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PERMUTATIONS AND COMBINATIONS

SYLLABUS

- Addition and multiplication principles for counting
- Concepts of permutation ($n!$ or nP_r) and combination (nC_r)
- Arrangements of objects in a line or in a circle, including cases involving repetition and restriction

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LECTURE 1

Lesson Outline

- Basic Principles of Counting
 - Addition & multiplication principles for counting
 - Combination
 - Combination of r objects from n distinct objects
 - Principle of complementation
-

1 BASIC PRINCIPLES OF COUNTING

1.1 Addition Principle for Mutually Exclusive Situations

If two operations are **mutually exclusive** (i.e. they cannot both be done at the same time) and one of them can be done in r ways and the other can be done in s ways, then one operation or the other can be done in **$r + s$ ways**. This result can be extended to more than 2 operations.

Example 1

A man travel by air or ship from Singapore to Tokyo. If there are 4 airlines and 3 shipping lines operating between the two cities, in how many ways can the man travel?

Solution

By the Addition Principle,
No. of ways =

Rationale

The man can either travel by air (4 ways) OR by sea (3 ways).

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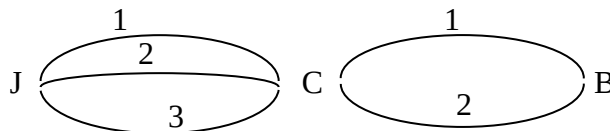
1.2 Multiplication Principle for Successive Operations

If there are r different ways of performing an operation, and after this operation is completed, a second operation can be performed in s different ways, then the two operations can be done one after the other in **$r \times s$ ways**.

Example 2

There are 3 buses operating between Jurong and Clementi and 2 buses operating between Clementi and Bedok. How many different ways can a man travel from Jurong to Bedok via Clementi by bus?

Solution



By the Multiplication Principle,
No. of ways =

Rationale

The man can travel from Jurong to Bedok via Clementi using one of the 6 routes:

$$J \xrightarrow{1} C \xrightarrow{1} B$$

$$J \xrightarrow{2} C \xrightarrow{1} B$$

$$J \xrightarrow{3} C \xrightarrow{1} B$$

$$J \xrightarrow{1} C \xrightarrow{2} B$$

$$J \xrightarrow{2} C \xrightarrow{2} B$$

$$J \xrightarrow{3} C \xrightarrow{2} B$$

■

Extension of result

This result can be extended to more than 2 operations. In general, if there are n_1 ways of performing the first operation, n_2 ways to perform a second operation, n_3 ways to perform a third operation, ending with n_k ways to perform the k^{th} operation, then there are $(n_1 \times n_2 \times n_3 \times \dots \times n_k)$ ways of performing all the k operations in succession.

Note

This multiplication rule only applies when the operations are **independent**, i.e., the choice made for one operation does not affect the choice made for any of the other operations.

2 COMBINATION

Definition

A combination of a set of objects is a **selection** of objects where the **order** of selection **does not matter**.

For example, we are interested in choosing 3 H2 subjects out of 5 subjects:

Mathematics (M), Economics (E), Biology (B),
Chemistry (C), History (H).

How many possible subject combinations are there?

Notice that we are **not** concerned with the order in which the subjects are selected.

So there are a total of 10 combinations, namely,

MEB, MEC, MEH, MBC, MBH, MCH, EBC, EBH, ECH, BCH

In other words, the number of combinations of 3 subjects selected from 5 subjects = ${}^5C_3 = 10$.

<i>More Examples</i>	<i>Number of Selections</i>
Select combinations of 3 letters from the 4 letters A, B, C, D	${}^4C_3 = 4$
Select 3 students from a class of 25 for AC Games	${}^{25}C_3 = 2300$
Draw 3 balls from a bag containing 5 differently coloured balls	${}^5C_3 = 10$
From an ordinary pack of 52 cards, draw 2 diamonds	${}^{13}C_2 = 78$
From the letters O,R,A,N,G,E, select 2 vowels	${}^3C_2 = 3$

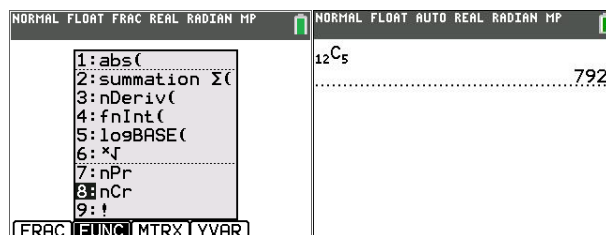
Note that in the above examples, we are not concerned with the order of items in the selection.

2.1 Combination of r Objects Taken From n Distinct Objects

Number of combinations of r objects taken from n distinct objects is given by

$${}^nC_r = \frac{n!}{(n-r)!r!}.$$

To calculate the value of nC_r , first key in the value of n , then press **[ALPHA] [WINDOW (F2)] [8]** and key in the value of r .



