

At $(1,1)$: $\quad a+b+c+d=1 \quad ---(2)$

At $(3,13)$: $\quad 27a+9b+3c+d=13 \quad ---(3)$

$$\frac{dy}{dx} = 3ax^2 + 2bx + c.$$

At $(1,1)$: $\quad 3a+2b+c=0 \quad ---(4)$

By GC, $a=1, b=-2, c=1, d=1$.

$\therefore$ Equation of the graph is $y = x^3 - 2x^2 + x + 1$. ■

Example 5

Mr. Spongebob went to the supermarket on 3 separate occasions to buy crabs, lobsters and bamboo clams. He observed that while the price of crabs and bamboo clams remained constant, the price of lobsters consistently increased by 10% compared to the immediate previous visit. The amount of crabs, lobsters and bamboo clams that he bought by weight for each visit as well as the total amount spent is shown in the table below.

	1st visit	2nd visit	3rd visit
Crab (kg)	3.20	5.60	4.50
Lobster (kg)	1.50	1.20	2
Bamboo Clam (kg)	7	6.50	6.50
Total amount paid in \$	277.50	347.00	395.18

What is the price per kilogram for crab, lobster and bamboo clam during Mr. Spongebob's third visit to the supermarket? [VJC 2009 Prelim]

Solution

Let the prices per kg for crab, lobster and bamboo clam be c, l and b respectively on the first visit. Therefore

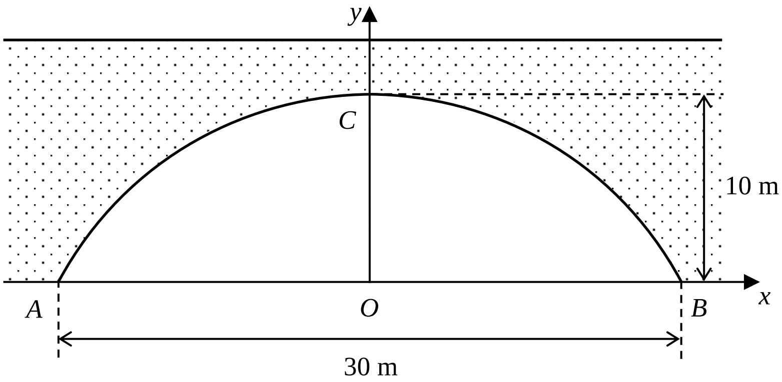
$$\begin{aligned} 3.2c + \quad 1.5l + \quad 7b &= 277.5 \quad ---(1) \\ 5.6c + 1.2(1.1l) + 6.5b &= 347 \quad ---(2) \\ 4.5c + 2(1.1^2 l) + 6.5b &= 395.18 \quad ---(3) \end{aligned}$$

By GC, $c = 36.198, l = 79.998\ (\Rightarrow 1.1^2 l = 96.798), b = 5.953$.

$\therefore$ The prices per kg for crab, lobster and bamboo clam during the 3rd visit are \$36.20, \$96.80 and \$5.95 respectively. ■

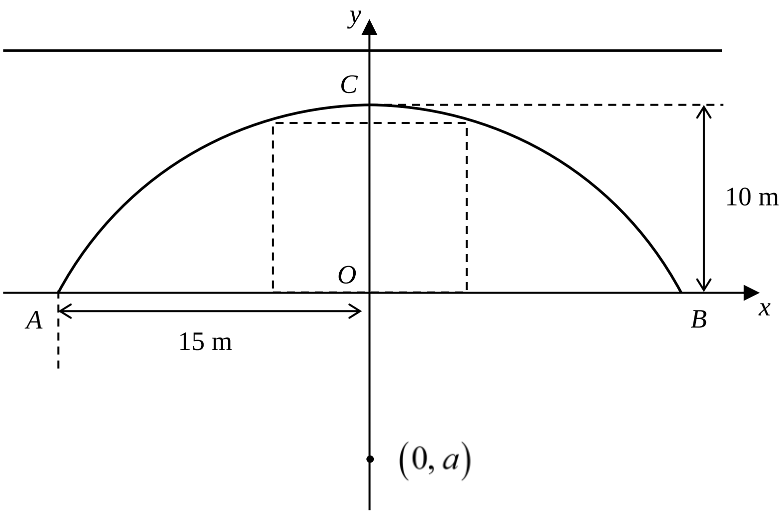
Example 6

The diagram below shows part of an arch bridge. The arch takes the shape of a circular arc. When the water level is at *AB*, the portion of the arch that is above the water level has a span of 30 m and a rise of 10 m. By taking coordinate axes O_x and O_y as shown, and the highest point of the arch to be directly above the origin *O*, find the equation of the circular arc.



A boat passes through the arch of the bridge when the water level is at *AB*. If the cross-section of the boat is to be modelled by a rectangle of width 10 m, find the greatest possible height of the boat above the water level. Justify your calculation.

Solution



To get a circle, the centre of the circle must be below *O*.

