# MATHEMATICS

**Paper 9758/02**
**Set III – Paper 2**

**Topic identification and short answers**

| Qn | Topic(s) | Part | Answers |
|---|---|---|---|
| **Section A: Pure Mathematics [40 marks]** | | | |
| **1** | Complex numbers (cartesian form, conjugate); System of linear equations | | $u = 3 + 2\mathrm{i}$<br>$w = 5 - \mathrm{i}$<br>$z = -4 + 3\mathrm{i}$ |
| **2** | Differentiation (connected rates of change) | | $s = \frac{1}{4}\sqrt{x^2 + 4^2}$ (or equivalent)<br>$T = \frac{25}{6}$ |
| **3** | Integration techniques (by substitution); Definite integrals (volume of revolution) | **(a)** | [shown] |
| | | **(b)** | $\frac{2}{3}\pi^2 - \frac{\sqrt{3}}{4}\pi$ units$^3$ |
| **4** | Vectors (two dimensions, collinearity) | **(a)** | [shown] |
| | | **(b)** | $\overrightarrow{OY} = \left(\frac{a}{a-c}\right)\mathbf{r} - \left(\frac{c}{a-c}\right)\mathbf{p}$<br>$\overrightarrow{OZ} = \left(\frac{b}{b-c}\right)\mathbf{r} - \left(\frac{c}{b-c}\right)\mathbf{q}$<br>[shown] |
| | | **(c)** | $c = \frac{ab}{2a-b}$ |
| **5** | Maclaurin series; Differential equations | **(a)** | [shown] |
| | | **(b)** | $y = 1 - 2k^2x^2 + \frac{2}{3}k^2\left(k^2 - 1\right)x^4 + \cdots$ |
| | | **(c)** | $y = \cos(2m\sin^{-1}x)$ |
| | | **(d)** | $k = m$ |

| **Section B: Probability and Statistics [60 marks]** | | | |
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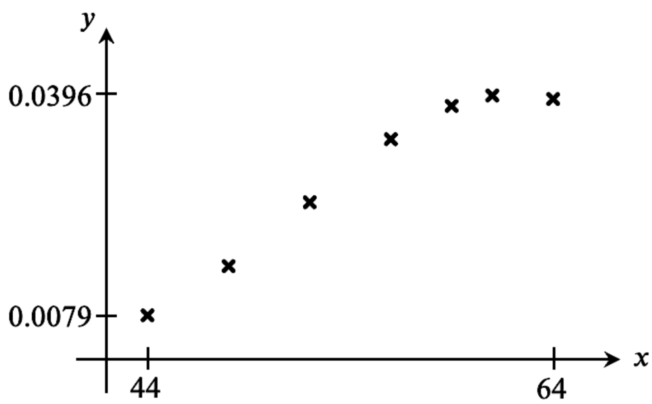
| **6** | Probability (permutations and combinations) | | Number of ways = 1016 |
| **7** | Hypothesis testing; Sampling | **(a)** | The probability of obtaining a sample which gives a test statistic that is as extreme or more extreme than the one being observed, assuming that the null hypothesis is true. |
| | | **(b)** | $H_0 : \mu = 30$ and $H_1 : \mu \neq 30$<br>$n = 100$<br>$\alpha = 5$<br><br>Reject the null hypothesis and conclude that there is sufficient evidence at 5% significance level that the average duration of uploaded videos differs from 30.0 seconds. |
| | | **(c)** | The second sample is large and yields $p$-value > 0.05 |
| **8** | Discrete random variables (expectations); Probability | **(a)** | [shown] |
| | | **(b)** | $a = 7$ |
| | | **(c)** | Least integer $y = 2$ |
| **9** | Linear regression; Normal distribution (normal curve) | **(a)** | <br>y<br>0.0396<br>0.0079<br>44<br>64<br>x |
| | | **(b)** | In the sketch, $\mathrm{f}(x)$ increases from $x = 59$ to $x = 61$, then decreases from $x = 61$ to $x = 64$. This implies that $\mathrm{f}(x)$ is maximum at either $x = 61$ or somewhere else in the range $59 < x < 64$. |
| | | **(c)** | $\mu = 62$<br>$a \approx 0.0399$<br>$b \approx -0.00499$ |
| | | **(d)** | $x \approx 78.6$ or $45.4$<br>Both estimates are equally reliable |
| | | **(e)** | $(71.8, 0.0247)$ or $(52.2, 0.0247)$ |

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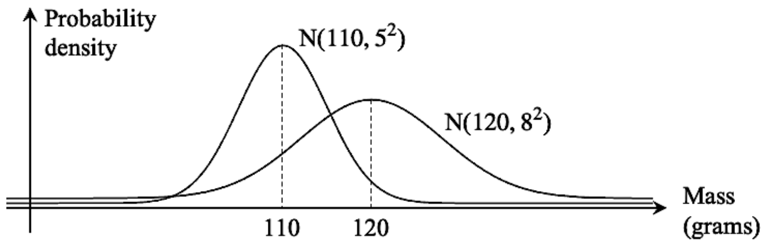
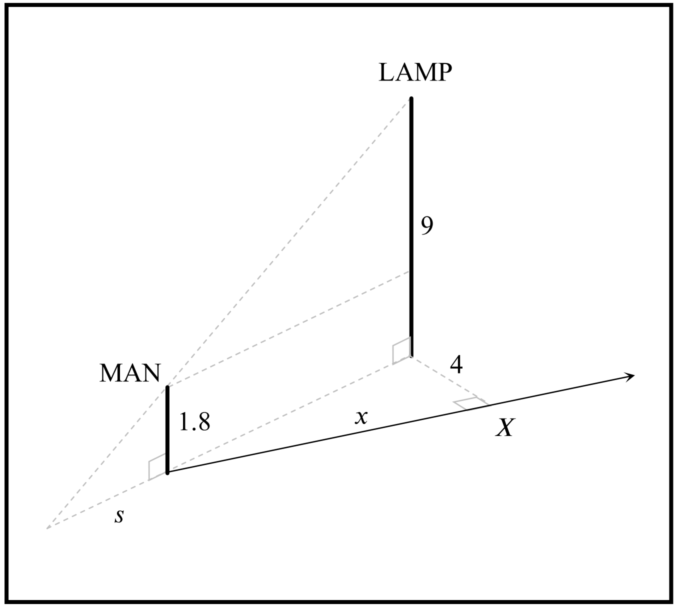
| **10** | Binomial distribution | **(a)** | The event that the stock price rises from one period to the next is <u>independent</u> of any other event where the price rises from any other period to the next. |
| | | **(b)** | [shown] |
| | | **(c)** | P(sell at period $2k$) $= \left(\frac{1}{2}\right)^{2k}\left[\binom{2k}{k+1}+\binom{2k}{k+2}\right]$<br>P(sell at period $2k+1$) $= \left(\frac{1}{2}\right)^{2k+1}\left[\binom{2k+1}{k+2}+\binom{2k+1}{k+3}\right]$ |
| | | **(d)** | $t_{\text{odd}} = 13$ or $15$<br>$t_{\text{even}} = 6$ or $8$<br><br>Sell at $t = 13$ or $15$ to get $r^3 S_0$ or $r^5 S_0$ with probability 0.2444<br>Sell at $t = 6$ or $8$ to get $r^2 S_0$ or $r^4 S_0$ with probability 0.3281<br>(Any suggestion acceptable <u>if justified using values above</u>.) |
| **11** | Normal distribution | **(a)** |  |
| | | **(b)** | Required probability $\approx 0.158$ |
| | | **(c)** | [shown]<br><br>Required least integer $m = 94$<br><br><u>A fruit of either kind (apple or pear) will almost absolutely exceed 94 grams</u>. To have a higher chance of distinguishing the fruit correctly, the vendor may use a larger value of $m$. |
| | | **(d)** | $k = 346$ |

**Suggested solutions and post-mortem**

| Qn | Suggested Solutions | Comments |
|---|---|---|
| | **Section A: Pure Mathematics [40 marks]** | |
| **1** **[5]** | Let $u = A + B\text{i}$, $w = C + D\text{i}$ and $z = E + F\text{i}$ (where $A, B, C, D, E, F \in \mathbb{R}$).<br><br>$u - w + z = -6 + 6\text{i}$<br>$(A + B\text{i}) - (C + D\text{i}) + (E + F\text{i}) = -6 + 6\text{i}$<br>$(A - C + E) + (B - D + F)\text{i} = -6 + 6\text{i}$<br>$\rightarrow A - C + E = -6$<br>$\rightarrow B - D + F = 6$<br><br>$3u + \text{i}w - 2\text{i}z = 16 + 19\text{i}$<br>$3(A + B\text{i}) + \text{i}(C + D\text{i}) - 2\text{i}(E + F\text{i}) = 16 + 19\text{i}$<br>$(3A - D + 2F) + (3B + C - 2E)\text{i} = 16 + 19\text{i}$<br>$\rightarrow 3A - D + 2F = 16$<br>$\rightarrow 3B + C - 2E = 19$<br><br>$\text{i}u^* - 2w^* - 3z = 4 - 8\text{i}$<br>$\text{i}(A - B\text{i}) - 2(C - D\text{i}) - 3(E + F\text{i}) = 4 - 8\text{i}$<br>$(B - 2C - 3E) + (A + 2D - 3F)\text{i} = 4 - 8\text{i}$<br>$\rightarrow B - 2C - 3E = 4$<br>$\rightarrow A + 2D - 3F = -8$<br><br>Solving simultaneously using GC,<br>$A = 3, B = 2 \rightarrow u = 3 + 2\text{i}$<br>$C = 5, D = -1 \rightarrow w = 5 - \text{i}$<br>$E = -4, F = 3 \rightarrow z = -4 + 3\text{i}$ | As apparent, this question assesses candidates' ability to form a correct system of linear equation in terms of cartesian complex numbers.<br><br>Great attentiveness to sign changes is required, especially those that arise due to multiplication by the imaginary unit and conjugation. The following parts on comparing complex number parts and solving the resulting system of six linear equations are the less arduous parts of the question.<br><br>This question is only made possible owing to the limitations of graphing calculators (that are approved within the H2 Mathematics syllabus), particularly the inability to accept complex numbers as direct inputs to the simultaneous equation solver. |

**2** **[6]**

Consider the three-dimensional diagram.
(All lengths in metres.)