Example 3

In how many ways can 5 basketball players be selected from a team of 12 players to participate in a friendly game

- (i) if there is no restriction?
- (ii) if the oldest has to be included?

Solution

- (i) No. of ways of selecting 5 players (no restriction) =
- (ii) No. of ways of selecting 5 players (include oldest) =

Rationale

Excluding the oldest from the selection, we are left with players to select

■

Practice Question 1

In how many ways can a group of 4 boys be selected from 10 boys if

- (i) there is no restriction?
- (ii) the eldest boy is included in the group?
- (iii) the eldest boy is excluded in the group?

Solution

	<i>Rationale</i>
(i) No. of ways of selecting 4 boys from 10 boys (no restriction) =	
(ii) No. of ways of selecting 4 boys from 10 boys (include eldest) =	Since the eldest boy has to be included in the group, we are left with boys to choose
(iii) No. of ways of selecting 4 boys from 10 boys (exclude eldest) =	Since the eldest boy has to be excluded, we are left with boys to choose

■

2.2 Principle of Complementation

If there are m ways for all the events to happen and n ways for event A to happen, then there are $(m - n)$ ways for event A' , the complement of A , to happen.

Example 4

A delegation of 5 pupils is to be chosen from a class of 4 girls and 6 boys. Find the number of possible delegations if the delegation consists of

- (i) 3 girls and 2 boys;
- (ii) at least 1 girl.

Solution

	<i>Rationale</i>
(i) No. of delegations (3 girls and 2 boys) =	By the Multiplication Principle, No. of delegations (3 girls and 2 boys) = No. of ways of selecting girls from × No. of ways of selecting boys from

(ii) Method 1

No. of delegations (at least 1 girl) =

Method 2

Case 1 (Exactly 1 girl):

=

Case 2 (Exactly 2 girls):

=

Case 3 (Exactly 3 girls):

=

Case 4 (Exactly 4 girls):

=

Total no. of delegations (at least 1 girl)

=

Using **Principle of Complementation**,

No. of delegations (at least 1 girl)

= No. of delegations without restriction

– No. of delegations with no girls

Choose girl from 4 and boys from 6

Choose girls from 4 and boys from 6

Choose girls from 4 and boys from 6

Choose girls from 4 and boy from 6

■

Important Note

The following is a common argument for **(ii)** which is to select 5 pupils with at least one girl.

First we choose one girl to make sure there is at least one girl. Then we choose any 4 persons from the remaining 9 to form the delegation. Therefore, number of delegations is

$${}^4C_1 \times {}^9C_4 = 504.$$

The answer is different from what we obtained earlier on. What is wrong with this method?

Example 5

In this question, a selection is defined to be a combination of letters. For example, 'ABC' is a selection similar to 'BAC' but is a different selection from 'ABD'.

- (a) Consider the word INTRUDE. How many different selections of 3 letters can be made if
- (i) there is no restriction on the 3 letters
 - (ii) the 3 letters are all consonants
 - (iii) the 3 letters consist of 1 consonant and 2 vowels
- (b) It is found that there is a missing letter and the word should be INTRUDER. Find how many different selections of 3 letters can now be made.

Of the 26 alphabets, 5 letters are vowels (A, E, I, O, U) and the rest are consonants.

Solution

	<i>Rationale</i>
(a)(i) No. of selections of 3 letters (no restriction) =	Choose 3 letters out of 7 distinct letters
(ii) No. of selections of 3 letters (all consonants) =	Choose 3 consonants out of (N, T, R, D)
(iii) No. of selections of 3 letters (1 consonant and 2 vowels) =	Choose 1 consonant out of (N, T, R, D) and 2 vowels out of (I, U, E)
(b) Case 1 (No 'R'):	Excluding the two 'R's, we are left with 5 letters to choose
Case 2 (1 'R'):	Since 1 'R' is included, we are left with 5 letters to choose
Case 3 (2 'R's):	Since two 'R's are included, we are left with 5 letters to choose more letter.
No. of selections of 3 letters = $20 + 15 + 6 = 41$	

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In any ordinary pack of 52 cards, there are 4 suits comprising 13 spades, 13 clubs, 13 hearts and 13 diamonds. Picture cards are J, Q and K.

Example 6

How many selections of 4 cards can be drawn from an ordinary pack of 52 cards if

- (i) there is no restriction?
- (ii) exactly one is club?
- (iii) at least one is a spade?
- (iv) all four are of different suits?

Solution

Alternatively, we can also add the cases where there are exactly 1, 2, 3 or 4 spades. However, there will be more calculations involved than the first method.

