RIVER VALLEY HIGH SCHOOL
2025 JC2 Preliminary Examination
Higher 3

NAME	

CLASS	2	4	J		

INDEX NUMBER	

MATHEMATICS 9820/01

Paper 1 **24 September 2025**

Additional Materials: Printed Answer Booklet
List of Formulae (MF27) **3 Hours**

READ THESE INSTRUCTIONS FIRST

Do not open this booklet until you are told to do so.
Write your name, class and index number in the space at the top of this page.

An answer booklet and a graph paper booklet will be provided with this question paper. You should follow the instructions on the front cover of the answer booklet. If you need additional answer paper or graph paper ask the invigilator for a continuation booklet or graph paper booklet.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

1 Let a, b, c be positive real numbers such that $abc = 36$.

(a) Show that $2a + b + 3c \geq 18$. [2]

(b) Deduce the minimum value of $\dfrac{4a^2}{b+3c} + \dfrac{b^2}{3c+2a} + \dfrac{9c^2}{2a+b}$ and find the values of a, b and c where this minimum value is attained. [5]

(c) Hence, or otherwise, prove that if p, q and r are positive real numbers such that $pqr = \dfrac{1}{36}$, then

$$\frac{1}{9p^3(r+3q)} + \frac{1}{36q^3(3p+2r)} + \frac{1}{4r^3(2q+p)} \geq 9.$$ [4]

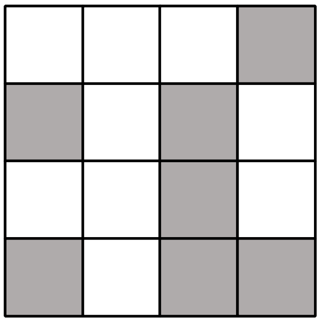
2 **(a)** The variables x and y are related by the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2xy}{x^2 - 2y^2}, \quad x \neq \sqrt{2}y \text{ for all } x, y \in \mathbb{R}.$$

Using the substitution $y = ux$, find the general solution of the differential equation. [7]

(b) Describe geometrically the set of points that satisfy the general solution obtained in part **(a)**. [4]

3 Consider 4-by-4 unit-square grids where each unit-square is either shaded or white.
For example, the following diagram illustrate an example of such a grid.



(a) **(i)** How many possible 4-by-4 unit-square grids are there? [1]

(ii) How many 4-by-4 unit-square grids are there which contains at least one entirely shaded 3-by-3 square? [4]

(b) Determine the minimum number of shaded unit-squares in a 4-by-4 unit-square grid to guarantee that it contains at least one entirely shaded 2-by-2 square. Justify your answer. [3]

(c) Define c_i, for $i = 1, 2, 3, 4$, to be the number of shaded unit squares in the ith column from the left of a given 4-by-4 unit-square grid.
For example, for the 4-by-4 unit-square above, $c_1 = 2, c_2 = 0, c_3 = 3, c_4 = 2$.
How many distinct possible tuples (c_1, c_2, c_3, c_4) are there if the given 4-by-4 unit-square grid has

(i) exactly 4 shaded unit-squares, [1]

(ii) at least 12 shaded unit-squares? [3]

4 Let S be the set of all integers from 1 to n and T be the set of all subsets of S, including the empty set $\varnothing$ and S itself.

(a) Explain why $|T| = 2^n$. [1]

The sets A_1 and A_2, which are not necessarily distinct, are chosen randomly and independently from the 2^n subsets in T.

(b) In this part of the question, we will calculate $\mathrm{P}\left(A_1 \cap A_2 = \varnothing\right)$ in 2 ways.

(i) Let i be an integer such that $1 \le i \le n$. Show that $\mathrm{P}\left(i \notin A_1 \cap A_2\right) = \dfrac{3}{4}$. [2]

(ii) Hence, find $\mathrm{P}\left(A_1 \cap A_2 = \varnothing\right)$. [1]

(iii) Let k be an integer such that $0 \le k \le n$. Find $\mathrm{P}\left(A_1 \cap A_2 = \varnothing \,\middle|\, |A_1| = k\right)$. [1]

(iv) Deduce that $\displaystyle\sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^{n+k}} = \left(\frac{3}{4}\right)^n$. [3]

(c) **(i)** Let k be an integer such that $0 \le k \le n$. Find $\mathrm{P}\left(|A_1 \cap A_2| = k\right)$. [2]

(ii) Find $\mathrm{E}\left(|A_1 \cap A_2|\right)$. [2]

5 **(a)** Let $\mathrm{f}(x)$ be a polynomial with integer coefficients.

(i) For integers α and β show that $\mathrm{f}(x) - \mathrm{f}(\alpha)$ has a factor of $x - \alpha$ and hence, or otherwise, show that

$$\mathrm{f}(\beta) - \mathrm{f}(\alpha) \equiv 0 \text{ (modulo } \beta - \alpha\text{)}.$$ [2]

(ii) Deduce that there does not exist a polynomial f with integer coefficients such that $\mathrm{f}(1) = 2, \mathrm{f}(3) = 5$. [1]

(iii) Explain why there does not exist a polynomial f with integer coefficients such that $\mathrm{f}(1) = 0,\ 0 < \mathrm{f}(2025) < 2024$. [2]

(b) A monic degree n polynomial is a polynomial where the coefficient of x^n is 1.

