

SUMMARY: Differentiation and Applications

Standard Formulae

Let u, v be functions of x , and a and b be constants.

Basic Rules	Examples
Addition/subtraction:	$\frac{d}{dx} \left(2x^5 + \frac{6}{x} - 3 \right) = 2(5x^4) + 6(-1 \times x^{-2}) - 0$
Product Rule: $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$	$\frac{d}{dx} \left[(x^2 + 1)(5x + 4) \right] = (x^2 + 1)(5) + (2x)(5x + 4)$
Quotient Rule: $\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$	$\frac{d}{dx} \left(\frac{4x^3}{x^2 + 2} \right) = \frac{(x^2 + 2)(12x^2) - 4x^3(2x)}{(x^2 + 2)^2}$
Chain Rule:	$\frac{d}{dx} \left[\frac{1}{2} (3x^2 + 1)^7 \right] = \frac{7}{2} (3x^2 + 1)^6 \times 6x$

Derivatives of Exponential Functions (Let a be a positive constant.)

	Examples
$\frac{d}{dx}(e^x) = e^x$	
$\frac{d}{dx}(e^{f(x)}) = f'(x)e^{f(x)}$	$\frac{d}{dx}(e^{\sin x}) = (\cos x)e^{\sin x}$
$\frac{d}{dx}(a^x) = a^x \ln a$	
$\frac{d}{dx}(a^{f(x)}) = f'(x)a^{f(x)} \ln a$ Note: Use $\ln y = \ln(a^x)$ with implicit differentiation if you do not want to memorise formula.	$y = 2^{\sin x} \Rightarrow \ln y = (\sin x) \ln 2$ Differentiating w.r.t. x , $\frac{1}{y} \frac{dy}{dx} = (\cos x) \ln 2$ $\Rightarrow \frac{dy}{dx} = (\cos x)(\ln 2)2^{\sin x}$

Derivatives of Logarithmic Functions

	Examples
$\frac{d}{dx}(\ln x) = \frac{1}{x}$	
$\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}$	$\frac{d}{dx} \left(\ln \left(\frac{x+1}{x-1} \right) \right) = \frac{d}{dx} (\ln(x+1) - \ln(x-1)) = \frac{1}{x+1} - \frac{1}{x-1}$
$\frac{d}{dx}(\log_a x) = \frac{1}{x} \log_a e$	
$\frac{d}{dx}(\log_a f(x)) = \frac{f'(x)}{f(x)} \log_a e$ Note: Change base from $\log_a$ to $\ln$ if you do not want to memorise formula.	$\frac{d}{dx}(\log_5(1+x^2)) = \frac{d}{dx} \left(\frac{\ln(1+x^2)}{\ln 5} \right) = \frac{1}{\ln 5} \left(\frac{2x}{1+x^2} \right)$

Derivatives of Trigonometric Functions (Note: x is in *radians*)

$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\sec x) = \sec x \tan x$ (MF27)
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

Derivatives of Trigonometric Functions using Chain Rule

	Examples
$\frac{d}{dx}(\sin^n x) = n(\sin^{n-1} x) \cos x$	$\frac{d}{dx}(\sin^3 x) = 3(\sin^2 x) \cos x$
$\frac{d}{dx}(\tan^n x) = n(\tan^{n-1} x) \sec^2 x$	$\frac{d}{dx}(\tan^2 x) = 2(\tan x) \sec^2 x$
$\frac{d}{dx}(\sec^n x) = n(\sec^{n-1} x) \sec x \tan x$	$\frac{d}{dx}(\sec^2 x) = 2(\sec x) \sec x \tan x = 2 \sec^2 x \tan x$

Differentiation Formula in MF27

Derivatives

$f(x)$	$f'(x)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\sec x$	$\sec x \tan x$

Derivatives of Inverse Trigonometric Functions

	Examples
$\frac{d}{dx}[\sin^{-1} f(x)] = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$	$\frac{d}{dx}(\sin^{-1} 2x) = \frac{2}{\sqrt{1-4x^2}}$
$\frac{d}{dx}[\cos^{-1} f(x)] = \frac{-f'(x)}{\sqrt{1-[f(x)]^2}}$	$\frac{d}{dx}[\cos^{-1}(\tan x)] = \frac{-\sec^2 x}{\sqrt{1-\tan^2 x}}$
$\frac{d}{dx}[\tan^{-1} f(x)] = \frac{f'(x)}{1+[f(x)]^2}$	$\frac{d}{dx}(\tan^{-1} e^x) = \frac{e^x}{1+e^{2x}}$