(i) No. of selections of 4 cards (no restriction) =

(ii) No. of selections of 4 cards (1 club) =

(iii) No. of selections of 4 cards (at least 1 spade) =

(iv) No. of selections of 4 cards (different suits) =

Rationale

Choose 1 club out of 13 **and** remaining 3 cards from 39 cards

Using **Principle of Complementation**,

No. of selections (at least 1 spade)

= No. of selections with no restriction

– No. of selections with no spade

Choose 1 spade out of 13, 1 heart out of 13, 1 club out of 13 **and** 1 diamond out of 13

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Practice Question 2

A box contains 3 red, 2 blue, 1 yellow and 1 white beads. How many different selections are there if 3 beads are chosen from the box?

Solution

Rationale

Case 1

Case 2

Case 3

Thus no. of ways of choosing 3 beads =

■

Example 7

A delegation of 3 people is to be formed from 5 married couples. Find the number of possible delegations if the delegation consists of

- (i) 1 married couple,
- (ii) no married couple.

Solution

- (i) No. of delegations (1 married couple) = ${}^5C_1 {}^8C_1 = 40$

- (ii) No. of delegations (no married couple)
 $= {}^5C_3 \times {}^2C_1 \times {}^2C_1 \times {}^2C_1$
 $= {}^5C_3 \times ({}^2C_1)^3 = 80$

Rationale

Choose 1 couple out of 5 **and** 1 person out of remaining 8 people

Choose any 3 couples out of 5 couples (5C_3 ways), **and** choose 1 person from each of the 3 couples chosen earlier (i.e. ${}^5C_3 \times ({}^2C_1)^3$ ways)

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Extra Practice

Annex: Practice Question 1, 2

LECTURE 2

Lesson Outline

- Permutation of n Distinct Objects
 - Permutation in a row
 - Relationship between permutation and combination
 - Permutation of r objects from n distinct objects (with or without replacement)
 - Permutation of n objects where **not all** are distinct
-

3 PERMUTATION OF n DISTINCT OBJECTS

A permutation of a set of objects is an arrangement of these objects (either taking some or all) in a **definite order**.

3.1 Permutation of n Distinct Objects in a Row

Consider the number of arrangements or permutations of n objects.

The 1st position can be filled by any of the n objects.

The 2nd position can be filled by any of the remaining $n-1$ objects.

The 3rd position can be filled by any of the remaining $n-2$ objects and so on.

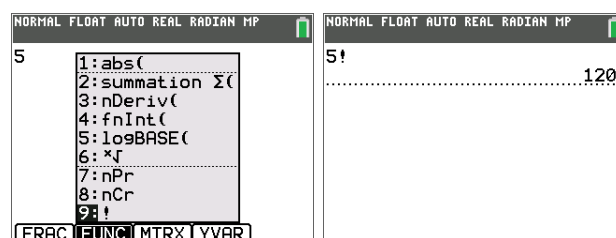
No. of ways of arranging n different objects in a row is

1 st	2 nd	3 rd		n^{th}
n	$n-1$	$n-2$	$\dots$	1

Total number of permutations of n **distinct** objects

$$= n \times (n-1) \times (n-2) \times \dots \times 1 = n!$$

To calculate the value of $n!$, first key in the value of n , then press **[ALPHA] [WINDOW (F2)] [9] [Enter]**



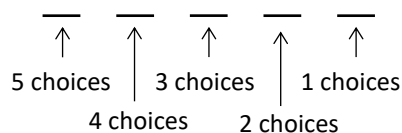
Example 8

Find the number of arrangements of all the letters of the word ENJOY.

Solution

No. of arrangements =

Rationale



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3.2 Relationship between Permutation and Combination

Find the number of permutations and combinations of 3 letters chosen from the set $\{A, B, C, D\}$.

No. of permutations
 $= {}^4P_3 = 4 \times 3 \times 2 = 24$.

<i>ABC</i>	<i>BAC</i>	<i>CAB</i>	<i>DAB</i>
<i>ACB</i>	<i>BCA</i>	<i>CBA</i>	<i>DBA</i>
<i>ABD</i>	<i>BAD</i>	<i>CAD</i>	<i>DAC</i>
<i>ADB</i>	<i>BDA</i>	<i>CDA</i>	<i>DCA</i>
<i>ACD</i>	<i>BCD</i>	<i>CBD</i>	<i>DBC</i>
<i>ADC</i>	<i>BDC</i>	<i>CDB</i>	<i>DCB</i>

No. of combinations = $\frac{4 \times 3 \times 2}{3!} = 4$

(since each combination has $3! = 6$ permutations)

Combination *ABC*

Corresponding Permutations:

ABC ACB BAC BCA CAB CBA

Combination *ABD*

Corresponding Permutations:

ABD ADB BAD BDA DAB DBA

Combination *ACD*

Corresponding Permutations:

