H3 MATHEMATICS (9820)
COMBINATORICS 3
RECURRENCE RELATIONS AND COUNTING TWO WAYS

Recurrence Relations

Example 1

Figure 1.1 shows a 9-step staircase. A boy wishes to climb the staircase up the highest step. Suppose that each time, the boy either climbs up one step or two steps. How many ways are there for the boy to climb up?

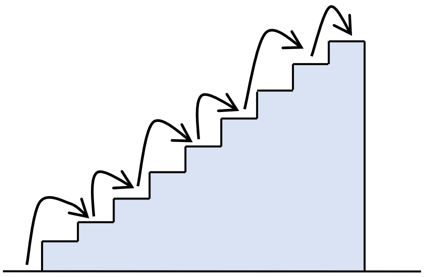


Figure 1

Using the problem-solving heuristic of "reduce the numbers", we can look at the number of ways to climb smaller staircases. Let u_n be the number of ways of climbing a n-step staircase. Then we can manually calculate say $u_1 = 1$, $u_2 = 2$, $u_3 = 3$. Unfortunately $u_4 = 5$ breaks the pattern, but $u_5 = 8$ lets us think of another pattern. Recalling what we know about the Fibonacci sequence, we realise that the crux is asking the question "What does the boy do on his first move?" and then considering the two cases:

Case 1 The first move covers 1 step: There are $n-1$ steps remaining. So there are u_{n-1} ways to climb.

Case 2 The first move covers 2 steps: There are $n-2$ steps remaining so u_{n-2} ways to climb.

Therefore, $u_n = u_{n-1} + u_{n-2}$ for $n \geq 3$. We also have to specify enough starting terms: $u_1 = 1$, $u_2 = 2$.

With these, GC can be used to calculate terms of the recurrence relation. H2 Further Mathematics teaches methods to solve some recurrence relations (solving means finding u_n in terms of n). The H3 syllabus does not expect us to know these methods.

Key steps:

1. Define the terms

2. Find the initial terms

3. Find the recurrence relation

Example 2

John has a large number of 10-cent, 20-cent and 50-cent coins. How many different coin sequences can he put in a vending machine to purchase a \$1.00 drink with no change leftover? Note that the sequence ‘1 50-cent coin, followed by 5 10-cent coins’ is different from the sequence ’5 10-cent coins, followed by 1 50-cent coin’.

Solution

Example 3 (Self-Reading)

A tower of n circular discs of different sizes is stacked on one of the 3 given pegs in decreasing size from the bottom, as shown in Figure 4. The task is to transfer the entire tower to another peg by a sequence of moves under the following conditions:

(i) Each move carries exactly one disc.

(ii) No disc can be placed on top of a smaller one.

What is the minimum number of moves required to accomplish the task?

