

6B

THREE-DIMENSIONAL VECTOR GEOMETRY

SYLLABUS

- Vector and cartesian equations of lines and planes
- Foot of the perpendicular and distance from a point to a line or to a plane
- Angle between two lines, between a line and a plane, or between two planes
- Relationships between
 - two lines (coplanar or skew)
 - a line and a plane
 - two planes

CONTENTS

1	Lines	3
1.1	Vector Equation of a Line	3
1.2	Parametric and Cartesian Equations of a Line	5
2	Relationships between Point and Line	9
2.1	Foot of Perpendicular from a Point to a Line	9
2.2	Perpendicular Distance from a Point to a Line	12
3	Relationships between Two Lines	16
3.1	Parallel, Intersecting or Skew Lines	16
3.2	Angle between Two Lines	23
4	Planes	25
4.1	Vector Equation of a Plane in Parametric Form	25
4.2	Vector Equation of a Plane in Scalar Product Form	27
4.3	Cartesian Equation of a Plane	31
4.4	Special Planes	32
5	Relationships between Point and Plane	34
5.1	Point on a Plane	34
5.2	Foot of Perpendicular from a Point to a Plane	35
5.3	Perpendicular Distance from a Point to a Plane	38
5.4	Distance of the Origin O from a Plane	39
5.5	Reflection of a Point in a Plane	40
6	Relationships between Line and Plane	41
6.1	Possible Configurations of a Line and a Plane	41
6.2	Intersection of a Line and a Plane	42
6.3	Angle between a Line and a Plane	43
6.4	Reflection of a Line in a Plane	45
7	Relationships between Two Planes	46
7.1	Two Parallel Planes	46
7.2	Intersection of Two Planes	48
7.3	Angle between Two Planes	52
	Annex A: Practice Questions on Lines and Planes	55

LECTURE 1

Lesson Outline

- Vector equation of a line
 - Parametric and Cartesian equations of a line
-

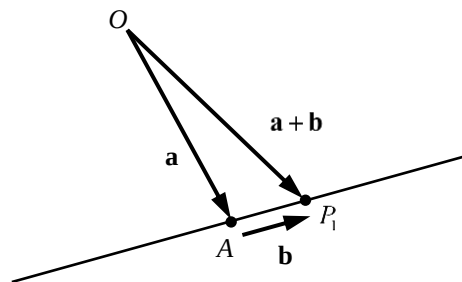
1 LINES

1.1 Vector Equation of a Line

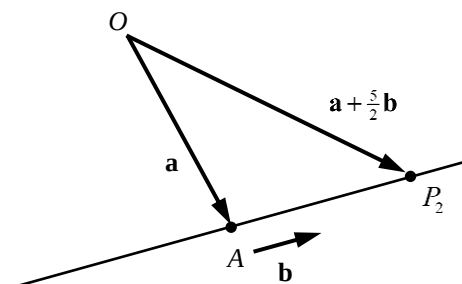
Consider an origin O , a point A with position vector $\mathbf{a}$, a vector $\mathbf{b}$ and a line l that passes through point A and is parallel to $\mathbf{b}$.

Let P_1 , P_2 and P_3 be three points on the line.

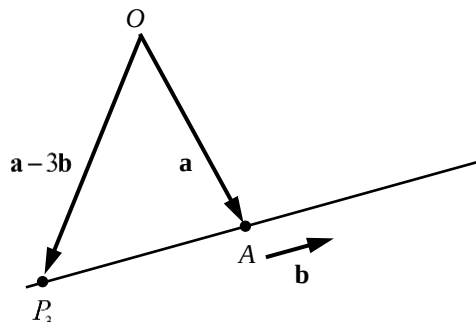
Suppose $\overrightarrow{AP_1} = \mathbf{b}$. Then $\overrightarrow{OP_1} = \overrightarrow{OA} + \overrightarrow{AP_1} = \mathbf{a} + \mathbf{b}$.