ACD ADC CAD CDA DAC DCA

Combination *BCD*

Corresponding Permutations:

BCD BDC CBD CDB DBC DCB

In general,

nP_r	=	nC_r	×	$r!$
Permutation of r objects from n distinct objects		Combination of r objects from n distinct objects		No. of arrangements of r distinct objects

<i>Combination Questions</i>	<i>Permutation Questions</i>
Find the number of ways of selecting 4 students from a class of 25 to be in the class committee. (${}^{25}C_4$ ways)	Find the number of ways of selecting a class committee comprising Class Rep, Assistant Class Rep, NE Rep and Treasurer from a class of 25 students. (${}^{25}C_4 \times 4! = {}^{25}P_4$ ways)
Find the number of selections of 3 letters from the letters E, N, J, O, Y (5C_3 ways)	Find the number of 3-letter code words from the letters E, N, J, O, Y (${}^5C_3 \times 3! = {}^5P_3$ ways)
Three Consolation prizes are to be awarded to 3 different members of a group of 10 people. Find the number of ways the prizes can be awarded. (${}^{10}C_3$ ways)	Three prizes are to be awarded to 3 different members of a group of 10 people. Find the number of ways the prizes can be awarded if there is a 1 st prize, a 2 nd prize and a 3 rd prize. (${}^{10}C_3 \times 3! = {}^{10}P_3$ ways)
20 students comprising 10 JC1 and 10 JC2 respectively are to be selected from entire cohort of 850 JC1 and 800 JC2 students. Find the number of selections. (${}^{850}C_{10} {}^{800}C_{10}$ ways)	20 students comprising 10 JC1 and 10 JC2 respectively are to be selected from entire cohort of 850 JC1 and 800 JC2 students. Find the number of ways of choosing if they are to be seated in a row. (${}^{850}C_{10} \times {}^{800}C_{10} \times 20!$ ways)

3.3 Permutation of r Objects From n Distinct Objects (without Replacement)

This is the number of ways of arranging r objects chosen from a group of n distinct objects ($r \leq n$). Since we want to arrange r objects only, there are only r number of positions.

1 st	2 nd	3 rd		r^{th}
n	$n-1$	$n-2$	...	$n-r+1$

Total number of permutations

$$\begin{aligned}
 &= n \times (n-1) \times (n-2) \times \dots \times (n-r+1) \\
 &= \frac{n \times (n-1) \times (n-2) \times \dots \times (n-r+1) \times (n-r) \times \dots \times 3 \times 2 \times 1}{(n-r) \times (n-r-1) \times \dots \times 3 \times 2 \times 1} \\
 &= \frac{n!}{(n-r)!} = {}^nP_r \text{ (notation)}
 \end{aligned}$$

Number of **permutations** of r objects from n **distinct** objects **without replacement** is

$${}^n P_r = \frac{n!}{(n-r)!}.$$

Note

Since definition of $0! = 1$, ${}^n P_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$.

Example 9

How many 5-letter code words can be formed using the letters from the word MATRICES?

Solution

Method 1

No. of 5-letter code words

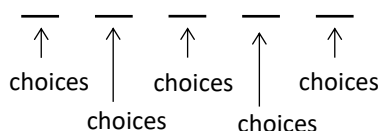
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Method 2

No. of 5-letter code words

=

Rationale



Firstly, we choose letters from the letters given.

Then arrange these letters.

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Example 10

If there are 10 winning entries for an essay competition, in how ways can the first, second and third prizes be awarded?

Solution

Method 1

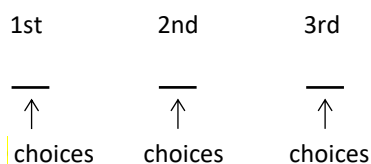
No. of ways

=

Method 2

No. of ways =

Rationale



Firstly, we select essays from the winning entries. Then arrange these essays for the first, second and third prize.

■

Example 11

How many even numbers greater than 6000 can be formed with the digits 3, 4, 5, 6, 8 if a digit cannot occur more than once in the number?

Solution

Case 1: No. of 5-digit even numbers =

Case 2: No. of 4-digit even numbers that are more than 6000

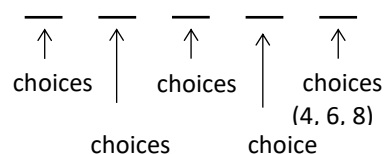
Hence, there are numbers greater 6000.

Rationale

Case 1: 5-digit even numbers greater than 6000

Note that all 5 digit even numbers are great than 6000.

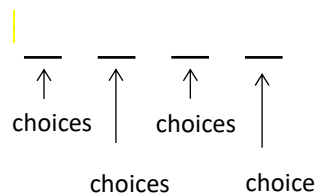
Choose the last digit first to make sure the number obtained is even.