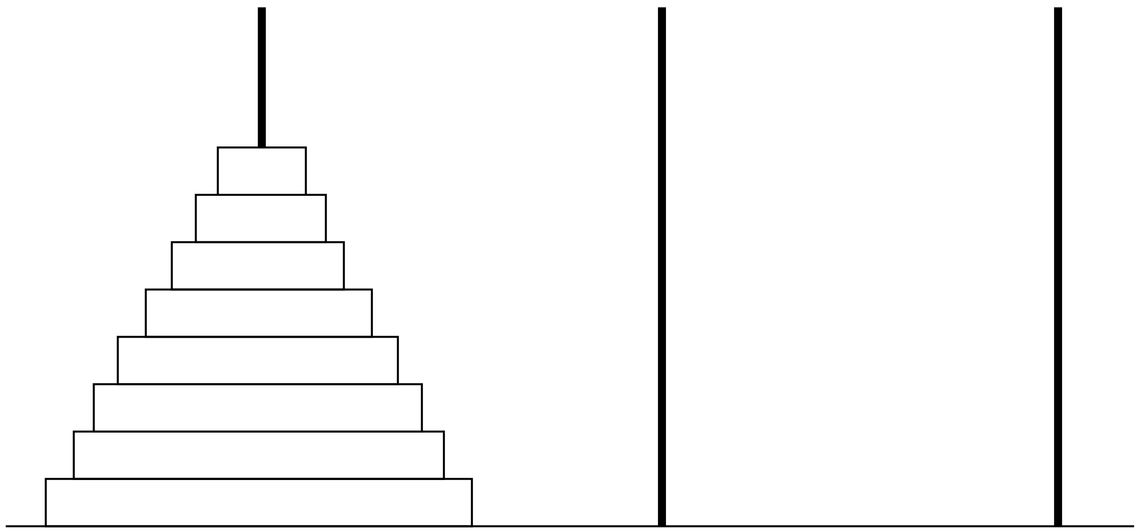


Figure 4

For $n \geq 1$, let b_n denote the minimum number of moves needed to accomplish the task with n discs. When $n = 1$, it is clear that one move is enough, so $b_1 = 1$. When $n = 2$, the following sequence of moves as shown in Figure 5 shows that 3 moves will complete the task. Thus $b_2 = 3$.

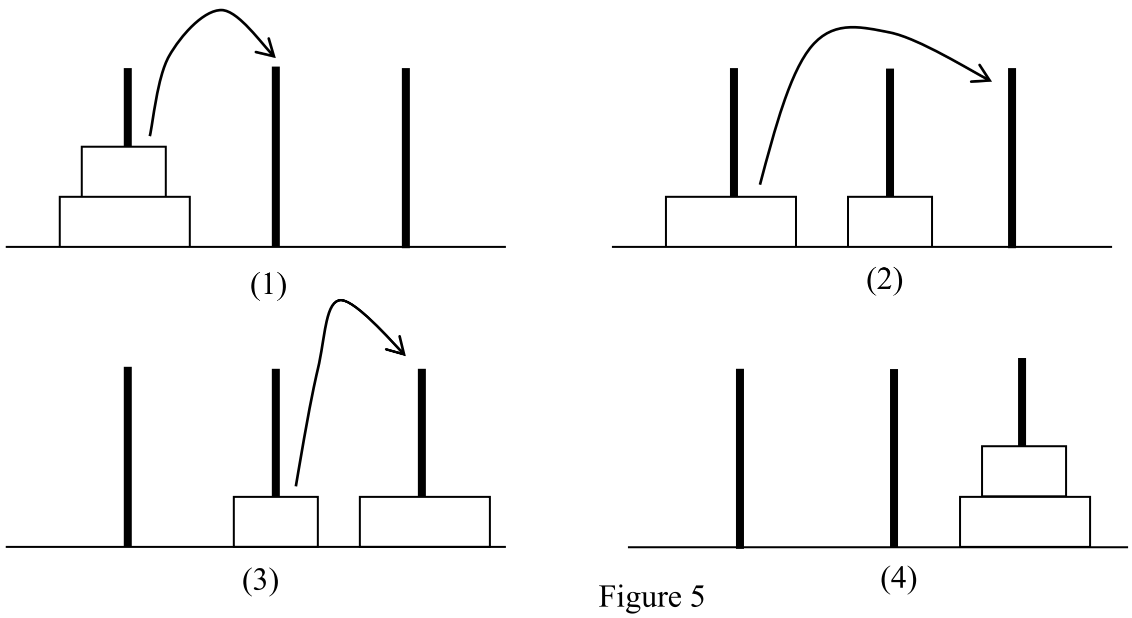


Figure 5

Consider the case when $n = 3$, the sequence of moves shown in Figure 6 shows that seven moves will do the job.