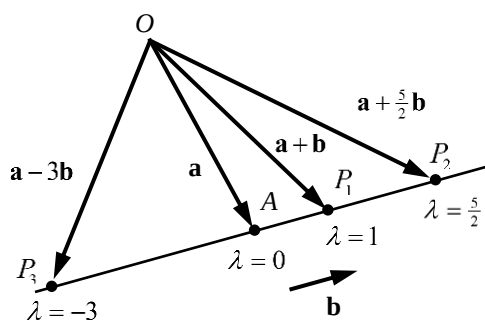
Suppose $\overrightarrow{AP_2} = \frac{5}{2}\mathbf{b}$. Then $\overrightarrow{OP_2} = \overrightarrow{OA} + \overrightarrow{AP_2} = \mathbf{a} + \frac{5}{2}\mathbf{b}$.



Suppose $\overrightarrow{AP_3} = -3\mathbf{b}$. Then $\overrightarrow{OP_3} = \overrightarrow{OA} + \overrightarrow{AP_3} = \mathbf{a} - 3\mathbf{b}$.



Combining the 3 cases, we have



In general, if P is any point on the line l such that $\overrightarrow{AP} = \lambda \mathbf{b}$, then the position vector $\mathbf{r}$ of P is given by

$$\mathbf{r} = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \mathbf{a} + \lambda \mathbf{b}.$$

Conversely, the vector $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$, where $\lambda \in \mathbb{R}$, is the position vector of a point P on the line passing through A and parallel to $\mathbf{b}$ such that $\overrightarrow{AP} = \lambda \mathbf{b}$. The vector $\mathbf{b}$ is called the **direction vector** of the line.

Hence, the **vector equation** of a line l passing through point A with position vector $\mathbf{a}$ and parallel to a vector $\mathbf{b}$ is

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}, \text{ where } \lambda \text{ is a parameter, } \lambda \in \mathbb{R}.$$

Here, $\mathbf{r}$ is the position vector of any point on the line l .

Note

- Expression of the equation of a line is **not** unique.
- If the equation of the line can be written as $\mathbf{r} = \lambda \mathbf{b}$, where λ is a parameter, $\lambda \in \mathbb{R}$, then the line passes through the origin O .

Example 1

- Find a vector equation of the line l_1 which is parallel to the vector $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and passes through the point $(5, -2, 4)$.
- Find a vector equation of the line l_2 passing through points $A(5, 2, 6)$ and $B(-3, 6, 2)$.
- Determine if the point $(7, -3, 7)$ lies on the lines l_1 and/or l_2 .

Solution

$$(a) \quad l_1 : \mathbf{r} = \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \lambda \in \mathbb{R}$$

$$(b) \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -3 \\ 6 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix} = 4 \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix}$$

$$l_2 : \mathbf{r} = \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}, \mu \in \mathbb{R}$$

(c) If $\lambda = 1$ in the equation for l_1 , then

$$\begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix} + \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix}$$

$$\text{Let } \mathbf{r} = \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix} \text{ in the equation of } l_2, \text{ we have } \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}.$$

As there is no solution for μ , the point $(7, -3, 7)$ does not lie on l_2 .

■

Extra Practice

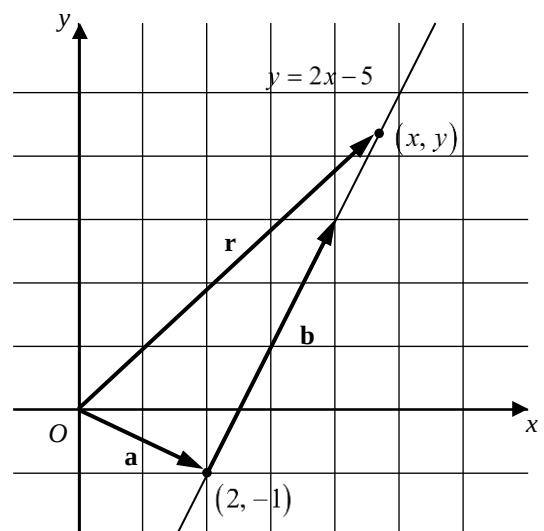
Annex A: Practice Question 1

1.2 Parametric and Cartesian Equations of a Line

Suppose we have a line that passes through the point $(2, -1)$ and is parallel to the vector $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

The vector equation of this line is $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \lambda \in \mathbb{R}$.