Consider two scenarios: scenario 1 where the last digit is '4' and scenario 2 where the last digit is either '6' or '8'. Then choose the 1st digit accordingly.

Scenario 1 (last digit is '4')

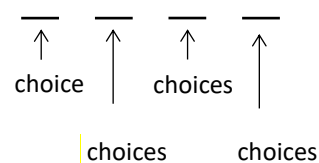
No. of 4-digit even numbers greater than 6000 =



Scenario 2 (last digit is '6' or '8')

No. of 4-digit even numbers greater than 6000

=



even

■

Practice Question 3

- (a) How many 3-digit odd numbers can be formed using the digits 1, 2, 3 and 4 if repetition of digits is not allowed?
- (b) How many of these 3-digit odd numbers are greater than 300?

Solution| *Rationale*

- (a) No. of 3-digit numbers

=

- (b) No. of 3-digit numbers that are greater than 300

=

■

3.4 Permutation of r Objects From n Distinct Objects (with Replacement)

This is the number of ways of arranging r objects chosen from a group of n distinct objects ($r < n$) when repetitions are allowed.

1 st	2 nd	3 rd		r^{th}
n	n	n	...	n

Number of **permutations** of r objects from n **distinct** objects **with replacement** is

$n^r.$

Example 12

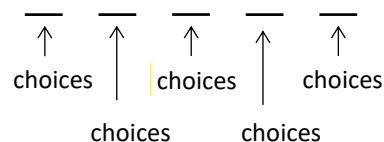
How many 5-letter code words can be formed using the letters from the word MATRICES, if each letter can be used more than once?

Solution

No. of 5-letter code words

=

Rationale



There are 8 choices for each of the letter in the code word as repetition is allowed. Therefore, there are possibilities.

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Example 13

In a library, books are labelled by using a 3-letter code chosen from 24 letters (excluding O and I) of the English alphabet followed by a 2-digit number.

- (a) How many books in the library can be labelled under this system?
 (b) Find the number of books that are labelled with three same letters and two same digits.

Solution

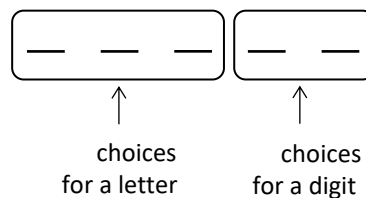
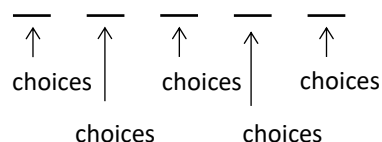
- (a) No. of books that can be labelled

=

- (b) No. of books that are labelled with the same three letters and same two digits

=

Rationale



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Practice Question 4

In how many ways can 5 books be given to 10 people if

- (a) they are to receive any number of books?
 (b) nobody can be given more than one book?

*Solution***(a)** No. of ways =**(b)** No. of ways
=*Rationale*

■

4 PERMUTATION OF n OBJECTS (NOT ALL DISTINCT)

Consider arranging 3 **distinct** objects ABC . There are $3! = 6$ ways of doing this, i.e.

$ABC \quad BAC \quad CAB \quad ACB \quad BCA \quad CBA$

Now what if two of the objects are identical, say, ABB , how many permutations are there that produce different outcomes?

To explain this, we first let the two B 's be B_1 and B_2 . Then there are $3! = 6$ ways of arranging them.

$AB_1B_2 \quad AB_2B_1 \quad B_1AB_2 \quad B_2AB_1 \quad B_1B_2A \quad B_2B_1A$

However the B 's are identical, therefore the permutations within them produces the same outcome.

$\underbrace{AB_1B_2 \quad AB_2B_1}_{ABB} \quad \underbrace{B_1AB_2 \quad B_2AB_1}_{BAB} \quad \underbrace{B_1B_2A \quad B_2B_1A}_{BBA}$

We see that we can group the $3! = 6$ combinations such that there are 2! permutations in each group.

Hence the number of different permutations is $\frac{3!}{2!} = 3$ ways,

$ABB \quad BAB \quad BBA$

In general, number of permutation of n objects of which n_1 are of one kind, n_2 of another kind, ..., and so on up to k kinds, is

$$\frac{n!}{n_1! n_2! n_3! \dots n_k!}$$

Example 14

- (a) How many 11-letter code words can be formed from the letters of the word INDEPENDENT?
- (b) How many 4-letter code words can be formed from the letters of the word INDEPENDENT?

Solution

(a) No. of 11-letter code words
=

Rationale

‘INDEPENDENT’ has 11 letters.
No. of arrangements of 11 letters from 11 letters is $11!$.

‘N’ and ‘E’ are repeated 3 times, and ‘D’ is repeated twice.