Let the coordinates of the centre be

Therefore the radius of the circle is

Hence, equation of circle is

Since (15,0) is on the circle, we have

Solving, we have

$a = -6.25$

Therefore equation of circle is

$$x^2 + (y + 6.25)^2 = (16.25)^2 .$$

Greatest height of the boat = value of y when $x =$

$^2 + (y + 6.25)^2 = (16.25)^2$

$\Rightarrow y = 9.21$ (3 s.f.) ■

Extra Practice

Annex A: Practice Questions 1 to 3

LECTURE 2

Lesson Outline

- Some basic rules and results for manipulating inequalities
- Polynomial inequalities

---

2 INEQUALITIES

2.1 Some Basic Rules and Results for Manipulating Inequalities

Let $x, y, a \in \mathbb{R}$.

For $x > y$	Examples
(i) $x + a > y + a$ $x - a > y - a$	
(ii) $ax > ay$, for $a > 0$ $ax < ay$, for $a < 0$, in particular $-x < -y$	
(iii) $\frac{x}{a} > \frac{y}{a}$, if $a > 0$ $\frac{x}{a} < \frac{y}{a}$, if $a < 0$	
(iv) $\frac{1}{x} < \frac{1}{y}$, if $x, y > 0$ or $x, y < 0$	
(v) $x^2 > y^2$, if $x, y > 0$ $x^2 < y^2$, if $x, y < 0$	
(vi) For $x, y > 0$, $\sqrt{x} > \sqrt{y}$.	
(vii) For $a > 1$, $a^x > a^y$	
(viii) For $a > 1$, $x, y > 0$ $\log_a x > \log_a y$	

Note

- In mathematics, the words **and** and **or** have slightly different meanings.
- When we say "$x>1$ and $x>3$", it means that both "$x>1$" and "$x>3$" are true, i.e.__________.
- When we say "$x>1$ and $x<3$", it means that both "$x>1$" and "$x<3$" are true, i.e.__________.
- When we say "$x<1$ or $x>3$", it means one or both of "$x<1$" and "$x>3$" are true, i.e.________________________.
- $xy>0 \Leftrightarrow \begin{cases} x>0 \text{ and } y>0 \\ \text{or} \\ x<0 \text{ and } y<0 \end{cases}$ and $xy<0 \Leftrightarrow \begin{cases} x>0 \text{ and } y<0 \\ \text{or} \\ x<0 \text{ and } y>0 \end{cases}$

2.2 Polynomial Inequalities

Graphical Method

In general, given a polynomial of degree *n*,

$$\mathrm{f}(x)=a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$$

if we can sketch $y=\mathrm{f}(x)$, we can solve inequalities of the form

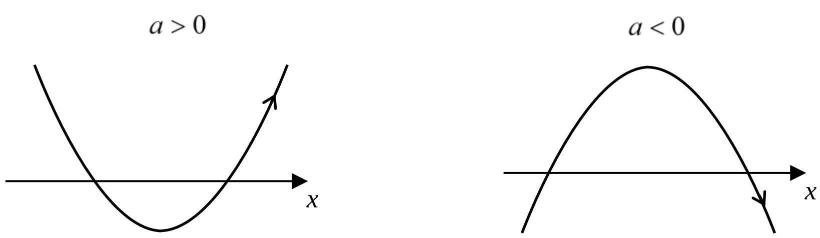
$$a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0 \geq 0 \left(\text{or} \leq 0\right).$$

The sketch of the polynomial curves depends on the sign of the **coefficient of the highest power,** i.e. a_n and the **multiplicities of the roots** (the number of times a root repeats) of the equation $\mathrm{f}(x)=0$.

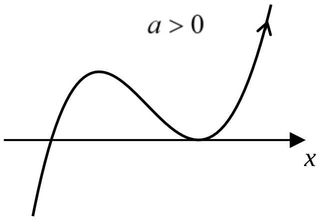
Coefficient of highest power

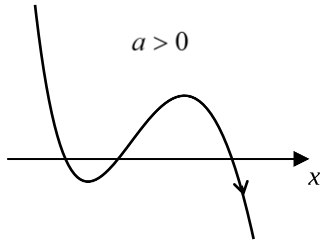
If the coefficient of the highest power $a_n > 0$, then as $x \to +\infty$, $\mathrm{f}(x) \to +\infty$. Similarly, if $a_n < 0$, then as $x \to +\infty$, $\mathrm{f}(x) \to -\infty$. This is illustrated in the common cases below.

Quadratic ($y=ax^2+bx+c$)

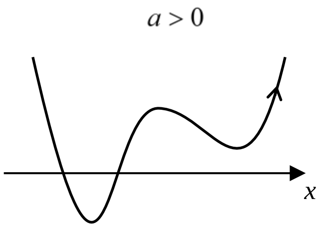


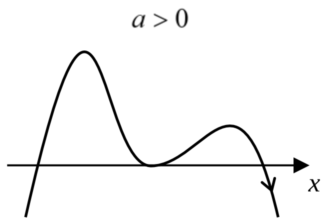
Cubic ($y = ax^3 + bx^2 + cx + d$)





Quartic ($y = ax^4 + bx^3 + cx^2 + dx + e$)





Notice the positions of the graphs when $x \to \infty$.