By similar triangles,

$$\rightarrow \frac{s}{s+\sqrt{x^2+4^2}} = \frac{1.8}{9} = \frac{1}{5}$$

$$\rightarrow 5s = s + \sqrt{x^2+4^2}$$

$$\therefore s = \frac{1}{4}\sqrt{x^2+4^2}$$

Differentiating with respect to time, $t$

$$\rightarrow \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{4}\left(\frac{1}{2}\right)\left(x^2+16\right)^{-\frac{1}{2}}(2x)\frac{\mathrm{d}x}{\mathrm{d}t}$$

$$\rightarrow \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{x}{4\sqrt{x^2+16}}(2) = \frac{x}{2\sqrt{x^2+16}}$$

$$\rightarrow \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = \frac{x^2}{4x^2+64}$$

$$\rightarrow 4\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 x^2 + 64\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = x^2$$

$$\rightarrow x^2 = \frac{64\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2}{1-4\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2}$$

Substituting rates of changes of $s$ to obtain $x$,

$$\frac{\mathrm{d}s}{\mathrm{d}t} = -0.3 \rightarrow x^2 = \frac{64(-0.3)^2}{1-4(-0.3)^2} = 9 \rightarrow x = -3 \quad \left(\because \frac{\mathrm{d}s}{\mathrm{d}t} < 0\right)$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 0.4 \rightarrow x^2 = \frac{64(0.4)^2}{1-4(0.4)^2} = \frac{256}{9} \rightarrow x = \frac{16}{3} \quad \left(\because \frac{\mathrm{d}s}{\mathrm{d}t} > 0\right)$$

$$\therefore T = \frac{\left(\frac{16}{3}+3\right)}{2} = \frac{25}{6}$$

The relationship between $s$ and $x$ is most obvious in a three-dimensional diagram, and hence candidates are not to rely only on the two-dimensional diagram provided by the question. Candidates should then more easily notice the required relationship through Pythagorean Theorem and similar triangles.

Upon differentiating with respect to time and finding the connected rates of change, candidates can deduce an expression for $x$ in terms of $\frac{\mathrm{d}s}{\mathrm{d}t}$ and hence use it to find the exact value for $x$ given the value of $\frac{\mathrm{d}s}{\mathrm{d}t}$. Candidates who are mindful of the sign of $\frac{\mathrm{d}s}{\mathrm{d}t}$ will notice that this will affect the sign of $x$ and in extension where it lies on the path with respect to $X$. In turn, they should be able to deduce the correct signs, following which the exact value of $T$ can be deduced.

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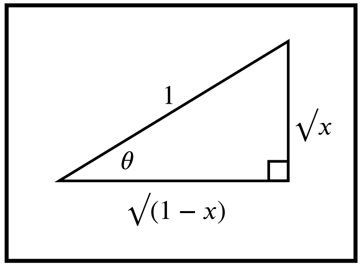
| **3**<br>**(a)**<br>**[4]** | $x = \sin^2\theta$<br>$\rightarrow \sqrt{x} = \sin\theta \rightarrow \theta = \sin^{-1}\sqrt{x}$<br>$\rightarrow x = 1 - \cos^2\theta \rightarrow \cos\theta = \sqrt{(1-x)}$<br>$\rightarrow \mathrm{d}x = 2\sin\theta\cos\theta\,\mathrm{d}\theta$<br><br>$\int \sqrt{\left(\frac{1-x}{x}\right)}\,\mathrm{d}x$<br>$= \int \sqrt{\left(\frac{1-\sin^2\theta}{\sin^2\theta}\right)}\,(2\sin\theta\cos\theta)\,\mathrm{d}\theta$<br>$= \int \sqrt{\left(\frac{\cos^2\theta}{\sin^2\theta}\right)}\,(2\sin\theta\cos\theta)\,\mathrm{d}\theta$<br>$= \int \left(\frac{\cos\theta}{\sin\theta}\right)(2\sin\theta\cos\theta)\,\mathrm{d}\theta$<br>$= \int 2\cos^2\theta\,\mathrm{d}\theta$<br>$= \int \cos 2\theta + 1\,\mathrm{d}\theta$<br>$= \frac{1}{2}\sin 2\theta + \theta + C$<br>$= \sin\theta\cos\theta + \theta + C$<br>$= \left(\sqrt{x}\right)\left(\sqrt{(1-x)}\right) + \sin^{-1}\sqrt{x} + C$<br>$= \sqrt{(x - x^2)} + \sin^{-1}\sqrt{x} + C$ | Integration by trigonometric substitution to yield an integrand in the form of a double-angle trigonometric function is routine within the scope of H2 Mathematics. Upon correctly deriving intermediary results from the substitution, candidates should successfully obtain the integration result $\frac{1}{2}\sin 2\theta + \theta + C$.<br><br>At this juncture, candidates may encounter difficulty in returning the result in terms of $x$. Successful attempts to this question will indicate some association between the substitution and right triangles, from which the relationship between $x$ and $\theta$ comes most naturally.<br><br>It is highly likely for unaware candidates to conclude their working prematurely and leave their final answer in terms of $\theta$. To curb this tendency, candidates may wish to immediately make a mental note to back-substitute upon seeing a substitution and an indefinite integral. |

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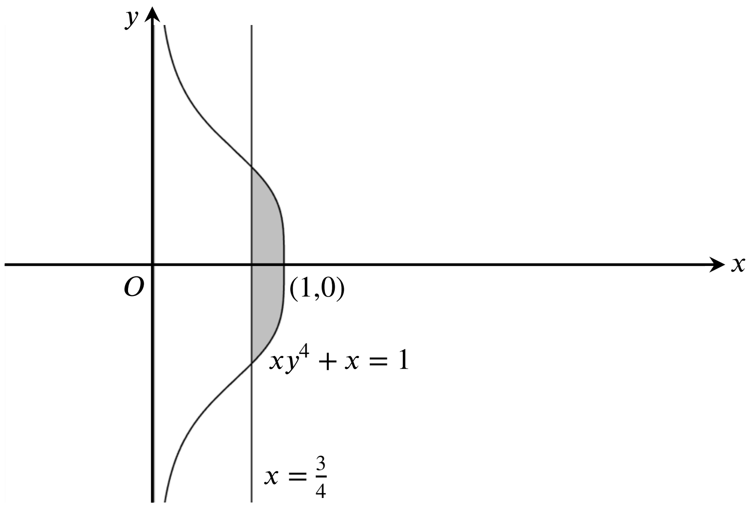
| **3 (b) [3]** | To find the limits of integration, consider sketching the curve $xy^4 + x = 1$ using the parametric equations $x = \dfrac{1}{t^4+1}$, $y = t$ for $t \in \mathbb{R}$<br><br>Hence, volume obtained when revolving the shaded region<br>$= \pi \displaystyle\int_{\frac{3}{4}}^{1} \sqrt{\left(\frac{1-x}{x}\right)}\,\mathrm{d}x$<br>$= \pi \left[\sqrt{(x-x^2)} + \sin^{-1}\sqrt{x}\right]_{\frac{3}{4}}^{1}$<br>$= \pi \left[0 + \pi - \left(\dfrac{\sqrt{3}}{4} + \dfrac{1}{3}\pi\right)\right]$<br>$= \dfrac{2}{3}\pi^2 - \dfrac{\sqrt{3}}{4}\pi$ units$^3$ | Although the bulk of this question concerns evaluating definite integrals, candidates may instead find it more difficult to define the region to be revolved, having been given a curve equation that cannot be directly expressed in the form $y = \mathrm{f}(x)$ expected by the graphing calculator.<br><br>To sketch curves with equations of the form $x = \mathrm{f}(y)$ using a graphing calculator, candidates may consider sketching the following instead: (1) a parameterised version of the equation in the form $y = t$ and $x = \mathrm{f}(t)$, or (2) a version of the curve reflected about the line $y = x$ to give $y = \mathrm{f}(x)$. While the latter strategy is much more intuitive and easier to visualise, there is larger room for human error. |