Parametric Differentiation

	Example
Use $\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$ (or $\frac{dy}{dx} = \frac{dy}{d\theta} \bigg/ \frac{dx}{d\theta}$)	$x = \sin 2t$, $y = \cos 2t$ $\frac{dx}{dt} = 2\cos 2t$, $\frac{dy}{dt} = -2\sin 2t$ $\Rightarrow \frac{dy}{dx} = \frac{-2\sin 2t}{2\cos 2t} = -\tan 2t$

Implicit Differentiation

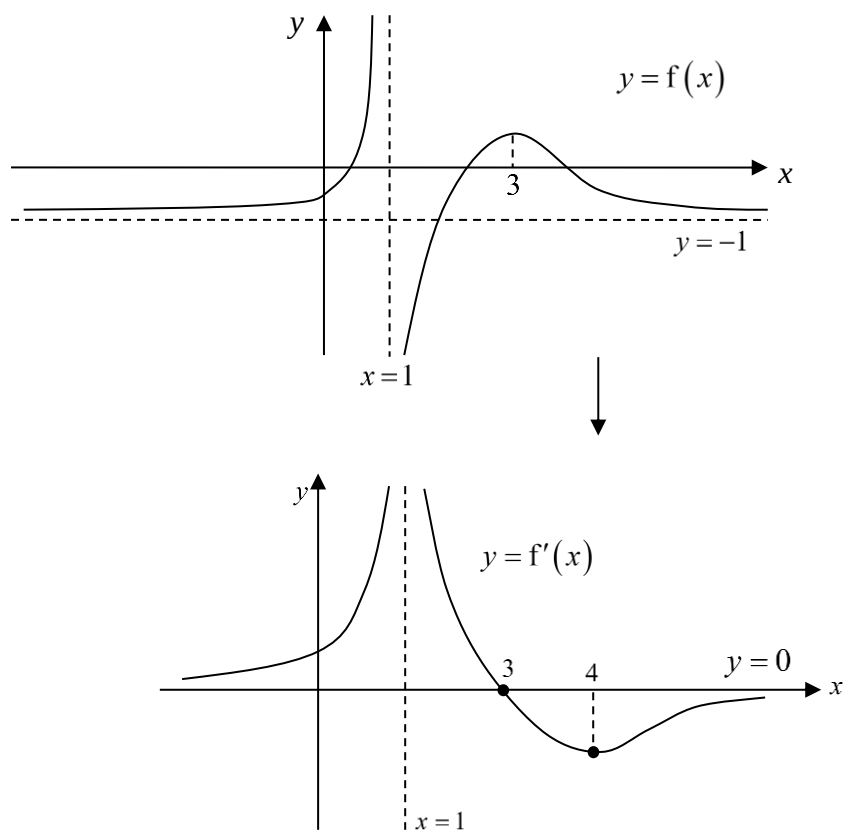
	Example
$\frac{d}{dx}(y^2) = 2y \frac{dy}{dx}$ $\frac{d}{dx}(x^2y) = x^2 \frac{dy}{dx} + y(2x)$	Given $x^2 + xy + y^2 = 2$, find $\frac{dy}{dx}$. $x^2 + xy + y^2 = 2$ Differentiating w.r.t. x , $2x + \left[x \frac{dy}{dx} + y \right] + 2y \frac{dy}{dx} = 0$ $\Rightarrow (2y + x) \frac{dy}{dx} = -y - 2x$ $\Rightarrow \frac{dy}{dx} = \frac{-(y + 2x)}{2y + x}$

Graph of $y=f'(x)$

	Graph of $y=f(x)$		Graph of $y=f'(x)$
	Mathematical Feature	Geometrical Feature	
1	Stationary points i.e., $\frac{dy}{dx}=0$ at $x=a$	Max, Min or stationary point of inflexion at $x=a$	Intercept at $x=a$
2(a)	Vertical asymptote at $x=p$		Vertical asymptote at $x=p$
2(b)	Horizontal asymptote at $y=q$		Horizontal asymptote at $y=0$
2(c)	Oblique asymptote at $y=mx+c$		Horizontal asymptote at $y=m$
	Start sketching the graph of $y=f'(x)$ from left to right, using the following checks:		
3(a)	$\frac{dy}{dx}>0$ in interval (a,b)	Curve increasing in interval (a,b)	Curve above x -axis
(b)	$\frac{dy}{dx}<0$ in interval (a,b)	Curve decreasing in interval (a,b)	Curve below x -axis
4(a)	$\frac{d^2y}{dx^2}>0$ in interval (a,b)	Curve is concave upwards in interval (a,b)	Curve is increasing in interval (a,b)
(b)	$\frac{d^2y}{dx^2}<0$ in interval (a,b)	Curve is concave downwards in interval (a,b)	Curve is decreasing in interval (a,b)
5	$\frac{d^2y}{dx^2}=0$ at $x=k$	Change in curvature of curve	There is a stationary point (max/min/inflexion) at $x=k$