To solve the inequality $\mathrm{f}(x) > 0$, we can obtain the range of values of x for which the graph of $y = \mathrm{f}(x)$ is above the x-axis. Similarly, to solve the inequality $\mathrm{f}(x) < 0$, we will obtain the range of values of x for which the graph of $y = \mathrm{f}(x)$ is below the x-axis.

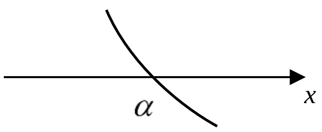
Multiplicities of roots

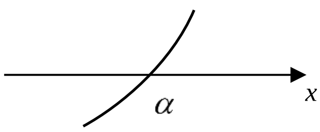
This is the number of times a root of the equation $\mathrm{f}(x) = 0$ repeats, where $\mathrm{f}(x)$ is a polynomial. Consequently, it is also the power of the factor when a polynomial is fully factorised.

For example, for $\mathrm{f}(x) = (x-2)(2x+1)^2(x+1)^3$, $x = 2$ is not repeated, $x = -\dfrac{1}{2}$ is repeated twice and $x = -1$ is repeated thrice.

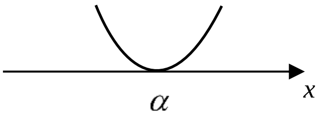
This causes the curve to take the shapes at the roots as shown in the following diagrams.

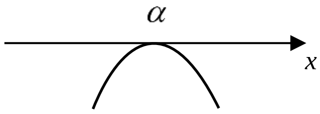
Not repeated: $(x - \alpha)$



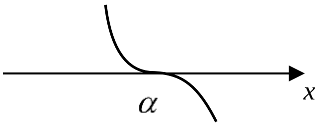


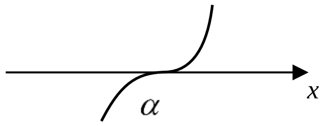
Repeated even number of times: $(x-\alpha)^2, (x-\alpha)^4$, etc



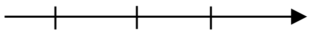
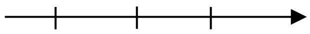
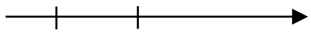
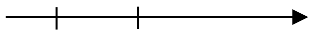


Repeated odd number of times: $(x-\alpha)^3, (x-\alpha)^5$, etc





Consider the following inequalities

Inequalities	**Graphs**	**Solutions for *x***
$(x-1)(x-2)(x-3)\geq 0$		
$(x-1)^2(x-2)(x-3)>0$		
$(1-x)(x-2)^3\geq 0$		
$(x-1)^2(x-2)^2>0$		

Alternatively, observe that the signs alternate between roots unless the root is a repeated one.

- If the coefficient of the highest power of x is positive, then the right-most region is positive, and subsequent regions alternate in signs from the right to the left.
- If the coefficient of the highest power of x is negative, then the right-most region is negative, and subsequent regions alternate in signs from the right to the left.

This observation leads us to the next method for solving inequalities.

Sign Test Method

Inequalities involving two or more linear factors can also be solved using the Sign Test method as illustrated in the example below.

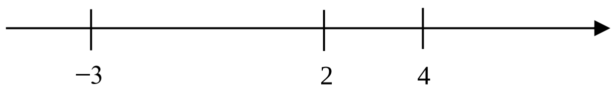
Example

Find the solution set of the inequality $(x^2+x-6)(4-x) \le 0$.

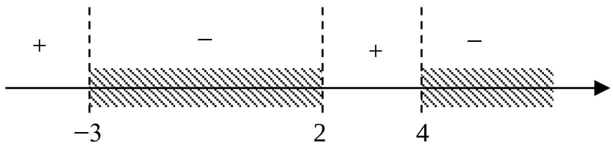
Step 1: Factorise the inequality as far as possible.

$$(x^2+x-6)(4-x) \le 0 \Rightarrow (x-2)(x+3)(4-x) \le 0$$

Step 2: Mark the critical values $x=-3, x=2, x=4$ on a number line.



Step 3: Test the sign of the function, $\mathrm{f}(x)=(x-2)(x+3)(4-x)$ in the intervals separated by the critical values, i.e. substitute values in the appropriate regions into the function.



The solution for the inequality corresponds to the interval where $\mathrm{f}(x) \le 0$. Hence, the solution set is $\{x \in \mathbb{R} : -3 \le x \le 2 \text{ or } x \ge 4\}$.

Example 7

Solve the inequality $\left(x^2+x+2\right)(1-x) > 0$ exactly.

Solution

$\therefore \left(x^2+x+2\right)(1-x) > 0$

■

Recall

Discriminant of a quadratic equation $ax^2+bx+c=0$ is b^2-4ac.