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| **4 (a) [2]** | By similar triangles, we have:<br>$\rightarrow \dfrac{\lvert\overrightarrow{PX}\rvert}{\lvert\overrightarrow{QX}\rvert} = \dfrac{a}{b}$<br>$\rightarrow b\overrightarrow{PX} = a\overrightarrow{QX}$<br>$\rightarrow b\left(\overrightarrow{OX} - \overrightarrow{OP}\right) = a\left(\overrightarrow{OX} - \overrightarrow{OQ}\right)$<br>$\rightarrow\ b\overrightarrow{OX} - b\overrightarrow{OP} = a\overrightarrow{OX} - a\overrightarrow{OQ}$<br>$\rightarrow a\overrightarrow{OQ} - b\overrightarrow{OP} = a\overrightarrow{OX} - b\overrightarrow{OX}$<br>$\rightarrow a\mathbf{q} - b\mathbf{p} = (a-b)\overrightarrow{OX}$<br>$\therefore \overrightarrow{OX} = \left(\dfrac{a}{a-b}\right)\mathbf{q} - \left(\dfrac{b}{a-b}\right)\mathbf{p}$ | It might not be particularly obvious from the diagram that $P$, $Q$ and $X$ are collinear. Candidates may choose to consider, for each circle, the quadrilateral formed by its centre, its two tangential points, and the point $X$, and therefore see at least the three angles which are shared between the quadrilaterals, resulting in their similarity. Candidates may from there deduce relevant length ratios involving $X$ with greater conviction. |
| **4 (b) [5]** | Substituting the result in **(a)** with suitable constants and vectors for $Y$ and $Z$,<br>$\overrightarrow{OY} = \left(\dfrac{a}{a-c}\right)\mathbf{r} - \left(\dfrac{c}{a-c}\right)\mathbf{p}$<br>$\overrightarrow{OZ} = \left(\dfrac{b}{b-c}\right)\mathbf{r} - \left(\dfrac{c}{b-c}\right)\mathbf{q}$<br><br>$\overrightarrow{XY}$<br>$= \overrightarrow{OY} - \overrightarrow{OX}$<br>$= \left(\dfrac{a}{a-c}\right)\mathbf{r} - \left(\dfrac{c}{a-c}\right)\mathbf{p} - \left(\dfrac{a}{a-b}\right)\mathbf{q} + \left(\dfrac{b}{a-b}\right)\mathbf{p}$<br>$= \dfrac{1}{(a-b)(a-c)}[a(a-b)\mathbf{r} - c(a-b)\mathbf{p} - a(a-c)\mathbf{q} + b(a-c)\mathbf{p}]$<br>$= \dfrac{1}{(a-b)(a-c)}[(ab - bc - ac + bc)\mathbf{p} - a(a-c)\mathbf{q} + a(a-b)\mathbf{r}]$<br>$= \dfrac{1}{(a-b)(a-c)}[a(b-c)\mathbf{p} + a(c-a)\mathbf{q} + a(a-b)\mathbf{r}]$<br>$= \dfrac{a}{(a-b)(a-c)}[(b-c)\mathbf{p} + (c-a)\mathbf{q} + (a-b)\mathbf{r}]$ | Given the result from **(i)**, the position vector for $Y$ and $Z$ is simply obtained by substituting suitable constants for the scalar quantity respectively.<br><br>Upon proving the collinearity of $X$, $Y$ and $Z$, there are chances candidates may stumble due to the flooding of constants. If need be, candidates may choose to abstract out reoccurring constants (i.e., letting $d = a - c$, and so on) to help reduce clutter and make the colinear relationship more noticeable. |

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| | **[continued]**<br><br>$\overrightarrow{XZ}$<br>$= \overrightarrow{OZ} - \overrightarrow{OX}$<br>$= \left(\frac{b}{b-c}\right)\mathbf{r} - \left(\frac{c}{b-c}\right)\mathbf{q} - \left(\frac{a}{a-b}\right)\mathbf{q} + \left(\frac{b}{a-b}\right)\mathbf{p}$<br>$= \frac{1}{(a-b)(b-c)}[b(a-b)\mathbf{r} - c(a-b)\mathbf{q} - a(b-c)\mathbf{q} + b(b-c)\mathbf{p}]$<br>$= \frac{1}{(a-b)(b-c)}[b(b-c)\mathbf{p} + (-ac+bc-ab+ac)\mathbf{q} + b(a-b)\mathbf{r}]$<br>$= \frac{1}{(a-b)(b-c)}[a(b-c)\mathbf{p} + b(c-a)\mathbf{q} + a(a-b)\mathbf{r}]$<br>$= \frac{b}{(a-b)(b-c)}[(b-c)\mathbf{p} + (c-a)\mathbf{q} + (a-b)\mathbf{r}]$<br>$= \frac{b(a-c)}{a(b-c)}\left\{\frac{a}{(a-b)(a-c)}[(b-c)\mathbf{p} + (c-a)\mathbf{q} + (a-b)\mathbf{r}]\right\}$<br>$= \frac{b(a-c)}{a(b-c)}\overrightarrow{XY}$<br><br>From the two results above,<br>$\rightarrow$ There exists $k \in \mathbb{R}$ such that $\overrightarrow{XZ} = k\overrightarrow{XY}$<br>$\rightarrow \overrightarrow{XZ} \mathbin{//} \overrightarrow{XY}$, with common point $X$<br>$\therefore X, Y$ and $Z$ are collinear | Through some algebraic manipulation, candidates should be able to bring out $\overrightarrow{XY}$ from the expression for $\overrightarrow{XZ}$. |
| **4 (c) [2]** | $Y$ is the midpoint of $XZ$<br>$\rightarrow \overrightarrow{XZ} = 2\overrightarrow{XY}$<br>$\rightarrow \frac{b(a-c)}{a(b-c)} = 2$<br>$\rightarrow ab - bc = 2ab - 2ac$<br>$\rightarrow 2ac - bc = ab$<br>$\rightarrow c = \frac{ab}{2a-b}$ | Candidates who are preconditioned to respond to the keyword “midpoint” by producing $\overrightarrow{OY} = \frac{1}{2}\left(\overrightarrow{OX} + \overrightarrow{OZ}\right)$ may suffer from longer working in this part. A succinct response would exploit the proportionality constant found in **(ii)**, from which the required expression for $c$ can be obtained through simpler manipulations. |

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| **5 (a) [2]** | Letting $n = 0$,<br>$\rightarrow (1-x^2)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\dfrac{\mathrm{d}y}{\mathrm{d}x} + 4k^2y = 0$<br><br>Tangent at $y$-intercept is $y = 1$. This implies that at $x = 0$,<br>$\rightarrow y = 1$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$<br>$\rightarrow \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4k^2 = 0$<br>$\therefore \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = -4k^2$ at $x = 0$ | This question pertaining recurrent differential equations aims to demonstrate the applicability of recurrence relations in concepts other than sequences and series. With unseen topic extensions such as these, H2 Mathematics assessments usually begin the question with a show part to acclimatize candidates with the nature of the topic.<br><br>Given the tangent at $x = 0$, candidates should be able to deduce $y$ and $\frac{\mathrm{d}y}{\mathrm{d}x}$. Noting that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is of interest, candidates should naturally aim to make all three expressions appear in one equation, which is done using the recurrence relation with $n = 0$. |
| **5 (b) [3]** | Letting $n = 1$ and $x = 0$,<br>$\rightarrow (1-0)\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3} - 3(0)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + (4k^2-1)\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$<br>$\rightarrow \dfrac{\mathrm{d}^3y}{\mathrm{d}x^3} + \left(4k^2-1\right)(0) = 0$<br>$\therefore \dfrac{\mathrm{d}^3y}{\mathrm{d}x^3} = 0$<br><br>Letting $n = 2$ and $x = 0$,<br>$\rightarrow (1-0)\dfrac{\mathrm{d}^4y}{\mathrm{d}x^4} - 5(0)\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3} + (4k^2-4)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$<br>$\rightarrow \dfrac{\mathrm{d}^4y}{\mathrm{d}x^4} + (4k^2-4)\left(-4k^2\right) = 0$<br>$\therefore \dfrac{\mathrm{d}^4y}{\mathrm{d}x^4} = 16k^4 - 16k^2$<br><br>$\therefore y = 1 + \dfrac{(-4k^2)}{2!}x^2 + \dfrac{(16k^4-16k^2)}{4!}x^4 + \cdots = 1 - 2k^2x^2 + \dfrac{2}{3}k^2\left(k^2-1\right)x^4 + \cdots$ | Upon successfully showing **(a)**, the expectation is that candidates can now work with the recurrence relation to deduce the value of $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ and $\frac{\mathrm{d}^4y}{\mathrm{d}x^4}$ using higher values of $n$.<br><br>Although Maclaurin expansions are among the major, highly rewarding topics in H2 Mathematics assessments, most marks for such questions are usually awarded for the routine calculus. As this question requires candidates to find the Maclaurin expansion for $y$ using the recurrence relation instead of further differentiation, fewer marks are awarded in this case. |