So we have to divide by the number of repetitions.

(b) No. of 4-letter code words:

Case 1: All letters are distinct

‘INDEPENDENT’ contains 6 distinct letters, I, N, D, E, P, T.
Choose 4 letters from 6 letters.

Case 2: 1 repeated letter and 2 other distinct letters

Choose 1 letter from ‘D, E, N’, 2 letters from the remaining 5.

Case 3: 2 repeated letters

Choose 2 letters from ‘D, E, N’.

Case 4: 1 letter repeated thrice and another letter

Choose 1 letter from ‘N, E’ and another letter from the remaining 5 letters.

Total no. of 4-letter code words
=

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Extra Practice

Annex: Practice Question 3

LECTURE 3

Lesson Outline

- Permutations with Conditions
 - Circular Permutations
-

5 PERMUTATION WITH CONDITIONS

General guideline: Handle the given restriction first.

<i>Possible Restrictions</i>	<i>Strategies</i>
Some objects must be together .	Treat those objects which must be together as a unit .
Some objects must be separated .	<i>Method 1</i> Consider the remaining objects that do not need to be separated and arrange these remaining objects first. Then insert those objects which must be separated. <i>Method 2 (Recommended only for 2 objects separated)</i> By the Principle of Complementation, No. of arrangements = Total no. of arrangements without restriction – No. of arrangements where the objects must be together.

Example 15

Three different Mathematics books and five other books are to be arranged on a shelf. Find the number of possible arrangements if

- (i) the three Mathematics books must be kept together.
 (ii) the three Mathematics books are separated.

Solution

- (i) No. of arrangements which the Mathematics books are kept together

=

- (ii) *Method 1*
 No. of arrangements which the Mathematics books are separated

=

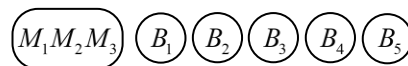
Method 2

No. of arrangements such that the Mathematics books are separated

$$= 8! - 6! \times 3! - 5! {}^6C_2 \times 2 \times 2 \times 3$$

$$= 14400$$

Rationale



Regard the 3 Mathematics books as one unit.

Arrange the 6 units in $6!$ ways.

Arrange the 3 Mathematics books in $3!$ ways.

We arrange the 5 non-mathematics books first.

— — — — —

Then insert the 3 Mathematics books.

↑ — ↑ — ↑ — ↑ — ↑ — ↑

The 3 Mathematics books can be inserted to 3 of the 6 possible positions, which give us 6C_3 .

Lastly, we arrange the 3 Mathematics books in $3!$ ways.

Total no. of arrangements

– no. of arrangements where all 3 books are together

– no. of arrangements where any 2 of the 3 books are together

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Example 16

In how many ways can 4 boys and 2 girls seat themselves in a row if

- (i) the 2 girls are to sit next to each other;
- (ii) the 2 girls are not to sit next to each other;
- (iii) the 2 girls are to be separated by 2 boys seated between them?

Solution

- (i) No. of ways with the 2 girls sit next to each other
 $= 5! \times 2! = 240$

- (ii) *Method 1*
 No. of ways with the 2 girls not to sit next to each other
 $= 4! \times {}^5C_2 \times 2 = 480$

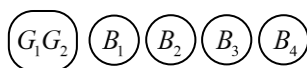
Method 2

By the **Principle of Complementation**,

No. of ways with the 2 girls not to sit next to each other
 $= 6! - 240 = 480$

- (iii) No. of ways with the 2 girls separated by 2 boys between them $= 4! (3 \times 2!) = 144$

Rationale

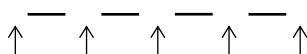


Regard the 2 girls as one unit.

Arrange the 5 units in $5!$ ways.

Arrange the 2 girls in $2!$ ways.

First, arrange the 4 boys in $4!$ ways then choose 2 out of 5 slots to insert the 2 girls.



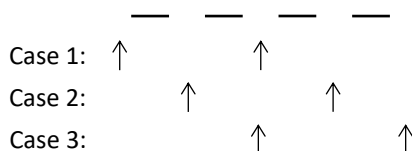
The number of arrangements of 6 people is $6!$. Thus no. of ways

$=$ no. of arrangements of 6 people
 $-$ no. of arrangements where the 2 girls sit together.

Firstly, arrange the 4 boys in $4!$ ways.



Then arrange the 2 girls such that they are separated by 2 boys.



The 2 girls can be arranged in the 3 ways as illustrated by the diagram above.

Lastly, the 2 girls can exchange their positions in $2!$ ways.