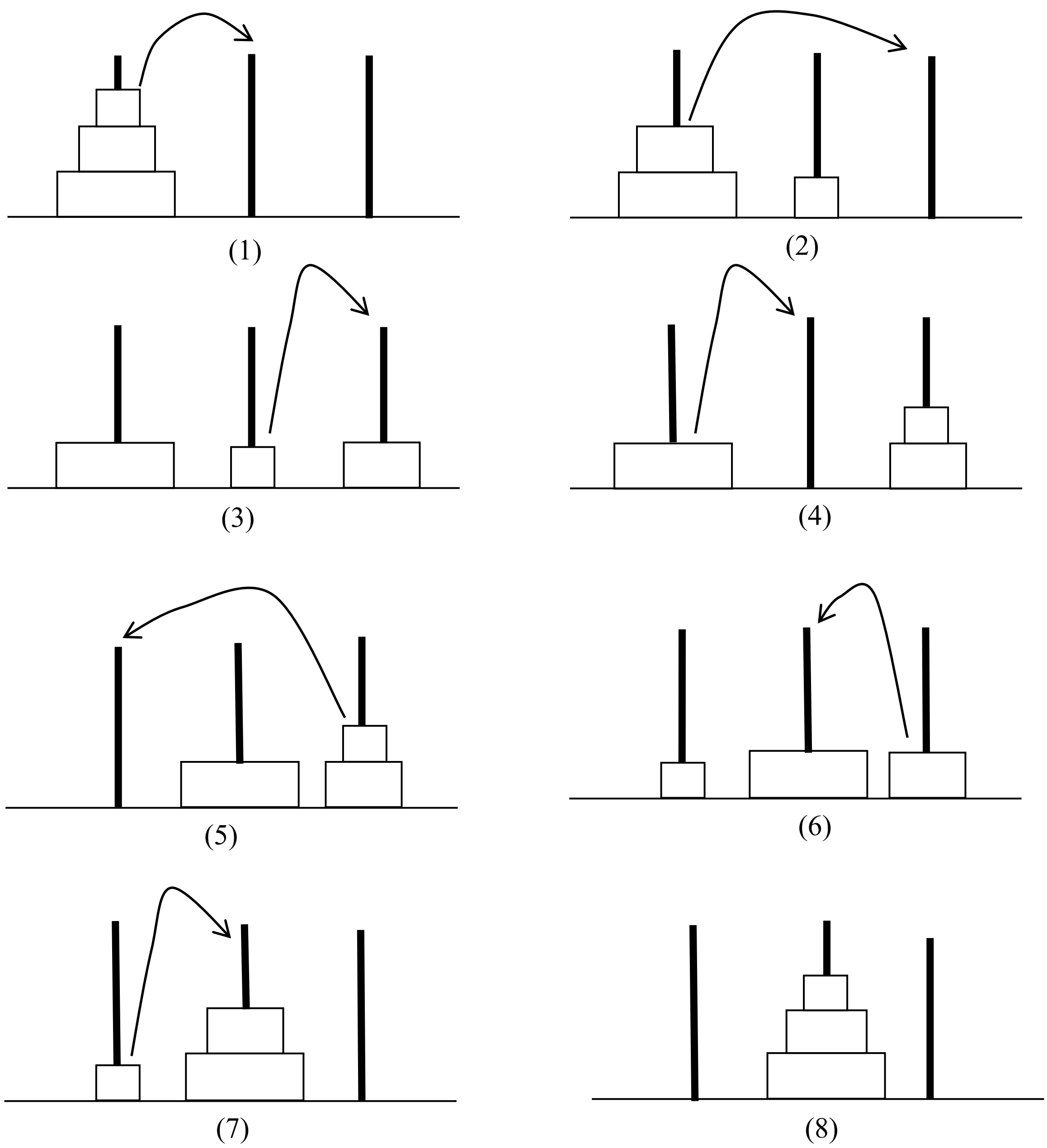


Figure 6

Is '7' the minimum number of moves required? As shown in Figure 6, before the largest disc can be moved to another peg, we have to transfer the entire tower of two smaller discs to a peg. We know that this requires b_2 moves. Next, we move the largest disc to the empty peg. Finally, we have to transfer the entire tower of two smaller discs and place it on the largest disc which requires another b_2 moves. Thus we need $b_2 + 1 + b_2$, i.e. $2b_2 + 1$ moves to accomplish this task. This result shows that $b_3 = 7$.

Imagine now we have a tower n (≥ 3) discs stacked on one of the 3 pegs as shown in Figure 7, the task of transferring the entire tower of n discs to another peg can be done via the following steps:

1. Transfer the top $n - 1$ discs to another peg.
2. Move the largest disc from the original peg to the only empty peg.
3. Transfer the entire tower of $n - 1$ smaller discs to the peg that the largest disc is currently placed.