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| **5 (c) [3]** | $\frac{\mathrm{d}y}{\mathrm{d}x} = -2m\left(\frac{1-y^2}{1-x^2}\right)^{\frac{1}{2}}$<br>$\rightarrow -\frac{1}{\sqrt{(1-y^2)}}\frac{\mathrm{d}y}{\mathrm{d}x} = 2m\frac{1}{\sqrt{(1-x^2)}}$<br>$\rightarrow \int -\frac{1}{\sqrt{(1-y^2)}}\,\mathrm{d}y = 2m\int \frac{1}{\sqrt{(1-x^2)}}\,\mathrm{d}x$<br>$\rightarrow \cos^{-1} y = 2m\sin^{-1} x + C$<br><br>When $x=0$, $y=1 \rightarrow C=0$<br>$\therefore y = \cos(2m\sin^{-1} x)$ | Solving this separable differential equation should not come across as difficult to candidates who are well-acquainted with integration results from the list of formulae.<br><br>As directed by the question, candidates should ensure that the integration results in an inverse cosine function. This can only be achieved by transferring the negative sign to the $y$-expression and afterwards integrating the negated expression with respect to $y$. |
| **5 (d) [5]** | **<u>Method 1 (recommended): Substitution and expansion</u>**<br>$\cos(2m\sin^{-1} x) = 1 - 2k^2x^2 + \frac{2}{3}k^2(k^2-1)x^4 + \cdots$<br><br>Substituting $x = \sin\theta$,<br>$\rightarrow \cos(2m\sin^{-1}(\sin\theta)) = 1 - 2k^2\sin^2\theta + \frac{2}{3}k^2(k^2-1)\sin^4\theta + \cdots$<br>$\rightarrow \cos(2m\theta) = 1 - 2k^2\left(\theta - \frac{1}{6}\theta^3 + \cdots\right)^2 + \frac{2}{3}k^2(k^2-1)(\theta - \cdots)^4 + \cdots$<br>$\rightarrow 1 - \frac{1}{2}(2m\theta)^2 + \frac{1}{24}(2m\theta)^4 + \cdots = 1 - 2k^2\left(\theta^2 - \frac{1}{3}\theta^4 + \cdots\right) + \frac{2}{3}k^2(k^2-1)(\theta^4 + \cdots) + \cdots$<br>$\rightarrow 1 - \frac{4m^2\theta^2}{2} + \frac{16m^4\theta^4}{24} + \cdots = 1 - 2k^2\theta^2 + \frac{2}{3}k^2\theta^4 + \frac{2}{3}k^4\theta^4 - \frac{2}{3}k^2\theta^4 + \cdots$<br>$\rightarrow 1 - 2m^2\theta^2 + \frac{2}{3}m^4\theta^4 + \cdots = 1 - 2k^2\theta^2 + \frac{2}{3}k^4\theta^4 + \cdots$<br><br>By comparison, $k = m$ | This question may prove challenging.<br><br>There are several ways to solve this problem; the suggested solution will discuss three possible methods that candidates would most likely go for.<br><br>**<u>Method 1: Substitution and expansion</u>**<br><br>This method is the most recommended as it requires the least deduction. By introducing a substitution to remove the nested trigonometric function, candidates may obtain more manageable expressions for $y$, which can be expanded using the standard expansions.<br><br>While the suggested answer expands both expressions of $y$ up to the fourth power, candidates may in fact choose to expand only up to the second term to obtain $1 - 2m^2\theta^2 + \cdots = 1 - 2k^2\theta^2 + \cdots$, from which it is acceptable to deduce that $k = m$, though it would be wise to expand further power for more reliability. |

**[Continued]**

**Method 2: Expansion of $\mathbf{f(x) = \sin^{-1} x}$**

$\sin^{-1} x$

$= \int \frac{1}{\sqrt{(1-x^2)}} \,\mathrm{d}x$

$= \int \left(1 + \left(-\frac{1}{2}\right)\left(-x^2\right) + \cdots\right) \mathrm{d}x$

$= \int \left(1 + \frac{1}{2}x^2 + \cdots\right) \mathrm{d}x$

$= x + \frac{1}{6}x^3 + \cdots$

(The same expansion for $\sin^{-1} x$ can be obtained otherwise via repeated differentiation.)

$\therefore y$

$= \cos\left(2m\left(x + \frac{1}{6}x^3 + \cdots\right)\right)$

$= 1 - \frac{1}{2!}\left(2m\left(x + \frac{1}{6}x^3 + \cdots\right)\right)^2 + \frac{1}{4!}\left(2m(x + \cdots)\right)^4 + \cdots$

$= 1 - \frac{4m^2}{2}\left(x^2 + \frac{1}{3}x^4 + \cdots\right) + \frac{16m^4}{24}\left(x^4 + \cdots\right) + \cdots$

$= 1 - 2m^2x^2 - \frac{2}{3}m^2x^4 + \frac{2}{3}m^4x^4 + \cdots$

$= 1 - 2m^2x^2 + \frac{2}{3}m^2\left(m^2 - 1\right)x^4 + \cdots$

From (**b**), $y = 1 - 2k^2x^2 + \frac{2}{3}k^2\left(k^2 - 1\right)x^4 + \cdots$

By comparison, $k = m$

**Method 2: Expansion of g($x$)**

Another possible alternative is obtaining the expansion for the inner function $\sin^{-1} x$, which will entail a great deal of calculus. Candidates choosing this method would most likely begin by differentiating $\sin^{-1} x$, and then proceed to either integrate the expansion of the derivative or continue via repeated differentiation.

As per the first method, expansion up to the second term is sufficient, though more terms will provide better confirmation. However, candidates choosing to do so using either method should take great care when expanding $\left(x + \frac{1}{6}x^3 + \cdots\right)^2$, as it is highly probable for candidates to overlook an $x^4$ term resulting from this multiplication.

**[Continued]**

**Method 3: Further differentiation**

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-2m\left(\frac{1-y^2}{1-x^2}\right)^{\frac{1}{2}}$$

$$\left(1-x^2\right)^{\frac{1}{2}}\frac{\mathrm{d}y}{\mathrm{d}x}=-2m\left(1-y^2\right)^{\frac{1}{2}}$$

**Method 3a: Immediate implicit differentiation**

$$\left(1-x^2\right)^{\frac{1}{2}}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}(-2x)\frac{\mathrm{d}y}{\mathrm{d}x}=-m\left(1-y^2\right)^{-\frac{1}{2}}(-2y)\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\left(1-x^2\right)^{\frac{1}{2}}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-x\left(1-x^2\right)^{-\frac{1}{2}}\frac{\mathrm{d}y}{\mathrm{d}x}=2my\left(1-y^2\right)^{-\frac{1}{2}}\left(-2m\left(\frac{1-y^2}{1-x^2}\right)^{\frac{1}{2}}\right)$$

$$\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-x\frac{\mathrm{d}y}{\mathrm{d}x}=-4m^2y$$

$$\therefore\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)+4m^2y=0$$

**Method 3b: Squaring both sides, followed by implicit differentiation**

$$\left(1-x^2\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=4m^2\left(1-y^2\right)$$

$$2\left(1-x^2\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)-2x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=4m^2(-2y)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

$$2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left[\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)+4m^2y\right]=0$$

$$\therefore\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)+4m^2y=0\quad\text{or}\quad\frac{\mathrm{d}y}{\mathrm{d}x}=0\quad(\text{reject }\because\text{ f is not constant})$$

From **(a)**, $(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-x\frac{\mathrm{d}y}{\mathrm{d}x}+4k^2y=0$

By comparison, $k=m$

**Method 3: Further differentiation**

Candidates who are more confident in implicit differentiation may choose to differentiate the second differential equation instead to obtain a new equation involving $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ using the second relationship, and afterwards compare this result with the one found in **(a)**.

For this method, some candidates may choose to square both sides to get rid of the root exponent before differentiating. Note that this will introduce a spurious solution which must be rejected with proper mathematical reason, or otherwise be penalised.

| Section B: Probability and Statistics [60 marks] | | |
|---|---|---|
| **6** **[4]** | We proceed case-by-case by noting the following:<br>• Needs exactly 7 rightward moves (→) and 7 downward moves (↓).<br>• A downward move can either be single (↓) or double (↓↓), but no more.<br>• There is no restriction on rightward moves.<br><br>We pivot to finding the number of ways of inserting ↓ and ↓↓ into seven consecutive rights →. | This question may prove challenging.<br><br>Like most A-Level H2 Mathematics questions on the topic of permutations and combinations, this question calls for candidates to see the problem from different perspectives, reframing the given situation suitably to formulate a response using existing probability techniques and counting principles.<br><br>Besides the need for different perspectives, there is not much that can be further discussed technique-wise in this question. Once candidates find that this situation can be modelled into an insertion problem, the answer should fall right into place. |

| Case | Number of ways |
|---|---|
| 7 ↓ and 0 ↓↓ | $\binom{8}{7}\times\frac{7!}{7!\,0!}=8$ |
| 5 ↓ and 1 ↓↓ | $\binom{8}{6}\times\frac{6!}{5!\,1!}=168$ |
| 3 ↓ and 2 ↓↓ | $\binom{8}{5}\times\frac{5!}{3!\,2!}=560$ |
| 1 ↓ and 3 ↓↓ | $\binom{8}{4}\times\frac{4!}{1!\,3!}=280$ |