- If $b^2-4ac<0$, the equation has no real roots.
- If $b^2-4ac=0$, the equation has two equal real roots.
- If $b^2-4ac>0$, the equation has two distinct real roots.

Self-Practice Exercise 1

Solve the following inequalities, leaving your answers in exact form.

(a) $3x^2-12x+5>0$

(b) $x^2-x+2>0$

(c) $2x^2+4x+1\geq 3\left(x^2+x+1\right)$

(d) $(3-x)(2x+1)\left(\sqrt{3}x-4\right)<0$

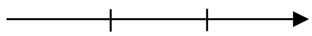
(e) $(x+2)^2(1-x)\geq 0$

(f) $(x+2)^2(1-x)>0$

Solution

(a) $3x^2-12x+5>0$

$=2\pm\dfrac{\sqrt{21}}{3}$



Ans: $x<2-\dfrac{\sqrt{21}}{3}$ or $x>2+\dfrac{\sqrt{21}}{3}$

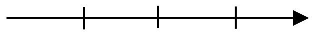
(b) $x^2-x+2>0$

Ans: All real values of *x*.

(c) $2x^2+4x+1\geq 3\left(x^2+x+1\right)$

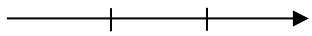
Answer: No solution.

(d) $(3-x)(2x+1)\left(\sqrt{3}x-4\right)<0$



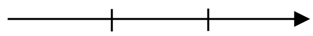
Ans: $-\dfrac{1}{2}<x<\dfrac{4}{\sqrt{3}}$ or $x>3$

(e) $(x+2)^2(1-x)\geq 0$



Ans: $x\leq 1$

(f) $(x+2)^2(1-x)>0$



Ans:

■

Example 8

It is given that $y = \dfrac{x^2+x+1}{x-1}$, $x \in \mathbb{R}$, $x \neq 1$. Without using a calculator, find the set of values that y can take. [5]

[N13/I/2]

Solution

$$y = \frac{x^2+x+1}{x-1}$$

Cross multiply, we get

$$y(x-1) = x^2+x+1$$

$\Rightarrow \boxed{\qquad\qquad\qquad\qquad\qquad\qquad}$

Since there are real values of x (except $x=1$ because of the denominator),

$\boxed{\qquad\qquad\qquad\qquad\qquad}$

$$(1-y)^2 - 4(1)(1+y) \geq 0$$

$$y^2 - 6y - 3 \geq 0$$

$\therefore y \leq 3-2\sqrt{3}$ or $y \geq 3+2\sqrt{3}$

Therefore solution set is $\boxed{\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$ ■

LECTURE 3

Lesson Outline

- Solve inequalities of the form $\frac{\mathrm{f}(x)}{\mathrm{g}(x)} > 0$ where $\mathrm{f}(x)$ and $\mathrm{g}(x)$ are linear expressions or quadratic expressions.

2.3 Inequalities Involving Rational Functions

Here we look at how to solve inequalities involving rational functions, i.e. $\frac{\mathrm{f}(x)}{\mathrm{g}(x)}$ where $\mathrm{f}(x)$ and $\mathrm{g}(x)$ are polynomials.

Step 1: Bring all the terms to the LHS to make the RHS 0 (or vice versa).

$$\frac{x+1}{x-1} > 2 \Rightarrow \frac{x+1}{x-1} - 2 > 0$$

Step 2: Combine the terms into a single fraction, i.e. $\frac{\mathrm{f}(x)}{\mathrm{g}(x)} > 0$.

$$\frac{x+1}{x-1} - 2 > 0 \Rightarrow \frac{x+1-2(x-1)}{x-1} > 0 \Rightarrow \frac{-x+3}{x-1} > 0$$

Step 3: Multiply by the square of the denominator, i.e. $\left(\mathrm{g}(x)\right)^2$ on both sides. This can be done without changing the inequality signs as $\left(\mathrm{g}(x)\right)^2$ is always positive.

Once we are used to this, we will skip the middle step.

$$\frac{\mathrm{f}(x)}{\mathrm{g}(x)} > 0 \Rightarrow \frac{\mathrm{f}(x)}{\mathrm{g}(x)}\left(\mathrm{g}(x)\right)^2 > 0 \Rightarrow \mathrm{f}(x)\,\mathrm{g}(x) > 0 .$$

We need to remove solutions where $\mathrm{g}(x) = 0$ because $\mathrm{g}(x)$ is in the denominator.

$$\frac{-x+3}{x-1} > 0 \Rightarrow \frac{(-x+3)}{x-1}(x-1)^2 > 0 \Rightarrow (-x+3)(x-1) > 0,\ x \neq 1$$

Note

- A common mistake when solving inequality involving rational function is to cross multiply, i.e. $\frac{x+1}{x-1} > 2 \not\Rightarrow x+1 > 2(x-1)$. This is because, depending on the value of *x*, the expression $x-1$ could be positive or negative, so the inequality will change if it is negative.