The minimum number of moves for step 1, 2 and 3 are b_{n-1}, 1 and b_{n-1} respectively. Hence the minimum number of moves for the whole task is $b_{n-1} + 1 + b_{n-1}$.

By the definition of b_n, we have $b_n = 2\, b_{n-1} + 1$.

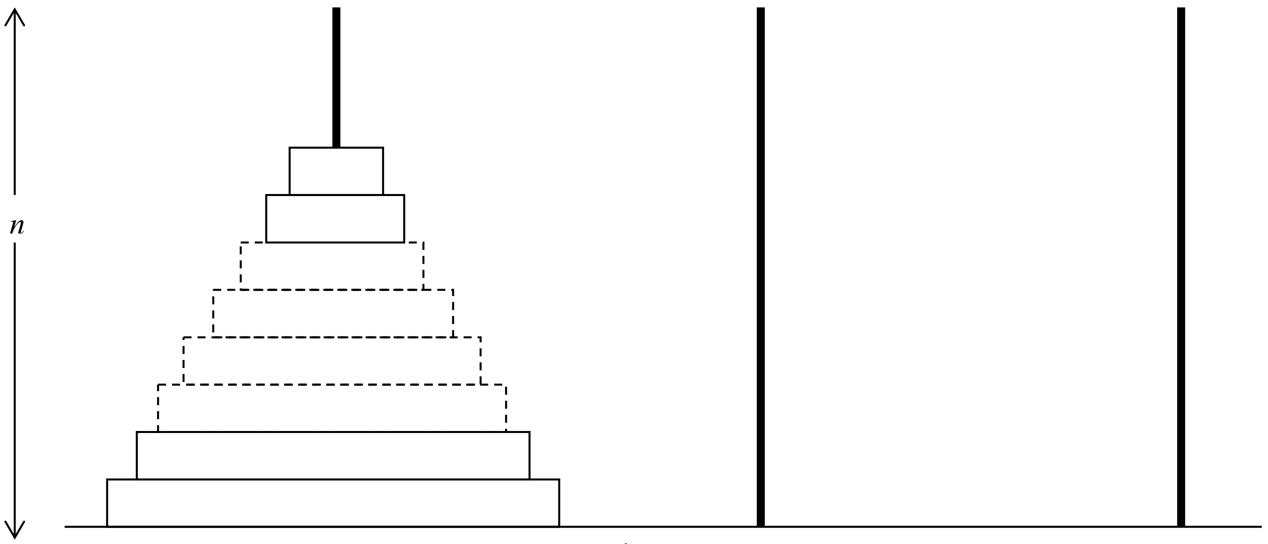


Figure 7

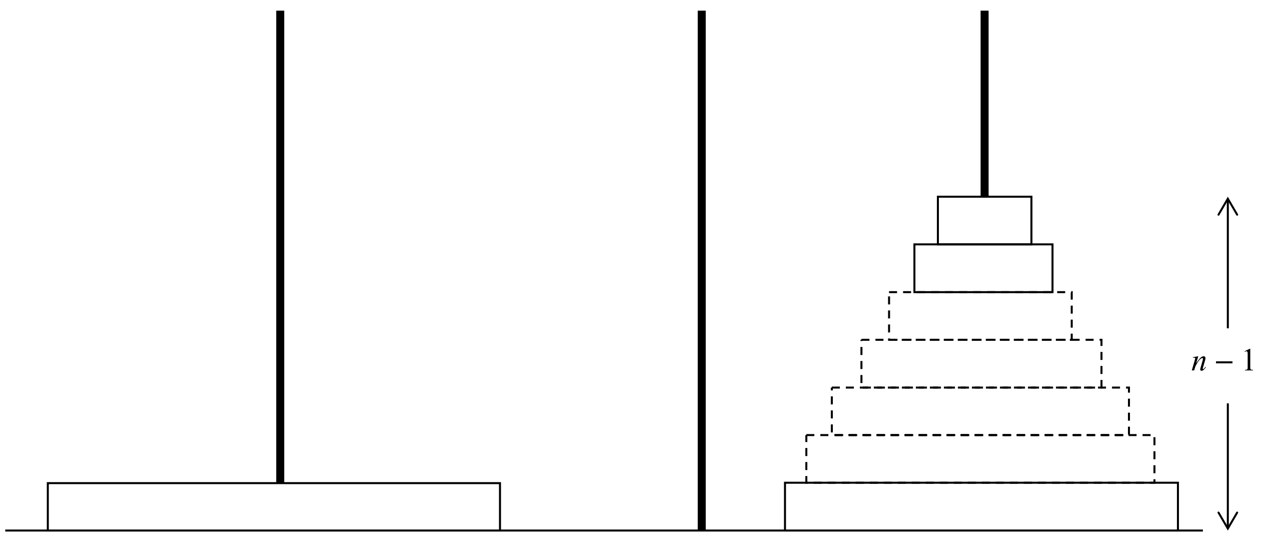


Figure 8

This problem described in Example 2 is known as the Tower of Hanoi which was first formulated and studied by Francois Edouard Anatole Lucas (1842-1891) in 1883.

Example 4 (2020/EJC/Prelim/Q2b)

Seven distinct colours are used to colour the sectors of a circle.

Let a_n denote the number of ways to colour a circle with n sectors such that each sector is coloured by one colour and any two adjacent sectors must be coloured by different colours.

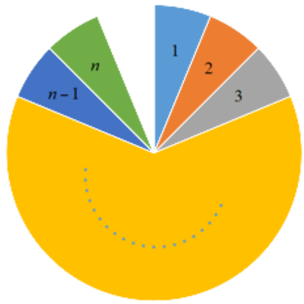
(i) Find the values of a_1 and a_2. [2]

(ii) Explain why $a_n + a_{n-1} = 7\left(6^{n-1}\right)$ for $n \geq 3,\ n \in \mathbb{Z}^+$. [3]

(iii) Use Mathematical Induction to show that

$$a_n = 6(-1)^n + 6^n \text{ for all } n \geq 3,\ n \in \mathbb{Z}^+. \quad [4]$$

Solution

(b)(i)	$a_1 = 7$, $a_2 = 42$
(b)(ii)	We use a combinatorial interpretation to explain $a_n = 7\left(6^{n-1}\right) - a_{n-1}$  Consider the number of ways of colouring a circle with n sectors. Number the sectors 1 to n. There are 7 colour choices for sector 1. For each of the remaining sectors, each must be a different colour from the one just before it, so there are 6 choices for each, which makes a total of $7\left(6^{n-1}\right)$ ways. However, not all of