$\therefore$ Required number of ways $= 8 + 168 + 560 + 280 = 1016$

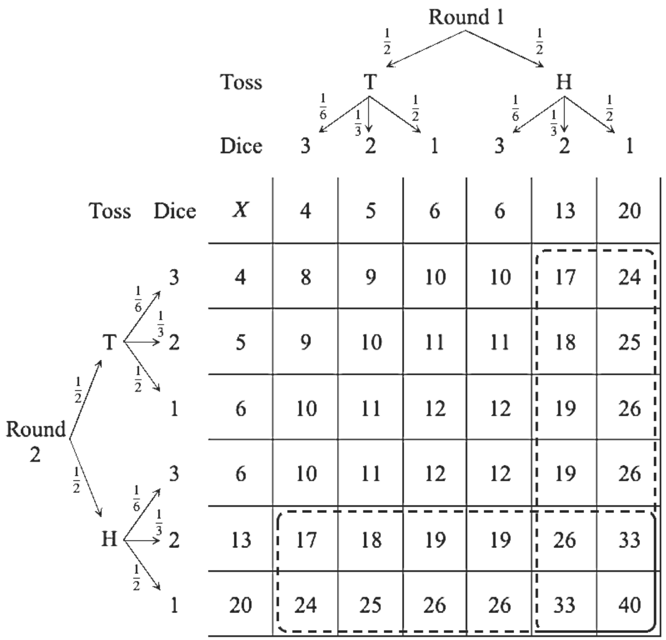
| | | |
|---|---|---|
| **7 (a) [1]** | In general, $p$-value is the probability that an obtained sample gives a test statistic that is <u>as extreme or more extreme</u> than the one being observed, assuming that the null hypothesis is true. | Candidates should engage with the fact that the $p$–value is calculated after obtaining a sample, **and** that it relates to the alternative hypothesis being true. |
| **7 (b) [6]** | Given that the null hypothesis is not rejected if $29.6325 < \bar{t} < 30.3675$, two deductions can be made:<br>• The hypothesis testing is a two-tailed test<br>• $\mu_0 = \frac{1}{2}(29.6325 + 30.3675) = 30$<br>$\therefore H_0 : \mu = 30$ and $H_1 : \mu \neq 30$<br><br>Let the left-tail $z$-value be $k$. Given the test is two-tailed and the distribution of the durations is unknown,<br>$\rightarrow \mathrm{P}(Z \leq k) = \dfrac{1}{2}(0.04004) = 0.0200202$<br>$\rightarrow k = \dfrac{\bar{t} - 30.0}{\frac{s}{\sqrt{n}}} = -2.05333189$ (GC)<br>$\rightarrow \bar{t} = 30.0 - 2.05333189\dfrac{s}{\sqrt{n}}$<br><br>Given $\Sigma(t - \mu_0) = -38.5$ and $\Sigma(t - \mu_0)^2 = 362.869375$, we express the unbiased estimates as follows:<br>$\rightarrow \bar{t} = \dfrac{-38.5}{n} + 30.0$<br>$\rightarrow s^2 = \dfrac{1}{n-1}\left(362.869375 - \dfrac{(-38.5)^2}{n}\right) = \dfrac{362.869375n - 1482.25}{n(n-1)}$<br>$\rightarrow s = \sqrt{\dfrac{362.869375n - 1482.25}{n(n-1)}}$<br><br>Substituting the unbiased estimates into $\bar{t} = 30.0 - 2.05333189\dfrac{s}{\sqrt{n}}$,<br>$\rightarrow \dfrac{-38.5}{n} + 30.0 = 30.0 - 2.05333189\sqrt{\dfrac{362.869375n - 1482.25}{n^2(n-1)}}$<br>$\rightarrow n \approx 100.00432764635$ (GC)<br><br>$\therefore$ The sample size is 100, since sample size $n$ is an integer in this context. | This question may prove challenging.<br><br>This question deconstructs the usual H2 Mathematics way of testing candidates on hypothesis testing – while candidates are usually provided hypotheses and sample size to deduce critical regions and $p$–value in the typical top-down style, here candidates are instead required to deduce from the bottom up.<br><br>Upon attempting this question, candidates may naturally begin by looking for the hypotheses. Those who recall that **critical region is associated with the hypotheses** should be able to deduce the type of test used, as well as the value of $\mu_0$ which is vital in hypothesis testing.<br><br>Having found the value of $\mu_0$, candidates may start deducing expressions for many variables:<br>• Using $\mu_0$ and the fact that the test is **two-tailed**, candidates can express $\bar{t}$ in terms of $s$ and $n$ through standardisation (invNorm).<br>• Using $\mu_0$ and the **summarised data values**, candidates can express $\bar{t}$ and $s$ in terms of $n$ respectively through sampling-related formulae.<br><br>With all three expressions, candidates may eventually perform substitution and finally obtain the integer value of the sample size $n$. |

| | | |
|---|---|---|
| | **[Continued]**<br><br>Finding other values relevant to the hypothesis and conclusion,<br>$\rightarrow \bar{t} = \dfrac{-38.5}{100} + 30.0 = 29.615$<br>$\rightarrow s^2 = \dfrac{1}{99}\left(362.869375 - \dfrac{(-38.5)^2}{100}\right) = 3.515625 \rightarrow s = 1.875$<br>$\rightarrow \alpha\% = 2\mathrm{P}\left(Z \leq \dfrac{29.6325 - 30.0}{\frac{1.875}{\sqrt{100}}}\right) = 0.04999565$<br>$\therefore \alpha = 5 \ (\because \alpha \in \mathbb{Z})$<br><br>Since the observed sample mean $\bar{t} = 29.615$ is outside the range $29.6325 < \bar{t} < 30.3675$, we reject the null hypothesis and conclude that there is sufficient evidence at 5% significance level that the average duration of uploaded videos differs from 30 seconds. | In the final stretch, upon writing the conclusion of the test in context, candidates may realise the need to consolidate the following values:<br>• $\mu_0$, the hypothesised population average,<br>• $\bar{t}$, the sample average, and<br>• $\alpha\%$, the integer significance level.<br><br>With previous deductions, candidates may easily evaluate some of these required values. Finding $\alpha$ is more deliberate, as it calls for $s$ and recalling the fact that the test is two-tailed.<br><br>**The level of detail in the test conclusion as written here in the suggested solution is expected**. Candidates should know how to properly write out the conclusion of a hypothesis testing in context at this point leading to the H2 Mathematics assessment. |
| **7 (c) [1]** | The second sample is large (i.e., $n \geq 30$) and yields $p$-value $> 0.05$ (or $p$-value $> 0.04999565$). | Candidates should engage with the fact that the sample must be large enough so that Central Limit Theorem applies and the $z$-test can be carried out, **and** that the $p$-value obtained from the sample will affect the conclusion. |

| | | |
|---|---|---|
| **8 (a) [3]** | $\mathrm{E}(X^2)$<br>$=\frac{1}{2}\left[\frac{1}{2}(a-1)^2+\frac{1}{3}(2a-1)^2+\frac{1}{6}(3a-1)^2\right]+\frac{1}{2}\left[\frac{1}{2}(a-1)^2+\frac{1}{3}(a-2)^2+\frac{1}{6}(a-3)^2\right]$<br>$=\frac{1}{4}\left[(a-1)^2+(a-1)^2\right]+\frac{1}{6}\left[(2a-1)^2+(a-2)^2\right]+\frac{1}{12}\left[(3a-1)^2+(a-3)^2\right]$<br>$=\frac{1}{4}\left[2(a-1)^2\right]+\frac{1}{6}\left(4a^2-4a+1+a^2-4a+4\right)+\frac{1}{12}\left(9a^2-6a+1+a^2-6a+9\right)$<br>$=\frac{1}{4}\left(2a^2-4a+2\right)+\frac{1}{6}\left(5a^2-8a+5\right)+\frac{1}{12}\left(10a^2-12a+10\right)$<br>$=\frac{1}{2}a^2-a+\frac{1}{2}+\frac{5}{6}a^2-\frac{4}{3}a+\frac{5}{6}+\frac{5}{6}a^2-a+\frac{5}{6}$<br>$=\frac{13}{6}a^2-\frac{10}{3}a+\frac{13}{6}$<br>$=a+\frac{13}{6}a^2-\frac{13}{3}a+\frac{13}{6}$<br>$=a+\frac{13}{6}(a-1)^2$ | This question test candidates on the evaluation of $\mathrm{E}(X^2)$ given a complex context – indeed, the modulus sign may understandably discourage many candidates.<br><br>When writing $X^2$ for different cases of $X=x$, candidates may also want to be privy that $\|x\|^2=x^2$. Upon careful consideration of the cases, and a painstaking session of algebraic manipulations, candidates may soon arrive at the result to be shown. |
| **8 (b) [2]** | $\mathrm{E}(X)=\frac{1}{2}\left(\frac{1}{2}\|a-1\|+\frac{1}{3}\|2a-1\|+\frac{1}{6}\|3a-1\|\right)+\frac{1}{2}\left(\frac{1}{2}\|a-1\|+\frac{1}{3}\|a-2\|+\frac{1}{6}\|a-3\|\right)$<br><br>$\mathrm{E}(X^2)=a+\frac{13}{6}(a-1)^2$ is an integer<br>$\rightarrow (a-1)^2$ is a multiple of 6<br>$\rightarrow a=1$ or $a=7$ ($\because a$ is a single-digit integer, i.e., $1\le a\le 9$, $a\in\mathbb{Z}$)<br><br>When $a=1$,<br>$\rightarrow \mathrm{E}(X^2)=1$ and $\mathrm{E}(X)=\frac{2}{3}$<br>$\rightarrow \mathrm{E}(X)$ is not an integer<br>$\rightarrow a\neq 1$<br><br>$\therefore a=7$ | The expression for $\mathrm{E}(X)$ can be easily written by adjusting accordingly the expression for $\mathrm{E}(X^2)$ found earlier.<br><br>Deducing $a$ is relatively simple and calls for rudimentary number theory. The single-digit requirement for $a$ narrows down its possible values to nine integers, and the fact that $\mathrm{E}(X^2)$ must also be an integer further eliminates those possibilities to just two integers.<br><br>The final deduction requires the use of $\mathrm{E}(X)$, as per directed by the question. |