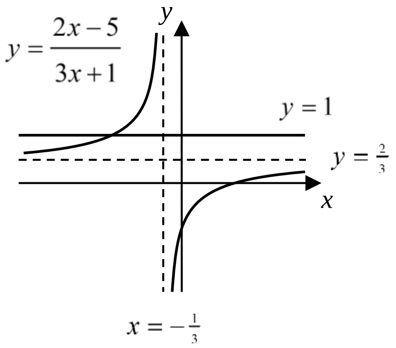
Example 9

Solve $\dfrac{2x-5}{3x+1} \geq 1$.

Solution

Graphical Method

Sketch the graphs of $y = \dfrac{2x-5}{3x+1}$ and $y = 1$ on the same diagram:



Solve for intersection of $y = \dfrac{2x-5}{3x+1}$ and $y = 1$.

$x = -6$

Therefore $-6 \leq x < -\dfrac{1}{3}$.

Algebraic Method

$\dfrac{2x-5}{3x+1} \geq 1,\ x \neq$

$$\frac{2x-5-(3x+1)}{3x+1} \geq 0$$

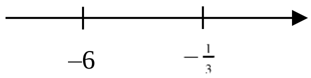
$$\frac{-x-6}{3x+1} \geq 0 \Rightarrow \frac{x+6}{3x+1} \leq 0$$

Multiplying both sides by $(3x+1)^2$, we get

$$\frac{x+6}{3x+1} \times (3x+1)^2 \leq 0$$

$$(x+6)(3x+1) \leq 0,\ x \neq -\frac{1}{3}$$

This step can be skipped if you understand that the solutions of $\frac{x}{y} \leq 0$ and $xy \leq 0$ are essentially the same, except that $y \neq 0$ for $\frac{x}{y} \leq 0$.



Therefore $-6 \leq x < -\dfrac{1}{3}$.

■

Example 10

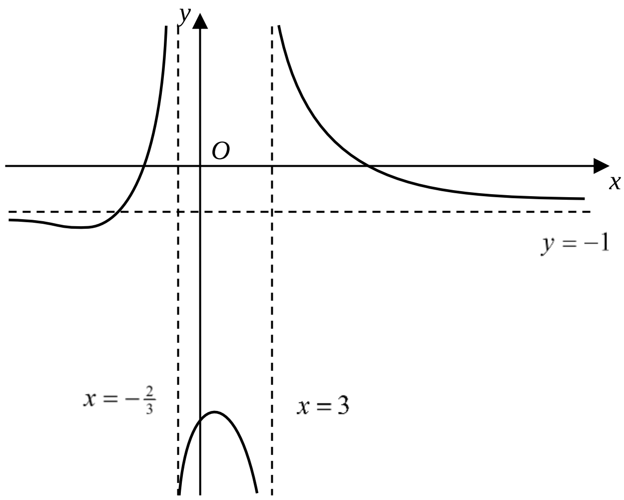
Solve $\dfrac{9}{x-3}-1\geq\dfrac{16}{3x+2}$.

Solution

Since the question did not ask for **exact** answers, the algebraic method is not recommended due to the tedious working involved.

Rearranging, we have $\dfrac{9}{x-3}-1-\dfrac{16}{3x+2}\geq 0$.

Thus we sketch the graph of $y=\dfrac{9}{x-3}-1-\dfrac{16}{3x+2}$.



From GC, the axial intercepts are $(-2.74, 0)$ and $(8.74, 0)$.

To solve $\dfrac{9}{x-3}-1-\dfrac{16}{3x+2}\geq 0$, note that $x\neq 3$ and $x\neq -\dfrac{2}{3}$, hence from the graph, $-2.74\leq x<-\dfrac{2}{3}$ or $3<x\leq 8.74$. ■

Self-Practice Exercise 2

Solve the following inequalities.

(a) $\dfrac{x^2+x-6}{4-x}\leq 0$

(b) $\dfrac{1}{x-2}<\dfrac{1}{x-3}$

Solution

Algebraic Method

(a)

(b)

Ans: **(a)** $-3 \le x \le 2$ or $x > 4$ **(b)** $x < 2$ or $x > 3$ ■

Extra Practice

Annex A: Practice Question 4

Example 11

Using the algebraic method, find the **solution set** of the inequality $\dfrac{1}{2x-1} \le 8x < x^2$.

Solution

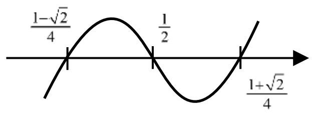
$\dfrac{1}{2x-1} \le 8x,\ x \ne \dfrac{1}{2}$ **and** $8x < x^2$

$$\frac{1}{2x-1} - 8x \le 0$$

$$\frac{1-8x(2x-1)}{2x-1} \le 0$$

$$(1-16x^2+8x)(2x-1) \le 0$$

$$(2x-1)(16x^2-8x-1) \ge 0$$

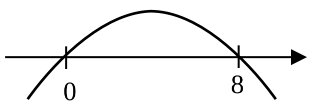


$$\therefore \frac{1-\sqrt{2}}{4} \le x < \frac{1}{2} \text{ or } x \ge \frac{1+\sqrt{2}}{4}$$

$$8x < x^2$$

$$8x - x^2 < 0$$

$$x(8-x) < 0$$



$\therefore x < 0$ or $x > 8$

We want x to satisfy **both** inequalities, i.e. $\frac{1}{2x-1} \le 8x$ **and** $8x < x^2$ at the same.