these are valid ways to colour the sectors, since there is one additional constraint namely that sector n must be a different colour from sector 1. So, we must subtract from this the number of ways of colouring such that sector 1 and sector n are the same colour. By combining sector 1 and sector n into a single sector, we observe that we get a colouring of the circle divided into $n-1$ sectors. Indeed, there are precisely a_{n-1} ways of colouring the circle with n sectors such that sector 1 and sector n are the same colour, but any other two adjacent sectors are different colours. Therefore, $a_n = 7\left(6^{n-1}\right) - a_{n-1}$.

(solution for induction part omitted)

<u>Counting Two Ways</u>

Counting two ways is a proof technique for showing that two expressions are equal by demonstrating two different ways of counting the same thing, one leading to each side of the equation.

(Note: This is very similar to the bijective principle, where in order to count a set, we establish a bijection with another set that is easier to count.)

Example 5

Explain why $\binom{n}{k} = \binom{n}{n-k}$ for all non-negative integers *n* and *k*, $k \le n$.

We'll count the number of ways to select *k* objects from of a set of *n* different objects in two ways:

Method 1: Choose the *k* objects directly.

There are $\binom{n}{k}$ ways of doing so.

Method 2: Choose the (*n*-*k*) objects to be excluded. Then take the *k* remaining objects.

There are $\binom{n}{n-k}$ ways of doing so.

Since these two methods count the same set, $\binom{n}{k} = \binom{n}{n-k}$.