**8 (c) [5]**

Consider the tree-table hybrid representation of the outcomes for $S$ (winning outcome is dotted).




| Toss | Dice | $X$ | 4 | 5 | 6 | 6 | 13 | 20 |
|---|---|---|---|---|---|---|---|---|
| T | 3 | 4 | 8 | 9 | 10 | 10 | 17 | 24 |
| T | 2 | 5 | 9 | 10 | 11 | 11 | 18 | 25 |
| T | 1 | 6 | 10 | 11 | 12 | 12 | 19 | 26 |
| H | 3 | 6 | 10 | 11 | 12 | 12 | 19 | 26 |
| H | 2 | 13 | 17 | 18 | 19 | 19 | 26 | 33 |
| H | 1 | 20 | 24 | 25 | 26 | 26 | 33 | 40 |

Let $X_1$ and $X_2$ be the outcome of Round 1 and Round 2 respectively, and the payback amount be $\$A$.

Using the diagram, $\mathrm{P}(A = y)$
$= \mathrm{P}(X_1 \in \{13, 20\} \cup X_2 \in \{13, 20\})$
$= \mathrm{P}(X_1 \in \{13, 20\}) + \mathrm{P}(X_2 \in \{13, 20\}) - \mathrm{P}(X_1 \in \{13, 20\} \cap X_2 \in \{13, 20\})$
$= 2\left(\frac{1}{2} \times \frac{5}{6}\right) - \left(\frac{1}{2} \times \frac{5}{6}\right)^2 = \frac{95}{144}$

To profit in the long run, $\mathrm{E}(A - 1) > 0$
$\rightarrow 0\left(1 - \frac{95}{144}\right) + y\left(\frac{95}{144}\right) - 1 > 0$
$\rightarrow y > \frac{144}{95} \approx 1.52$

$\therefore$ Least integer $y = 2$

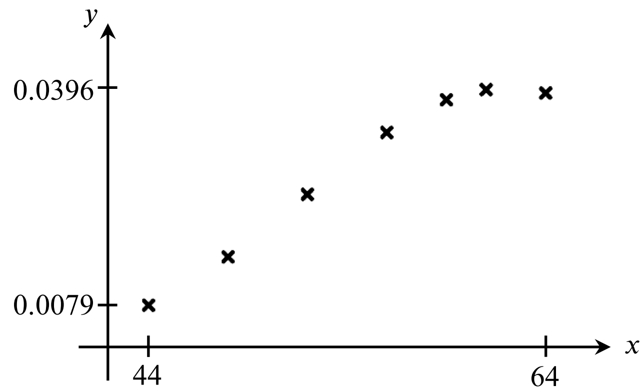
This question may prove challenging.

To begin attempting this question, candidates must first realise the event space for the given context. Once candidates notice how to obtain the number of all possible values of $S = X_1 + X_2$, it should naturally follow that **a table of outcomes is required to map out all possible outcomes that results from the two events**. This table would then inform candidates which outcomes are desirable.

Upon filling out the table with possible values of $X_1$ and $X_2$ as well as their sums $S$, candidates can then continue calculating the probabilities of each sum happening. While it may be time-consuming to fill out each cell with the related probabilities, **candidates may choose to present the information in more convenient ways** – the suggested solution provides one such presentation that are acceptable.

Once candidates have all the information – the possible values of $S$ and the probability of each happening – candidates can the begin to sum up the probabilities of all desirable events. Again, summing up probabilities cell-by-cell will be very time consuming. As such, candidates may consider **using Venn diagram to group desirable events and simplify calculations**.

In the final stretch, candidates deduce long-term results using expectation calculations. Great care must be taken to interpret "profit" as **the difference between the payback amount and the \$1 initial payment**.

<table>
<tr>
<td>9<br>(a)<br>[1]</td>
<td>
</td>
<td>When drawing the scatter diagram, candidates should be mindful to clarify the distance between points with close values. In this case, for instance, there are three points with $y = 0.0381$, $0.0396$ and $0.0391$ which require proper vertical distinction. Otherwise, the diagram would seem to insinuate that they all have the same $y$-value, which is incorrect and therefore penalisable.</td>
</tr>
<tr>
<td>9<br>(b)<br>[1]</td>
<td>On a normal distribution curve $y = \mathrm{f}(x)$, the maximum point occurs when $x = \mu$. In the sketch, $\mathrm{f}(x)$ increases from $x = 59$ to $x = 61$, then decreases from $x = 61$ to $x = 64$. This implies that <u>$\mathrm{f}(x)$ is maximum at either $x = 61$ or somewhere else in the range $59 < x < 64$</u>. Either way, this implies $59 < \mu < 64$.</td>
<td>This question requires candidates to recall that the population mean of a continuous random variable corresponds to the maximum of its distribution curve.</td>
</tr>
<tr>
<td>9<br>(c)<br>[4]</td>
<td>$y = ae^{b(x-\mu)^2} \rightarrow \ln y = \ln a + b(x-\mu)^2$<br><br>
Considering the product moment correlation coefficient for different values of $\mu$,<br><br>
<table>
<tr><th>$\mu$</th><th>$r$</th></tr>
<tr><td>60</td><td>−0.9950770639</td></tr>
<tr><td>61</td><td>−0.9990560121</td></tr>
<tr><td>62</td><td>−0.9999988238</td></tr>
<tr><td>63</td><td>−0.9993803383</td></tr>
</table><br>
Using the table, $\mu = 62$ as the value $r = -0.9999988238$ is closest to $-1$, which gives the strongest negative linear correlation between $\ln \mathrm{f}(x)$ and $(x-\mu)^2$ as compared to the other possible values of $\mu$.<br><br>
When $\mu = 62$,<br>
$\rightarrow \ln y = -3.222488544 - 0.0049923557(x-62)^2$<br>
$\rightarrow \ln a = -3.222488544$ and $b = -0.0049923557$<br>
$\therefore a \approx 0.0399$ and $b \approx -0.00499$</td>
<td>To determine the most appropriate value of $\mu$, candidates should first understand that they are aiming for a $\mu$-value which gives the best model, and therefore the basis of comparison for its appropriateness would entail a discussion regarding product moment correlation coefficients, as hinted by the question. There are only a few cases of $r$-values to be considered, having known that $\mu$ is an integer and $59 < \mu < 64$.<br><br>
Candidates may wish to check out similar questions that appeared in past H2 Mathematics papers:<br>
• 2012/P2/Q8(iv)-(vii)</td>
</tr>
</table>

| | | |
|---|---|---|
| **9 (d) [3]** | Using $\ln y = -3.222488544 - 0.0049923557(x-62)^2$,<br>$\rightarrow \ln 0.0100 = -3.222488544 - 0.0049923557(x-62)^2$<br>$\rightarrow (x-62)^2 = \dfrac{3.222488544 - \ln 0.0100}{0.0049923557}$<br>$\rightarrow x = 62 \pm \sqrt{\left(\dfrac{3.222488544 + \ln 0.0100}{-0.0049923557}\right)}$<br>$\rightarrow x \approx 78.64211$ or $45.35789$<br>$\therefore x \approx 78.6$ or $45.4$<br><br>**Both estimates are equally reliable** because <u>both are obtained from a $y$-value within the data range</u> $(0.0079 < y < 0.0391)$, which is not an extrapolation, and <u>the product moment correlation coefficient $r$ of the linear regression model used is very close to $-1$</u> $(r = -0.9999988238 \approx -1)$, which implies that the model used to obtain the estimation is a good fit. | Candidates are reminded to use the values of $a$ and $b$ in the previous part which have not been rounded to three significant figures, as they would otherwise affect the accuracy of the results in this part.<br><br>For the reliability argument, candidates must mention both interpolation and the strength of the model. Moreover, this question tests on candidates' understanding that **an estimate is unreliable not because the estimate is outside the data range, but because the given input is** – the estimate $x \approx 78.6$ is thus not unreliable. |
| **9 (e) [2]** | Since <u>the regression line</u> exactly passes through <u>the average of the transformed data</u>,<br>$\rightarrow (x-62)^2 = 95.71428571$ (GC) and $\ln y = -3.700328299$ (GC)<br>$\rightarrow x = 62 \pm \sqrt{(95.71428571)}$ and $y = \mathrm{e}^{-3.700328299}$<br>$\therefore$ The possible data points are $(71.8, 0.0247)$ or $(52.2, 0.0247)$ | This question requires candidates to recall that the regression line passes through the average of the dataset used to produce it. As such, candidates should take care to calculate average values of **the transformed dataset** and use those values in **the linearized model** to deduce the required data point.<br><br>Any evidence of using the original dataset and model to deduce the required data point would indicate a grave misconception and thus cannot be awarded any mark. |