For intersection of solution sets,

Hence, the solution set is ■

Example 12

Show that $x^2 - x + 1$ is always positive for all real x. Hence, or otherwise, find the range of values of a if the inequality $\dfrac{x^2 + ax - 2}{x^2 - x + 1} < 2$ is satisfied for all real x. [ACJC 1999 Prelim]

Solution

$x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} > 0$ for all real values of x.

$$\frac{x^2 + ax - 2}{x^2 - x + 1} < 2$$

Since $x^2 - x + 1$ is always positive,

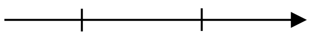
For the graph of $y = x^2 + (-2 - a)x + 4$ has no x-intercept, therefore

Discriminant

$\Rightarrow (-2 - a)^2 - 4(1)(4) < 0$

$\Rightarrow 4 + 4a + a^2 - 16 < 0$

$\Rightarrow a^2 + 4a - 12 < 0$



$\therefore -6 < a < 2$ ■

LECTURE 4

Lesson Outline

- Concept of $|x|$
- Use of the relations
  - $|x-a|<b \iff a-b<x<a+b$
  - $|x-a|>b \iff x<a-b$ or $x>a+b$

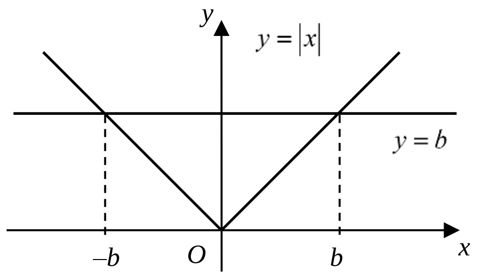
  in the course of solving inequalities.

2.4 Inequalities involving the modulus function

Definition of the modulus function

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

Important Properties of the Modulus Function



For a constant $a, b \in \mathbb{R}^+$ and variables $x, y \in \mathbb{R}$, the following holds:

(i) $|x|<b \Leftrightarrow -b<x<b$

In general,

$$|x-a|<b \Leftrightarrow -b<x-a<b \Leftrightarrow a-b<x<a+b$$

(ii) $|x|>b \Leftrightarrow x<-b$ or $x>b$

In general,

$$|x-a|>b \Leftrightarrow x-a<-b \text{ or } x-a>b \Leftrightarrow x<a-b \text{ or } x>a+b$$

(iii) $|x|<|y| \Leftrightarrow |x|^2<|y|^2 \Leftrightarrow x^2<y^2$

(iv) $|x^2|=|x|^2=x^2$

(v) $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$, $y \neq 0$

(vi) $|xy| = |x||y|$

(vii) $\sqrt{x^2} = |x|$

Note

Inequalities involving modulus function are generally easier solved by graphical method.

Example 13

Solve the following inequalities:

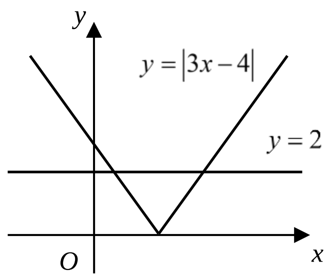
(a) $|3x-4| > 2$ **(b)** $\left|\dfrac{2x+3}{x-1}\right| < 1$ **(c)** $|2x+3| - |x+4| < 2$

Solution

(a) $|3x-4| > 2$

Graphical Method

Method 1: Graph $y = |3x-4|$ and $y = 2$ then solve for the intersection points with or without GC.



$|3x-4| = 2$

$\Rightarrow 3x - 4 = \pm 2$

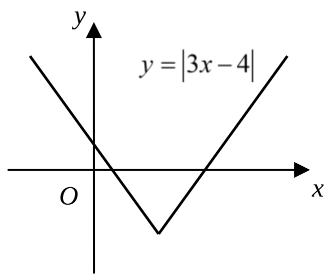
$\Rightarrow x = \dfrac{2}{3}$ or $x = 2$

$\therefore x < \dfrac{2}{3}$ or $x > 2$

Method 2:

$|3x-4| > 2 \Rightarrow |3x-4| - 2 > 0$.

Graph $y = |3x-4| - 2$. Then find the x-intercepts using GC.