Example

The diagram below shows the graph of $y=f(x)$. The graph has a horizontal asymptote at $y=-1$, a vertical asymptote at $x=1$, a maximum point at $x=3$, and a point of inflexion at $x=4$.



Applications of Differentiation

(A) Tangents & Normals

- The equation of the *tangent* to a curve $y = f(x)$ at $P(x_1, y_1)$ is $y - y_1 = m(x - x_1)$, where m is the value of $\frac{dy}{dx}$ when $x = x_1$.
- The equation of the *normal* to a curve $y = f(x)$ at $P(x_1, y_1)$ is $y - y_1 = -\frac{1}{m}(x - x_1)$, where m is the value of $\frac{dy}{dx}$ when $x = x_1$.
- Consider a curve C with the following parametric equations:
 $x = 1 + \sin t$, $y = \cos t$ $0 \leq t \leq \pi$.
The following are **equivalent statements**
 - (i) Find the equation of the tangent at the point with parameter p .
 - (ii) Find the equation of tangent at $t = p$.
 - (iii) Find the equation of tangent at the point $(1 + \sin p, \cos p)$.

Example

A curve is defined by the parametric equations $x = 3t + 2$ and $y = t + \frac{3}{t}$, where t is a non-zero parameter. Find the equation of the tangent at the point P with parameter p and hence write down the equation of tangent at the point where $x = 11$.

Solution:

$$\frac{dx}{dt} = 3, \frac{dy}{dt} = 1 - \frac{3}{t^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \left(1 - \frac{3}{t^2}\right) \times \frac{1}{3} = \frac{1}{3} - \frac{1}{t^2}$$

Step 1

Find $\frac{dy}{dx}$ in terms of t .

Step 2

At the **point P with parameter p** , sub $t = p$ to find x, y and $\frac{dy}{dx}$ in terms of p .

When $t = p$, $\frac{dy}{dx} = \frac{1}{3} - \frac{1}{p^2}$, $x = 3p + 2$, $y = p + \frac{3}{p}$

Equation of tangent at $\left(3p + 2, p + \frac{3}{p}\right)$ is

$$y - \left(p + \frac{3}{p}\right) = \left(\frac{1}{3} - \frac{1}{p^2}\right)[x - (3p + 2)]$$

Step 3

Use the formula $y - y_1 = m(x - x_1)$ or $y = mx + c$ at the point P where $t = p$

When $x = 11$, $3p + 2 = 11 \Rightarrow p = 3$

Thus $y = 3 + \frac{3}{3} = 4$, and $\frac{dy}{dx} = \frac{1}{3} - \frac{1}{3^2} = \frac{2}{9}$

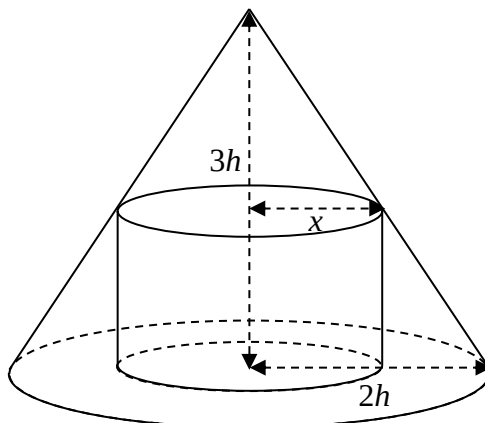
Important to find the specific value of p when $x = 11$ as this allows you to find values of y and $\frac{dy}{dx}$

Equation of tangent when $x = 11$ is $y - 4 = \left(\frac{2}{9}\right)(x - 11) \Rightarrow y = \frac{2x}{9} + \frac{14}{9}$

(B) Maxima/Minima

Example

A closed circular cylinder of radius x is inscribed in a right circular cone of radius $2h$ and vertical height $3h$, where h is a constant. One circular end of the cylinder lies on the base of the cone and the circumference of the other circular end is in contact with the inner surface of the cone.