Example 6

Explain why $\sum_{i=0}^{n} \binom{n}{i} = 2^n$ for all positive integers *n*.

Example 7

Using a combinatorial argument, explain why $\sum_{i=k}^{n}\binom{n}{i}\binom{i}{k}=2^{n-k}\binom{n}{k}$ for all non-negative integers n and k, $k \leq n$.

We'll count the number of ways of first shortlisting some $(\geq k)$ people from a group of n people, before finalizing our committee of k people from the shortlist.

LHS: For each $k \leq i \leq n$, first select a group of i people to form the shortlist, before selecting k people from the shortlist to form the committee.

There are $\binom{n}{i}$ ways to form the shortlist and $\binom{i}{k}$ to form the committee from the shortlist. So there are $\binom{n}{i}\binom{i}{k}$ ways of first forming a shortlist of i people before obtaining a committee of k people. The number of people in the shortlist can vary from k to n, so there are $\sum_{i=k}^{n}\binom{n}{i}\binom{i}{k}$ ways of doing so.

RHS: We choose the committee first, then form the shortlist.

There are $\binom{n}{k}$ ways to form a committee of k people. For each of the remaining $n-k$ people not in the committee, there are two possibilities: either they are in the shortlist or not, making 2^{n-k} ways. So there are a total of $2^{n-k}\binom{n}{k}$ ways.

Example 8 (continuation of Example 1)

Let F_n be the number of ways for a boy to climb a staircase of *n* steps, where he can climb up one or two steps at a time.

(a) [Insert Example 1]

(b) Show that $F_m F_n + F_{m+1} F_{n+1} = F_{m+n+2}$ for all positive integers *m*, *n*.

Solution

(b) The RHS is the number of ways to climb a staircase of $m+n+2$ steps. We will count this in an alternative way to obtain the LHS.

Tutorial Questions

1. Find a recurrence relation for the number of binary sequences b_n of length n with no consecutive 0's, and state its initial conditions.

2. A $2 \times n$ rectangle is to be paved by 1×2 identical blocks. Let a_n represent the number of ways this can be done. Find a recurrence relation for a_n, and state its initial conditions.

3. Find the number of ways to arrange n integers 1, 2, 3, …, n such that each integer is either bigger than all the preceding integers, or smaller than all the preceding integers. For example, a possible way to arrange 5 integers is 3, 4, 5, 2, 1.

4. A ternary sequence is a sequence of digits where each digit can only be 0, 1 or 2.

Find a recurrence relation for the number of ternary sequences of length n where

(a) no consecutive digits that are the same, x_n,

(b) no consecutive digits sum to 3, y_n.

5. Let x_n be the number of n-digit ternary sequences whose sum of digits is even. Find a recurrence relation with initial conditions for x_n.

6. Explain why $\sum_{r=1}^{n} r\binom{n}{r} = n2^{n-1}$ for all positive integers n.

7. Explain why $\sum_{r=k}^{n} \binom{r}{k} = \binom{n+1}{k+1}$ for all positive integers n, k with $n \geq k$ [Hockey Stick identity]

8. Explain why $\sum_{r=1}^{n} r(n-r) = \sum_{r=1}^{n} \binom{r}{2} = \binom{n+1}{3}$ for all positive integers n.

9. Explain why $\sum_{r=0}^{n} \binom{n}{r}^2 = \binom{2n}{n}$ for all positive integers n.

10. Prove that $\sum_{k=0}^{r} \binom{n-m}{k}\binom{m}{r-k} = \binom{n}{r}$ whenever $0 \leq m, r \leq n$. [Vandermonde's Identity. Note the identity in the previous question is a special case of this.]

11. **N2019/I/4(i)**

An n-digit number uses no digits other than 1, 2 and 3. It does not have any 2s adjacent to each other and it does not have any 3s adjacent to each other. Let there be T_n such numbers, with X_n of these having first digit 1 and Y_n having first digit 2.