Let p be an odd prime. Let $\mathrm{h}(x)$ be a monic degree n polynomial with integer coefficients.

We call an integer γ is a *root modulo p* of $\mathrm{h}(x)$ if $\mathrm{h}(\gamma) \equiv 0$ (modulo p).

Moreover, if $\alpha \not\equiv \beta$ (modulo p), we call integers α, β *incongruent*.

(i) Find two incongruent roots modulo 7 for $x^2 + x + 1$. [1]

(ii) For $n \ge 1$, show that if γ is a root modulo p of $\mathrm{h}(x)$, then there exist a monic degree $n-1$ polynomial $\mathrm{h}_0(x)$ with integer coefficients, such that

$$\mathrm{h}(x) \equiv (x - \gamma)\mathrm{h}_0(x) \text{ (modulo } p\text{)}.$$ [3]

(iii) Prove, by induction, that any monic degree n polynomial $\mathrm{h}(x)$ has at most n incongruent roots modulo p for $n \ge 0$. [5]

Use the information in the mathematical text to answer Question 6. You should read the whole mathematical text before you start answering the questions.

On the irrationality of π

We denote $f^{(k)}(x)$ to be the *k*th derivative of $f(x)$ with respective to *x* and $f^{(0)}(x) = f(x)$.

A number is called *rational* if it can be simplified to the form $\frac{a}{b}$ where *a* is an integer and *b* is a positive integer where *a* and *b* do not share any prime factors. A number is called irrational if it is not a rational number. For example, the numbers $0.1, \frac{4}{3}, -\frac{39}{13} = -3$ are rational numbers but $\sqrt{2}$ and π are not.

However, showing that π is irrational, is not trivial.
In 1947, the Canadian-American mathematician Ivan Morton Niven produced an elementary yet elegant proof that π is irrational. His methods could be traced back to French mathematician Charles Hermite's 1873 paper on proving irrationality of e^r, where *r* is a rational number. We will investigate a modified version of Niven's proof of the irrationality of π.

We suppose that $\pi = \frac{a}{b}$ for positive integers *a*, *b*, for contradiction.
In his proof, for each positive integer *n*, Niven considered the polynomial

$$f(x) = \frac{x^n (a - bx)^n}{n!} = \frac{b^n \left[x(\pi - x)\right]^n}{n!}.$$

Using binomial theorem, we can write this polynomial in the form

$$f(x) = \frac{1}{n!}\sum_{r=n}^{2n} c_r x^r ,$$

where c_r are integers for all *r*.
By considering the expansion, we can show that $f^{(k)}(0)$ is an integer, for $0 \le k \le 2n$. This would then imply that $f^{(k)}(\pi)$ is an integer, for $0 \le k \le 2n$, as well.

Niven also define another function $F(x)$ as,

$$F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - \dots + (-1)^n f^{(2n)}(x).$$

This is carefully chosen so that the identity $\int_0^{\pi} f(x)\sin(x)\, dx = F(0) + F(\pi)$ holds for all *n*.

For $0 < x < \pi$, we have $0 < f(x) < \frac{\pi^n m^n}{n!}$ for some positive real constant *m*.
We can then develop the bound

$$0 < \int_0^{\pi} f(x)\sin(x)\, dx < \frac{2\pi^n m^n}{n!} \text{ for all } n \ge 1.$$

It is a known fact that if $\lim_{n\to\infty} u_n = 0$, this implies that for sufficiently large *n*, we have $u_n < 1$.

By selecting a sufficiently large *n*, we will attain the required contradiction to prove that π is irrational.

6 In this question you are asked to prove that π is irrational.

For the entirety of the question, we will assume that $\pi = \dfrac{a}{b}$, where a, b are positive integers, for contradiction.

(a) **(i)** Consider any c_r in the sum $\mathrm{f}(x) = \dfrac{1}{n!}\sum_{r=n}^{2n} c_r x^r$.

Briefly explain why c_r is an integer. [2]

(ii) Hence, explain why $\mathrm{f}^{(k)}(0)$ is an integer for $0 \le k \le 2n$. [2]

(b) By showing that $\mathrm{f}(x) = \mathrm{f}(\pi - x)$, prove that $\mathrm{f}^{(k)}(\pi)$ is an integer for $0 \le k \le 2n$. [3]

(c) **(i)** Prove that $\mathrm{F}''(x) = \mathrm{f}(x) - \mathrm{F}(x)$. [2]

(ii) Deduce that $\int_0^{\pi} \mathrm{f}(x)\sin(x)\,\mathrm{d}x = \mathrm{F}(0) + \mathrm{F}(\pi)$. [3]

(d) **(i)** Prove that for some positive real constant m, $0 < \mathrm{f}(x) < \dfrac{\pi^n m^n}{n!}$ for $0 < x < \pi$. [2]

(ii) Hence, show that $0 < \int_0^{\pi} \mathrm{f}(x)\sin(x)\,\mathrm{d}x < \dfrac{2\pi^n m^n}{n!}$. [1]

(iii) By considering the series expansion of e^x, or otherwise, explain why $\lim\limits_{n\to\infty} \dfrac{2\pi^n m^n}{n!} = 0$. [2]

(e) Prove that π is irrational. [3]

End of Paper