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Extra Practice

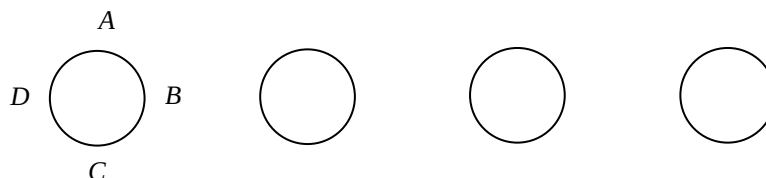
Annex: Practice Question 4

6 CIRCULAR PERMUTATIONS

Consider the different ways 4 men can sit at a round table.

If 4 men are arranged in a row, there are $4!$ permutations.

But in a table arrangement, the following 4 arrangements are the same.



Thus the 4 different arrangements correspond to 1 arrangement in a circle.

Therefore the number of ways in which the 4 men can sit at a table is $\frac{4!}{4} = 3!$

In general, n distinct objects can be arranged in a circle in

$$\frac{n!}{n} = (n-1)! \text{ ways.}$$

Example 17

Eight people are seated at a round table.

- (a) Find the number of possible arrangements if
- there are no restrictions,
 - Ben and Jerry are seated next to each other,
 - Ben and Jerry are separated.
- (b) Find the number of possible arrangements to (a)(i), (ii) and (iii) if the chairs are numbered.

Solution

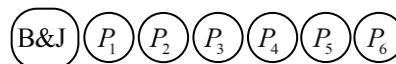
(a)(i) No. of arrangements for 8 people to be seated at the round table

=

(ii) No. of arrangements with Ben and Jerry sitting next to each other

=

Rationale



Regard Ben and Jerry as a unit.

Arrange the 7 units in a circle in $(7-1)!$ ways.

Ben and Jerry can exchange their seats in $2!$ ways.

- (iii) No. of arrangements with Ben and Jerry sitting separately
=
- (b)(i) No. of arrangements when the chairs are numbered
= $(8-1)! \times 8 = 40320$
- (ii) No. of arrangements with Ben and Jerry sitting next to each other
= $6! \times 2! \times 8 = 11520$
- (iii) No. of arrangements with Ben and Jerry sitting separately
= $(5040 - 1440) \times 8 = 28800$

By the **Principle of Complementation**,

No. of arrangements where Ben and Jerry sit separately = Total no. of arrangements – no. of arrangements where Ben and Jerry sit together.

The approach for each part in (b) is the same as that in (a).

As the seats are numbered, we have to multiply the answers in (a) by the number of seats. This is because for every arrangement in (a), there are 8 different starting points.

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Example 18

3 boys, 2 girls and a puppy sit at a round table. In how many ways can they be arranged if the puppy is to be seated

- (i) between the 2 girls,
(ii) between any 2 boys?

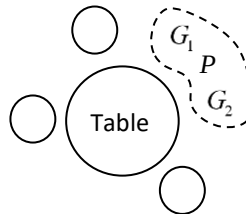
Solution

- (i) No. of arrangements if the puppy is to be seated between 2 girls
= $(4-1)! \times 2! = 12$

Rationale

Let B_1, B_2, B_3, G_1, G_2 and P denote the 3 boys, 2 girls and a puppy respectively.

Regard the 2 girls and the puppy as a unit i.e. G_1PG_2 .



Now, arrange the 4 units in $(4-1)!$ ways. Lastly, the 2 girls can exchange their positions.

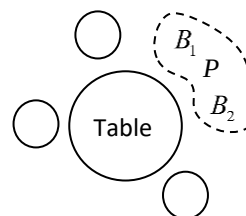
- (ii) No. of arrangements if the puppy is to be seated between any 2 boys

$$= {}^3C_2 \times 2! \times (4-1)!$$

$$= 36$$

Firstly, we choose 2 boys from the 3 boys whom the puppy is seated in between.

Then, regard the 2 boys and the puppy as a unit e.g. $B_1 P B_2$



Now, arrange the 4 units in $(4-1)!$ ways.

Lastly, the 2 boys can exchange their positions.

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More Examples

Example 19

A tourist has one of each of the coins of denominations 5 cents, 10 cents, 20 cents, 50 cents and \$1. In how many ways can he tip the hotel boy?

Solution

Method 1

$$\text{No. of ways} = 2^5 - 1 = 31$$

Method 2

$$\begin{aligned} \text{No. of ways} &= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 \\ &= 31 \end{aligned}$$

Rationale

For each of the 5-cent, 10-cent, 20-cent, 50-cent and \$1 coin, he either uses it or does not use it to tip the boy (i.e. $2 \times 2 \times 2 \times 2 \times 2 = 32$ ways)

Subtract 1 way where he does not use any of the denominations.

Case 1: Choose 1 out of the 5 denominations

Case 2: Choose 2 out of the 5 denominations

Case 3: Choose 3 out of the 5 denominations

Case 4: Choose 4 out of the 5 denominations

Case 5: Choose 5 out of the 5 denominations

■

Example 20

The number 75600 can be expressed in prime factors $2^4 \times 5^2 \times 7 \times 3^3$. How many positive integers are factors of 75600, excluding 1 and itself?