Prove that, for $n \geq 2$,

(a) $Y_n = X_{n-1} + Y_{n-1}$, [1]

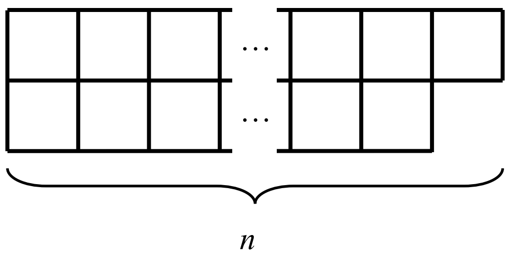
(b) $X_n = X_{n-1} + 2Y_{n-1}$, [2]

(c) $X_{n+1} = 2X_n + X_{n-1}$. [2]

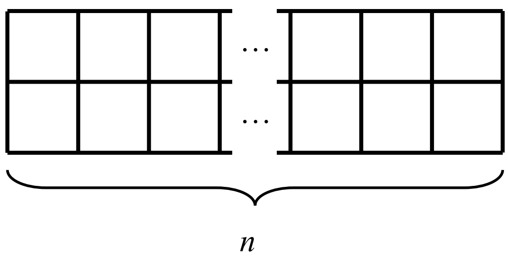
12. (2019/EJC/Prelim/Q6 part) You have an unlimited supply of 1×1, 1×2 and 2×2 tiles. Tiles of the same size are indistinguishable.

Consider the tilings of a $2 \times n$ path. (The 1×2 tiles can be rotated in the tilings.)

Let P_n be the number of tilings of



Let Q_n be the number of tilings of



(ii) Show that $P_{n+1} = P_n + Q_n$ for $n \geq 1$. Explain your reasoning clearly. [2]

(iii) Show that $Q_{n+1} = 2P_{n+1} + 2Q_{n-1}$ for $n \geq 2$. Explain your reasoning clearly. [4]

(iv) Use **(ii)** and **(iii)** to show that $P_{n+2} + 2P_{n-1} = 3P_{n+1} + 2P_n$ for $n \geq 2$. [2]

13. **(2020/HCI/Prelim/Q5)**

Casper has n fence posts, each of different width, arranged in a straight line in front of his house. To surprise his wife, Ashley, Casper decides to paint the posts using three different colours red, white and green in such a way that no adjacent fence posts have the same colours. Let r_n be the number of ways that Casper can paint the fence posts if he paints the first and last post red, and let s_n be the number of ways that Casper can paint the posts if he paints the first post red but the last post with either of the other two colours.

(i) Explain why $r_{n+1} = s_n$. Hence determine the value of $r_{n+1} + r_n$ for all $n \in \mathbb{Z}^+$. [2]

(ii) Prove that for all $n \in \mathbb{Z}^+$,

$$r_n = \frac{2^{n-1} + 2(-1)^{n-1}}{3}.$$ [4]

(iii) Find the number of ways of painting the n fence posts, given that $n \geq 3$, with the three different colours red, white and green in such a way that no adjacent fence posts have the same colours, if Casper wants to arrange the posts in a circle surrounding a newly built pond outside his house. Give your answer in terms of n. [3]

14. **(2020/RI/Prelim/Q7)**

Define by $B(n)$ the number of partitions of n into sums of powers of 2. For example, $B(6) = 6$, since

$$\begin{aligned} 6 &= 1+1+1+1+1+1 \\ &= 1+1+1+1+2 \\ &= 1+1+2+2 \\ &= 1+1+4 \\ &= 2+2+2 \\ &= 4+2 \end{aligned}$$

Prove that

(i) $B(2n+1) = B(2n)$ for all $n \geq 1$. [2]

(ii) $B(2n) = B(2n-1) + B(n)$ for all $n \geq 1$. [3]

(iii) $B(n)$ is even for all $n \geq 2$. [4]

We say that a partition is even if the number of parts is even, and a partition is odd otherwise. For example, $1+1+1+1+1+1$, $1+1+2+2$ and $4+2$ are even partitions of 6, whereas $1+1+1+1+2$, $1+1+4$ and $2+2+2$ are odd partitions of 6.

(iv) Show that there is an equal number of even and odd partitions of n for all $n \geq 2$. [3]