If x is made to vary, find the maximum volume of the cylinder in terms of h .

[4]

Solution:

Let y be the height of the cylinder.

$$V = \pi x^2 y$$

By similar triangles,

$$\frac{3h - y}{3h} = \frac{x}{2h}$$

$$\Rightarrow 3h - y = \frac{3}{2}x$$

$$\Rightarrow y = 3h - \frac{3}{2}x$$

$$\begin{aligned} \text{Hence } V &= \pi x^2 y \\ &= 3\pi x^2 \left(h - \frac{1}{2}x\right) \end{aligned}$$

For maximum V ,

$$\frac{dV}{dx} = 3\pi h(2x) - \frac{3}{2}\pi(3x^2) = 0 \Rightarrow x(6h - \frac{9}{2}x) = 0$$

$$\Rightarrow x = 0 \text{ (rejected)} \quad \text{or} \quad x = \frac{4}{3}h$$

Check that when $x = \frac{4}{3}h$, $\frac{d^2V}{dx^2} = 6\pi h - 12\pi h = -6\pi h < 0$.

$$V = \pi \left(\frac{4}{3}h\right)^2 \left[3h - \left(\frac{3}{2}\right)\left(\frac{4}{3}\right)h\right] = \pi \left(\frac{16}{9}h^2\right)(h) = \frac{16}{9}\pi h^3$$

Hence V is maximum at $\frac{16}{9}\pi h^3$.

Step 1

Since V is to be maximum, we start off by expressing V in terms of 2 variables, x and y .

Step 2

Identify the constraint to reduce the no. of variables to 1. Possible constraints include similar triangles, Pythagoras Theorem, fixed quantities such as fixed surface area. In this question, the constraint is similar triangles where x and y are related, hence V can be expressed in terms of x only.

Step 3

Solve $\frac{dV}{dx} = 0$ to find x .

Step 4

Show that when $x = \frac{4}{3}h$,

$\frac{d^2V}{dx^2} < 0$ i.e. V is maximum

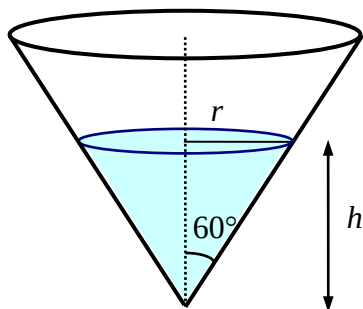
Step 5

Find maximum V using the value of x found.

(C) Rate of Change

Example

An inverted right circular conical tank has a 60° semi-vertex angle. Water is leaking out of the tank from the vertex at a rate of $3 \text{ cm}^3 \text{ s}^{-1}$. Find the rate of decrease of the depth of water when the depth is 4 cm.



To find $\frac{dh}{dt}$, use

$$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$$

Step 1

Form a chain rule.

Given $\frac{dV}{dt} = -3$

Step 2

Identify the rate of change of variable **given**.

Let us find $\frac{dh}{dV}$.

Volume of water in the cone,

$$V = \frac{1}{3}\pi r^2 h \text{ -----(1)}$$

Step 3

Identify the rate of change of variable **to be found**.

To find $\frac{dh}{dV}$, express V in terms of one variable (eliminate r).

$$\text{Using } \tan 60^\circ = \frac{r}{h} \Rightarrow \sqrt{3} = \frac{r}{h} \Rightarrow r = \sqrt{3}h \text{ ---- (2)}$$

$$\begin{aligned} \text{Sub. (2) into (1): } V &= \frac{1}{3}\pi(\sqrt{3}h)^2 h \\ &= \pi h^3 \\ \therefore \frac{dV}{dh} &= 3\pi h^2 \end{aligned}$$

Step 4

Use the Chain rule and substitute the respective rates of change of variables at $h = 4$.

$$\text{So } \frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt} = \frac{3}{\pi h^2} \times (-3) = -\frac{1}{\pi h^2}$$

$$\text{When } h = 4, \quad \frac{dh}{dt} = -\frac{1}{\pi(4)^2} = -\frac{1}{16\pi}$$

Thus when the depth is 4 cm, the rate of decrease of the depth of water is $\frac{1}{16\pi} \text{ cm s}^{-1}$.