Since $\mathbf{r}$ is the position vector of a point on the line, we can also write $\mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix}$, where (x, y) are the coordinates of a point on the line.



Since $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, we have $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2+2\lambda \\ -1+4\lambda \end{pmatrix}$,

so

$$\begin{aligned} x &= 2 + 2\lambda, \\ y &= -1 + 4\lambda. \end{aligned}$$

This is called the **parametric equation** of the line, since x and y are both given in terms of a third variable λ .

We can make λ the subject in both equations, to get

$$\lambda = \frac{x-2}{2}, \lambda = \frac{y+1}{4}.$$

Therefore

$$\frac{x-2}{2} = \frac{y+1}{4}.$$

This is the **Cartesian equation** of the line. By making y the subject it can also be rearranged into the more familiar form $y = 2x - 5$.

Note that we have three ways of expressing the relationship between the coordinates of the points on the line, namely the vector equation, the parametric equation, and the cartesian equation.

Since the vector equation of a line is **not** unique, neither are the parametric equation and the cartesian equation.

We can perform the same sequence of operations in three dimensions:

Example

Equation of line l which passes through the point $(5, -2, 4)$ and parallel

to $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ can be represented in the following ways:

Vector Equation	$\mathbf{r} = \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, where $\lambda \in \mathbb{R}$.
Parametric Equation	Let $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then $\begin{aligned} x &= 5 + 2\lambda \\ y &= -2 - \lambda \\ z &= 4 + 3\lambda \end{aligned}$, where $\lambda \in \mathbb{R}$.
Cartesian Equation	Make λ the subject for each of the parametric equations, i.e., $\frac{x-5}{2} = \frac{y+2}{-1} = \frac{z-4}{3}$.

In general, if we have a vector equation of a line given by

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}, \lambda \in \mathbb{R},$$

where $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, to convert this to the parametric and

cartesian equations, we let $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \Rightarrow \begin{cases} x = a_1 + \lambda b_1 \\ y = a_2 + \lambda b_2 \\ z = a_3 + \lambda b_3 \end{cases} \\ \Rightarrow \lambda &= \frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}. \end{aligned}$$

Example 2

- (a) Find cartesian and parametric equations of the line passing through the points $A(-1, 2, 3)$ and $B(4, -5, 6)$.
- (b) Convert the following cartesian equations of the lines to vector equations:

$$(i) \frac{x-3}{2} = \frac{y+1}{3} = \frac{2-z}{4} \quad (ii) 2x-1=3z, y=4.$$

Solution

(a)

(b)(i)**(ii)** Let $\lambda = 2x - 1 = 3z$, $y = 4$, therefore

$$2x - 1 = \lambda \Rightarrow x = \frac{1}{2} + \frac{1}{2}\lambda$$

$$y = 4$$

$$3z = \lambda \Rightarrow z = \frac{1}{3}\lambda$$

Hence

$$\begin{aligned} \mathbf{r} &= \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}\lambda \\ 4 \\ \frac{1}{3}\lambda \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} \\ 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} \\ 4 \\ 0 \end{pmatrix} + \lambda' \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} \end{aligned}$$

■

Extra Practice

Annex A: Practice Question 2, 3

LECTURE 2

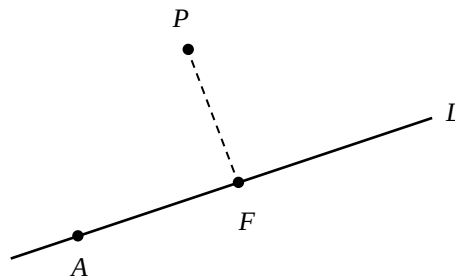
Lesson Outline

- Foot of perpendicular from a point to a line
 - Perpendicular distance from a point to a line
-

2 RELATIONSHIPS BETWEEN POINT AND LINE

2.1 Foot of Perpendicular from a Point to a Line

Given equation of line L , $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, and point P with position vector $\overrightarrow{OP}$, let F be the foot of perpendicular from point P to line L .