| | | |
|---|---|---|
| **10 (a) [1]** | The event that the stock price rises from one period to the next is independent of any other event where the price rises from any other period to the next. | A staple in A–Levels, questions on binomial distribution condition should be a low-hanging fruit for all candidates. |
| **10 (b) [2]** | Let $m$ be the number of times the price goes up.<br><br>For the price of the stock $S$ to be ideal after 10 periods,<br>$\rightarrow S = r^m\left(\frac{1}{r}\right)^{10-m}S_0 = r^{2m-10}S_0 = r^nS_0$, where $n$ is either 2, 3, 4, or 5<br>$\rightarrow 2m - 10 = 2, 3, 4, \text{ or } 5$<br>$\rightarrow m = \frac{10+2}{2}, \frac{10+3}{2}, \frac{10+4}{2} \text{ or } \frac{10+5}{2} = 6, \frac{13}{2}, 7 \text{ or } \frac{15}{2}$.<br>Since $m \in \mathbb{Z}$, the price of stock $S$ is ideal when it rises 6 or 7 times.<br><br>Let $X\sim\text{B}\left(\frac{1}{2}, 10\right)$ denote the event that a price rises from a period to the next.<br>$\therefore$ Required probability $= \text{P}(X = 6) + \text{P}(X = 7) = \frac{105}{512} + \frac{15}{128} = \frac{165}{512}$ | This part aims to prime candidates for the remainder of question by establishing the mathematical idea of achieving the “ideal price”.<br><br>The first concept to grasp is that the resulting price $S$ at period 10 depends on the number of times $m$ the price rises (or drops). With this understanding, candidates could express $S$ in terms of $m$ and equate it with the ideal prices to find possible values of $m$.<br><br>Secondly, with the given information in the question, candidates could intuit the number of periods as the number of trials and deduce the probability of success from the keyword “equal probability”. Using a suitable binomial distribution with the deduced possible values of $m$ for which the price is ideal, candidates may then calculate relevant probabilities to obtain the final total probability.<br><br>Candidates may wish to check out similar questions that appeared in past H2 Mathematics papers:<br>• 2013/P1/Q10(i)-(iii)<br>• 2016/P1/Q9(i)<br>• 2018/P1/Q10 |

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| **10 (c) [4]** | Let $m$ be the number of times the price goes up.<br><br>For the price of the stock $S$ to be ideal after $2k$ periods,<br>$\rightarrow 2m - 2k = 2, 3, 4, \text{ or } 5$<br>$\rightarrow m = k + 1, k + 1.5, k + 2 \text{ or } k + 2.5$<br>Since $m \in \mathbb{Z}$, the price of stock $S$ is ideal when it rises $(k + 1)$ or $(k + 2)$ times.<br><br>Let $X_{2k} \sim \mathrm{B}\left(\frac{1}{2}, 2k\right)$ denote the event that a price rises from a period to the next. Hence, $p_{2k}$<br>$= \mathrm{P}(X_{2k} = k + 1) + \mathrm{P}(X_{2k} = k + 2)$<br>$= \binom{2k}{k+1}\left(\frac{1}{2}\right)^{k+1}\left(\frac{1}{2}\right)^{2k-(k+1)} + \binom{2k}{k+2}\left(\frac{1}{2}\right)^{k+2}\left(\frac{1}{2}\right)^{2k-(k+2)}$<br>$= \left(\frac{1}{2}\right)^{2k}\left[\binom{2k}{k+1} + \binom{2k}{k+2}\right]$<br><br>Similarly, for $2k + 1$ periods, the probability $p_{2k+1}$<br>$= \binom{2k+1}{k+2}\left(\frac{1}{2}\right)^{k+2}\left(\frac{1}{2}\right)^{2k+1-(k+2)} + \binom{2k+1}{k+3}\left(\frac{1}{2}\right)^{k+3}\left(\frac{1}{2}\right)^{2k-(k+3)}$<br>$= \left(\frac{1}{2}\right)^{2k+1}\left[\binom{2k+1}{k+2} + \binom{2k+1}{k+3}\right]$ | This part generalizes the concept in **(b)** for two cases. Candidates could at this point recognise that the parity of period number affects the ideal price that can be achieved – to be exact, odd periods yield $r^3 S_0$ and $r^5 S_0$ while even periods yield $r^2 S_0$ and $r^4 S_0$ – it should thus make sense why the probability expression must be distinguished based on the period parity.<br><br>After obtaining one of the two required expressions, candidates may simply proceed with a suitable substitution to obtain the other expression, though candidates are welcomed to replicate the deduction process for thoroughness. |

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<td>10<br>(c)<br>[5]</td>
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Let $p_t$ be the probability of selling at an ideal price at period $t$.

For $t_{\text{odd}}$, find $k$ for which $p_{2k+1} \geq p_{2k+3}$

$\rightarrow p_{2k+1} - p_{2k+3} = \left(\frac{1}{2}\right)^{2k+1}\left[\binom{2k+1}{k+2} + \binom{2k+1}{k+3}\right] - \left(\frac{1}{2}\right)^{2k+3}\left[\binom{2k+3}{k+3} + \binom{2k+3}{k+4}\right] \geq 0$

| $k$ | $p_{2k+1} - p_{2k+3}$ | Is $p_{2k+1} - p_{2k+3} \geq 0$? |
|---|---|---|
| 5 | $-0.003$ | No |
| 6 | 0 | Yes |
| 7 | 0.0016 | Yes |

Using GC, $p_{2k+1} \geq p_{2k+3}$ when $k \geq 6$

$\rightarrow p_{13} \geq p_{15} \geq p_{17} \geq \cdots \rightarrow p_{13}$ is maximum for odd periods

$\rightarrow$ Also, when $k = 6$, $p_{2k+1} - p_{2k+3} = 0 \rightarrow p_{13} = p_{15}$

$\therefore t_{\text{odd}} = 13$ or 15

Similarly, for $t_{\text{even}}$, find $k$ for which $p_{2k} \geq p_{2k+2}$

$\rightarrow p_{2k} - p_{2k+2} = \left(\frac{1}{2}\right)^{2k}\left[\binom{2k}{k+1} + \binom{2k}{k+2}\right] - \left(\frac{1}{2}\right)^{2k+2}\left[\binom{2k+2}{k+2} + \binom{2k+2}{k+3}\right] \geq 0$

| $k$ | $p_{2k} - p_{2k+2}$ | Is $p_{2k} - p_{2k+2} \geq 0$? |
|---|---|---|
| 2 | $-0.016$ | No |
| 3 | 0 | Yes |
| 4 | 0.0059 | Yes |

Using GC, $p_{2k} \geq p_{2k+2}$ when $k \geq 3$

$\rightarrow p_6 \geq p_8 \geq p_{10} \geq \cdots \rightarrow p_6$ is maximum for even periods

$\rightarrow$ Also, when $k = 3$, $p_{2k} - p_{2k+2} = 0 \rightarrow p_6 = p_8$

$\therefore t_{\text{even}} = 6$ or 8

For odd periods, ideally sell at $t = 13$ or 15 to get $r^3 S_0$ or $r^5 S_0$ with probability $p_{13} = p_{15} = 0.2444$

For even periods, ideally sell at $t = 6$ or 8 to get $r^2 S_0$ or $r^4 S_0$ with probability $p_6 = p_8 = 0.3281$

(Any suggestion is acceptable <u>only when justified using the values above</u>.)

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This part may prove challenging.

Firstly, candidates are to engage with the keyword "maximum probability" to recognise that for binomially distributed events, **the number of trials $n$ with maximum success probability must have higher probability than that for fewer trials** ($n-1, n-2, \ldots$) **and more trials** ($n+1, n+2, \ldots$).

However, due to the context of the question, candidates need to also **separate odd-numbered trials from even-numbered, which in turn affects how candidates use the probability expression in (c)**. It is sufficient to find the required $n$ from just one direction – this suggested solution uses success probabilities of more trials.

Either of the parities are reasonable suggestion for the best sell period – explaining its appeal requires more effort. Any justification substantiated with calculation of related probability will be awarded the mark. **The ideal explanation would engage with fact that there are greater rewards for greater risk –** while odd periods yield more, the wait is long and the chances are low; meanwhile, even periods yield less but only after a short wait and with higher chances.