Solution

$$75600 = 2^4 \times 5^2 \times 7 \times 3^3$$

No. of integers

$$= 5 \times 3 \times 2 \times 4 - 1 - 1$$

$$= 118$$

Rationale

For factor 2, choose either $2^0, 2^1, 2^2, 2^3$ or 2^4 (5 ways)

For factor 5, choose either $5^0, 5^1$ or 5^2 (3 ways)

For factor 7, choose either 7^0 or 7^1 (2 ways)

For factor 3, choose either $3^0, 3^1, 3^2$ or 3^3 (4 ways)

Subtract 2 numbers (1 and 75600)

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Example 21

4 students Albert, Ben, Chris, Danny are to be divided equally into 2 groups. Find the number of groupings if

- (i) the groups are labelled as Group 1 and Group 2 respectively.
- (ii) the groups are not labelled.

Solution

- (i) No. of groupings (groups are labelled as Group 1 and Group 2)
- $$= {}^4C_2 \times {}^2C_2 = 6$$

Rationale

The 6 groupings are

Group 1	Group 2
AB	CD
AC	BD
AD	BC
BC	AD
BD	AC
CD	AB

By choosing 2 out of 4 students for Group 1, Group 2 is formed simultaneously.

(Since the remaining 2 students will form a group automatically.)

(ii) No. of groupings (groups are not labelled)

$$= \frac{{}^4C_2}{2!} = 3$$

The groupings are

1st grouping: AB and CD

2nd grouping: AC and BD

3rd grouping: AD and BC

Note that when the groups are **not** labelled, the following grouping in (i):

Group 1	Group 2
---------	---------

AB	CD
----	----

CD	AB
----	----

becomes the same grouping.

Thus, because of the repetition of groupings in (i), we need to divide 4C_2 by $2!$.

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Extra Practice

Annex: Practice Question 5

ANNEX

Practice Questions on Permutations and Combinations

- 1 A team of 4 students is to be chosen from a group of 6 boys and 9 girls. Find the number of ways this can be done if
 - (i) there is no restriction,
 - (ii) if a particular boy is to be included in the team,
 - (iii) if a particular girl is to be excluded from the team,
 - (iv) if two of the girls cannot be in the team together,
 - (v) if the team must contain exactly one boy,
 - (vi) if the team must contain at least one boy,
 - (vii) if the team must contain at least one boy and one girl.

- 2 The following boxes contain balls that are indistinguishable apart from colour. Find the number of different selections of 3 balls from each of these boxes (without replacement, and where order is not important).
 - (i) 1 ball for each of the 7 colours of the rainbow
 - (ii) 3 red balls and 3 yellow balls
 - (iii) 3 red balls, 2 yellow balls and 1 black ball
 - (iv) 2 red balls, 2 yellow balls, 2 black balls and 2 white balls

- 3 Find the number of permutations of the letters of the word if

(i) all the letters are used	(ii) 3 letters are used.
(a) GRAPHS	(b) EQUATIONS
(d) CALCULUS	(e) SEQUENCES
	(c) FUNCTIONS

- 4 5 thin books and 4 thick books are to be arranged on a bookshelf. All the books are different titles. Find the number of ways of arranging them in order
 - (i) if there are no restrictions,
 - (ii) if the thin and thick books must alternate,
 - (iii) if no two thick books are to be placed next to each other,
 - (iv) if the thin books and thick books are to be grouped together respectively,
 - (v) if the thin books are to be grouped together,
 - (vi) if one particular thick book must be placed at either end of the bookshelf.

- 5 A committee comprising a female manager, 4 male employees and 3 female employees are to be seated at a round table for meeting. Find the number of arrangements around the table
 - (i) if there are no restrictions,
 - (ii) if the male employees sit together,
 - (iii) if the males and females are seated alternately,
 - (iv) if the manager must be seated next to a particular female employee.

Answers

- 1 **(i)** 1365 **(ii)** 364 **(iii)** 1001 **(iv)** 1287 **(v)** 504 **(vi)** 1239 **(vii)** 1224
- 2 **(i)** 35 **(ii)** 4 **(iii)** 6 **(iv)** 16
- 3 **(a)** 720; 120 **(b)** 362 880; 504 **(c)** 181 440; 357
 (d) 5040; 96 **(e)** 30 240; 751
- 4 **(i)** 362 880 **(ii)** 2880 **(iii)** 43 200 **(iv)** 5760 **(v)** 14 400 **(vi)** 80 640
- 5 **(i)** 5040 **(ii)** 576 **(iii)** 144 **(iv)** 1440