The intercepts are $\left(\dfrac{2}{3}, 0\right)$ and $(2, 0)$

$\therefore x < \dfrac{2}{3}$ or $x > 2$

Algebraic Method

$|3x-4| > 2 \Rightarrow 3x - 4 \quad -2$ or $3x - 4 \quad 2$

$\Rightarrow 3x < 2$ or $3x > 6$

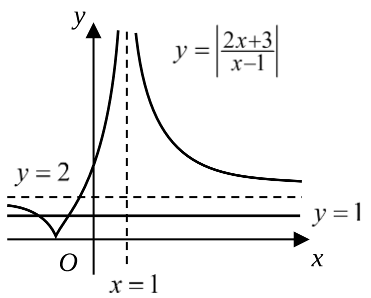
$\therefore x < \dfrac{2}{3}$ or $x > 2$

(b) $\left|\dfrac{2x+3}{x-1}\right|<1$

Graphical Method

$y=\dfrac{2x+3}{x-1}=2+\dfrac{5}{x-1}$

Sketch the graphs of $y=\left|2+\dfrac{5}{x-1}\right|$ and $y=1$ on the same diagram:



To find intersection points without GC, solve

$\dfrac{2x+3}{x-1}=\pm 1 \Rightarrow x=-4$ or $x=-\dfrac{2}{3}$

Therefore solution is

$-4<x<-\dfrac{2}{3}$

Algebraic Method

$\left|\dfrac{2x+3}{x-1}\right|<1$

$|2x+3|<|x-1|$

Squaring both sides, (this is only possible if both sides are positive)

$(2x+3)^2<(x-1)^2$

$(2x+3)^2-(x-1)^2<0$

$((2x+3)+(x-1))((2x+3)-(x-1))<0$

$(3x+2)(x+4)<0$

$-4<x<-\dfrac{2}{3}$

(c) $|2x+3|-|x+4|<2$

Graphical Method

Method 1

$|2x+3|-|x+4|<2$

$\Rightarrow |2x+3|<|x+4|+2$

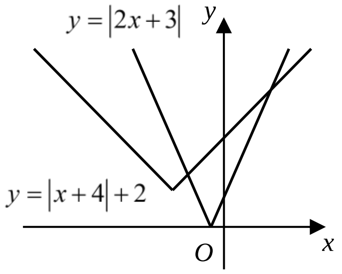
Sketch $y=|2x+3|$ and $y=|x+4|+2$ on the same diagram and find the intersections.

Method 2

$|2x+3|-|x+4|<2$

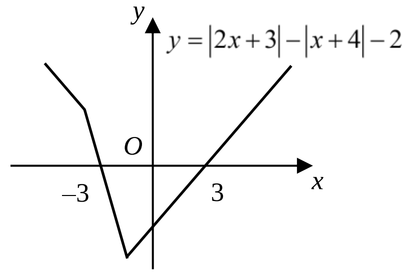
$\Rightarrow |2x+3|-|x+4|-2<0$

Sketch $y=|2x+3|-|x+4|-2$ and find the *x*-intercepts using GC.



The intersection points are $(-3,3)$ and $(3,9)$.

Hence solution is



The x-intercepts are $(-3,0)$ and $(3,0)$.

Hence solution is ■

Example 14

Using the algebraic method, solve $x+1>|2x+1|$.

Solution

Use the results $|x|<a \Rightarrow -a<x<a$ and $|x|>a \Rightarrow x>a$ or $x<-a$ when solving an inequality involving modulus algebraically.

What if the inequality sign changes, i.e. $x+1<|2x+1|$?

$x+1>|2x+1|$

$\Rightarrow 3x>-2$ $\qquad$ $x<0$

$\Rightarrow x>-\dfrac{2}{3}$ $\qquad$ $x<0$

$\Rightarrow -\dfrac{2}{3}<x<0$ ■

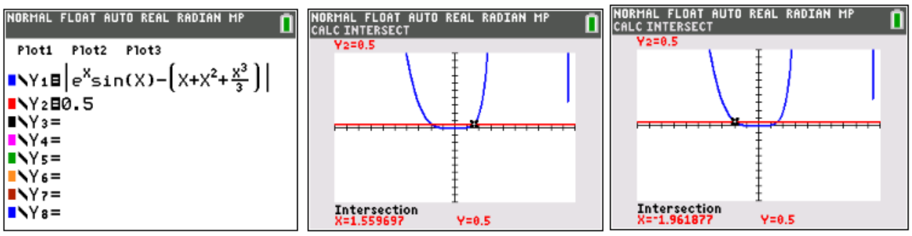
Example 15

Let $\mathrm{f}(x)=\mathrm{e}^x\sin x$ and $\mathrm{g}(x)=x+x^2+\dfrac{x^3}{3}$. Find, for $-3\le x\le 3$, the set of values of x for which the value of $\mathrm{g}(x)$ is within ± 0.5 of the value of $\mathrm{f}(x)$.

[N08/ II/1 (modified)]

Solution

Using the GC, sketch $y=\left|\mathrm{e}^x\sin x-\left(x+x^2+\dfrac{x^3}{3}\right)\right|$ and $y=0.5$ and then find the intersections of the two curves.