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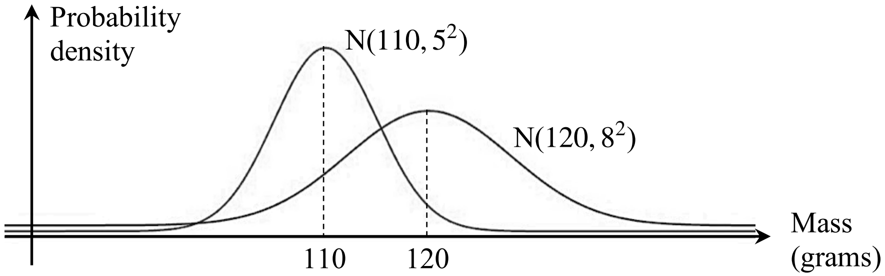
| **11 (a) [2]** |  | As the marks suggests, there are two features of interest in the combined distribution sketch: (1) the distribution with the lower standard deviation will have a **higher peak**, and (2) hence, a **lower asymptote**. |
| **11 (b) [4]** | Let $M_A$ and $M_P$ be the mass, in grams, of a randomly chosen apple and pear respectively.<br><br>For the mean of all fruits combined, $M_{\text{combined}}$,<br>$\rightarrow M_{\text{combined}} = \frac{1}{10}\left(\sum_{i=1}^{6} M_{A_i} + \sum_{i=1}^{4} M_{P_i}\right)$<br>$\rightarrow M_{\text{combined}} \sim \mathrm{N}\left(\frac{1}{10}(6 \cdot 110 + 4 \cdot 120), \frac{1}{10^2}(6 \cdot 5^2 + 4 \cdot 8^2)\right) \rightarrow \mathrm{N}\left(114, \frac{406}{100}\right)$<br><br>For the average of mean of each fruit type, $M_{\text{separated}}$,<br>$\rightarrow M_{\text{separated}} = \frac{1}{2}\left(\frac{1}{6}\sum_{i=1}^{6} M_{A_i} + \frac{1}{4}\sum_{i=1}^{4} M_{P_i}\right) = \frac{1}{12}\sum_{i=1}^{6} M_{A_i} + \frac{1}{8}\sum_{i=1}^{4} M_{P_i}$<br>$\rightarrow M_{\text{separated}} \sim \mathrm{N}\left(\frac{1}{12}(6 \cdot 110) + \frac{1}{8}(4 \cdot 120), \frac{1}{12^2}(6 \cdot 5^2) + \frac{1}{8^2}(4 \cdot 8^2)\right) \rightarrow \mathrm{N}\left(115, \frac{121}{24}\right)$<br><br>For the difference of the two means,<br>$\rightarrow M_{\text{separated}} - M_{\text{combined}} \sim \mathrm{N}\left(115 - 114, \frac{121}{24} - \frac{406}{100}\right) \rightarrow \mathrm{N}\left(1, \frac{589}{600}\right)$<br><br>$\therefore$ Required probability<br>$= \mathrm{P}(\lvert M_{\text{separated}} - M_{\text{combined}} \rvert > 2)$<br>$= 1 - \mathrm{P}(\lvert M_{\text{separated}} - M_{\text{combined}} \rvert \leq 2)$<br>$= 1 - \mathrm{P}(-2 \leq M_{\text{separated}} - M_{\text{combined}} \leq 2)$<br>$\approx 0.158$ | This question requires candidates to properly model the two averaging methods into a suitable normal distribution. Without great care, candidates may stumble and produce incorrect parameter values.<br><br>Candidates are expected to engage with the keyword "difference" and identify the use of modulus operator. Another thing of note here is the need to use complementary probabilities.<br><br>This exercise aims to highlight the conundrum of accuracy when using different averaging methods – the probability is **large** for the two averages to have even a **small** difference. |

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|---|---|---|
| **11 (c) [5]** | Let $X$ and $Y$ denote the mass, in grams, of a randomly chosen fruit $X$ and fruit $Y$ respectively.<br><br>Modelling the situation,<br>$\rightarrow \mathrm{P}(X > m) > \mathrm{P}(Y > m)$<br>$\rightarrow \mathrm{P}\left(Z > \frac{m-\mu_1}{\sigma_1}\right) > \mathrm{P}\left(Z > \frac{m-\mu_2}{\sigma_2}\right)$<br>$\rightarrow \frac{m-\mu_1}{\sigma_1} < \frac{m-\mu_2}{\sigma_2}$<br>$\rightarrow \sigma_2 m - \sigma_2\mu_1 < \sigma_1 m - \sigma_1\mu_2$<br>$\rightarrow \sigma_1\mu_2 - \sigma_2\mu_1 < (\sigma_1 - \sigma_2)m$<br>$\therefore \frac{\sigma_1\mu_2 - \sigma_2\mu_1}{\sigma_1 - \sigma_2} < m$<br><br>To find the least integer $m$ for the fruit to more likely be a pear than an apple,<br>$\rightarrow \frac{\sigma_1\mu_2 - \sigma_2\mu_1}{\sigma_1 - \sigma_2} = \frac{8 \cdot 110 - 5 \cdot 120}{8 - 5} = \frac{280}{3} \approx 93.3 < m$<br>$\therefore$ Required least integer $m = 94$.<br><br>Using $m = 94$, we have that for pears $\mathrm{P}(X > m) \approx 0.999$, while for apples $\mathrm{P}(Y > m) \approx 0.999$. This means that <u>it is almost certain that a fruit of either kind (apple or pear) will exceed 94 grams</u>. To have a higher chance of distinguishing the fruit correctly, the vendor may use a larger value of $m$.<br><br>(For instance, with $m = 110$, we have that for pears $\mathrm{P}(X > m) \approx 0.894$, while for apples $\mathrm{P}(Y > m) = 0.5$, which implies that it is $0.894 - 0.5 \approx 40\%$ more likely that a fruit reading more than 110 grams on the scale is a pear than an apple.) | This part may prove challenging.<br><br>Candidates should first recognise that it is impossible to obtain anything meaningful by directly comparing probabilities of two different normal distributions, and that there is a need to "standardise" these two instances. Along this line of thought, standard normal distribution should come to mind. With this, candidates should be able to transform the probability comparison into a more meaningful $z$-value comparison, which eventually leads to the required results.<br><br>The latter part on explaining the use of a higher weight benchmark requires candidates to consider the applicability of theoretical results. Candidates are expected to use the value they obtained in context to assess whether it is fit for use. In this case, candidates will find that using $m = 94$, while theoretically valid, is unrealistic – even at first glance, it is almost guaranteed that fruits with expected masses 110 grams and 120 grams will be heavier than 94 grams. |

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<td>11<br>(d)<br>[4]</td>
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Consider the following four cases of the total masses of the three fruits:

| Case | Number of fruits | Distribution | `[GC keystroke]` Probability of the scale reading $\leq k$ |
|---|---|---|---|
| 1 | 3 apple(s), 0 pear(s) | $\mathrm{N}(330, 75)$ | `normalcdf(-1E99, k, 330, √(75))` |
| 2 | 2 apple(s), 1 pear(s) | $\mathrm{N}(340, 114)$ | `normalcdf(-1E99, k, 340, √(114))` |
| 3 | 1 apple(s), 2 pear(s) | $\mathrm{N}(350, 153)$ | `normalcdf(-1E99, k, 350, √(153))` |
| 4 | 0 apple(s), 3 pear(s) | $\mathrm{N}(360, 192)$ | `normalcdf(-1E99, k, 360, √(192))` |

Let $p_n$ be the probability of the scale reading $\leq k$ grams for Case $n$.

Modelling the situation given 3 fruits,
$\rightarrow \mathrm{P}(\text{more apples than pears} \mid \text{scale} \leq k) > 0.75$
$\rightarrow \dfrac{\mathrm{P}(\text{more apples than pears} \cap \text{scale} \leq k)}{\mathrm{P}(\text{scale} \leq k)} = \dfrac{p_1 + p_2}{p_1 + p_2 + p_3 + p_4} > 0.75$
$\rightarrow p_1 + p_2 > 0.75\,(p_1 + p_2 + p_3 + p_4)$
$\rightarrow 4p_1 + 4p_2 > 3p_1 + 3p_2 + 3p_3 + 3p_4$
$\rightarrow p_1 + p_2 - 3(p_3 + p_4) > 0$

| $k$ | $p_1 + p_2 - 3(p_3 + p_4)$ |
|---|---|
| 345 | 0.191 |
| 346 | 0.0925 |
| 347 | −0.016 |

Using GC, $k = 346$

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This part may also prove challenging.

To begin, candidates may note the following from the keywords in the question to strategise their response:

- **Cases**: the keyword "three fruits" and the fact that there are two types of fruits – apples and pears – should immediately signal the need to consider cases. Accordingly, the normal distributions of the total mass the of three fruits differ for each case.
- **Probability**: The keyword "if" should also signal the need for considering conditional probability. The keyword "at most" should also be carefully interpreted.
- **Presentation**: the keyword "integer" should immediately hint the appeal to tabulated presentation using GC.

Candidates can only afford so much simplification through manual handiwork up to a point where GC is considerably more effective.

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