From graph, the solution is $-1.96 < x < 1.56$. ■

Note

We could also rewrite the inequality as $|\mathrm{f}(x)-\mathrm{g}(x)|-0.5<0$. In which case, we only need to plot $y=|\mathrm{f}(x)-\mathrm{g}(x)|-0.5$ and then look for the region under the *x*-axis.

Example 16

Solve $\dfrac{6}{x+1}<x$. Hence solve the following inequalities:

(a) $\dfrac{6}{|x|+1}<|x|$

(b) $\dfrac{6}{\mathrm{e}^x+1}<\mathrm{e}^x$

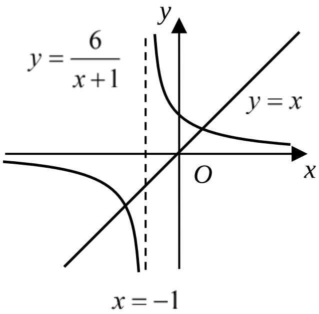
(c) $\dfrac{6}{x^2}<x^2-1$

(d) $\dfrac{6x}{x+1}<\dfrac{1}{x}$

Solution

Graphical Method

Sketch the graphs of $y=\dfrac{6}{x+1}$ and $y=x$ on the same diagram.



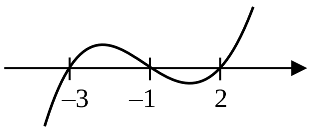
The intersection points are $(-3,-3)$ and $(2,2)$.

From graph,

$-3<x<-1$ or $x>2$.

Algebraic Method

$$(6-x^2-x)(x+1)<0$$
$$(x^2+x-6)(x+1)>0$$
$$(x+1)(x+3)(x-2)>0$$



$\therefore -3<x<-1$ or $x>2$

(a) $\dfrac{6}{|x|+1} < |x|$

Replace x by

$\therefore x > 2$ or $x < -2$

(b) $\dfrac{6}{e^x+1} < e^x$

Replace x by

$\therefore \underbrace{-3 < e^x < -1}_{\text{no solution}}$ or $e^x > 2$

$\therefore x > \ln 2$

When solving inequalities involving $\frac{1}{x}$ such as (d), it is recommended to solve it by sketching the graph of $y = \frac{1}{x}$.

(c) $\dfrac{6}{x^2} < x^2 - 1$

Replace x by

$\therefore x > \sqrt{3}$ or $x < -\sqrt{3}$

(d) $\dfrac{6x}{x+1} < \dfrac{1}{x}$

Replace x by

$\therefore -1 < x < -\dfrac{1}{3}$ or $0 < x < \dfrac{1}{2}$

■

Extra Practice

Annex A: Practice Question 5

ANNEX A

Practice Questions on Equations and Inequalities

Equations

1 Solve the following system of linear equations

(i) $3x+y+14z=68,\ 2x+9y+125z=409,\ 77y+34z=256$

(ii) $a+b+c=75000,\ 0.02a+0.04b+0.08c=5000,\ a=2b$

2 Given that

$$\begin{aligned} a+b+c+d &= 22 \\ a-d &= 3 \\ 100a-90b-9c-d &= 228 \\ 999a+90b-90c-999d &= 3267 \end{aligned}$$

evaluate the product *abcd*.

3 Given that the curve with equation $y=ax^2+bx+c$ passes through the points with coordinates $(-1.5, 4.5)$, $(2.1, 3.2)$ and $(3.4, 4.1)$, find the values of *a*, *b* and *c*. Give your answers correct to 3 decimal places. [3]

[9740 N11/I/2(modified)]

Inequalities

4 Solve the following inequalities.

(a) $3x(x-5)<2(2x-3)$ **(b)** $x^3<6x-x^2$ **(c)** $\frac{4-x}{x-2}>3$

(d) $\frac{2x-3}{x-5}\geq\frac{3}{2}$ **(e)** $\frac{1}{x-1}<\frac{1}{x+1}$ **(f)** $\frac{15}{x-2}\geq 3x-2$

5 Solve $\frac{x+1}{x-1}<4$. Hence or otherwise, solve

(i) $\frac{|x|+1}{|x|-1}<4$ **(ii)** $\left|\frac{x+1}{x-1}\right|<4$

Answers

1 **(i)** $x=8,\ y=2,\ z=3$ **(ii)** $a=12500,\ b=6250,\ c=56250$

2 $abcd=720$

3 $a=0.215,\ b=-0.490,\ c=3.281$

4 **(a)** $\frac{1}{3}<x<6$ **(b)** $x<-3$ or $0<x<2$ **(c)** $2<x<\frac{5}{2}$

(d) $x\leq -9$ or $x>5$ **(e)** $-1<x<1$ **(f)** $x\leq -1$ or $2<x<\frac{11}{3}$

5 $x<1$ or $x>\frac{5}{3}$ **(i)** $x<-\frac{5}{3}$ or $-1<x<1$ or $x>\frac{5}{3}$ **(ii)** $x<\frac{3}{5}$ or $x>\frac{5}{3}$