To find the position vector of F

Method 1: Equation of line

Step 1: Since F is a point on the line L ,

$$\overrightarrow{OF} = \mathbf{a} + \lambda\mathbf{b} \text{ for some real value of } \lambda.$$

Step 2: Since PF is perpendicular to L (and therefore $\mathbf{b}$)

$$\overrightarrow{PF} \cdot \mathbf{b} = 0 \Rightarrow (\overrightarrow{OF} - \overrightarrow{OP}) \cdot \mathbf{b} = 0 \text{ and solve for } \lambda$$

Step 3: Substitute λ into the expression of $\overrightarrow{OF}$ to find the position vector of F .

Method 2: Projection

(Recall Chapter 6A Section 5.4 Projection of a Vector in a Given Direction)

Step 1: Since $\overrightarrow{AF}$ is the vector component of $\overrightarrow{AP}$ parallel to $\mathbf{b}$,

$$\overrightarrow{AF} = (\overrightarrow{AP} \cdot \hat{\mathbf{b}}) \hat{\mathbf{b}}$$

Step 2: Find $\overrightarrow{OF}$ using $\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AF}$

Example 3

Given a line L which has vector equation, $\mathbf{r} = \begin{pmatrix} 7 - \lambda \\ 1 + 3\lambda \\ -5 + \lambda \end{pmatrix}$ and a point P with

coordinates $(3, -6, 1)$, find the coordinates of the foot of the perpendicular from point P to line L . Hence find the shortest distance from point P to the line L .

Solution

Method 1

$$\text{Let } \overrightarrow{OA} = \begin{pmatrix} 7 \\ 1 \\ -5 \end{pmatrix}.$$

Let F be foot of the perpendicular from point P to line L .

Step 1: Since point F lies on line L ,

Step 2: $PF \perp L \Rightarrow \overrightarrow{PF} \cdot \mathbf{b} =$

Step 3: Substitute λ into the expression of $\overrightarrow{OF}$.

Method 2

$$\overrightarrow{AF} = (\overrightarrow{AP} \cdot \hat{\mathbf{b}}) \hat{\mathbf{b}} \text{ where } \hat{\mathbf{b}} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$$

$$= \left[\begin{pmatrix} -4 \\ -7 \\ 6 \end{pmatrix} \cdot \frac{1}{\sqrt{11}} \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \right] \frac{1}{\sqrt{11}} \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$$

$$= \frac{(4 - 21 + 6)}{11} \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$$

$$\therefore \overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AF} = \begin{pmatrix} 7 \\ 1 \\ -5 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \\ -6 \end{pmatrix}$$

$$\text{Thus } \overrightarrow{PF} = \overrightarrow{OF} - \overrightarrow{OP} = \begin{pmatrix} 5 \\ 4 \\ -7 \end{pmatrix}.$$

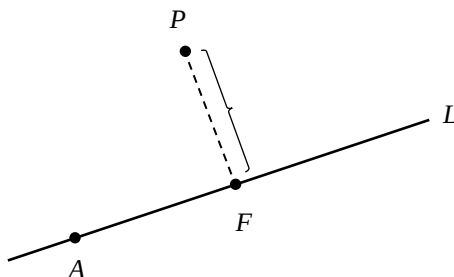
$$\text{Shortest distance} = |\overrightarrow{PF}| = \left| \begin{pmatrix} 5 \\ 4 \\ -7 \end{pmatrix} \right| = \sqrt{90} = 3\sqrt{10} \text{ units.}$$

■

2.2 Perpendicular Distance from a Point to a Line

In the diagram, the line L has equation $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$.

A is a point on L such that $\overrightarrow{OA} = \mathbf{a}$ and F is the foot of the perpendicular from a point P to L . PF is called the perpendicular (shortest) distance from P to the line L .



To find the perpendicular distance PF

Method 1: Cross Product

(Recall Chapter 6A, Section 6.3 Perpendicular Distance from a Point to a Line)

If the question only asks for perpendicular distance, we can use the formula

Method 2: Finding foot of perpendicular

If the question first asks for position vector of the foot of perpendicular from point P to line L , then we can use this position vector to deduce the perpendicular distance PF :

Step 1: Find $\overrightarrow{OF}$, where F is the foot of perpendicular from point P to line L . (Recall Chapter 6B Section 2.1 Foot of Perpendicular from a Point to a Line)

Step 2: Find $\overrightarrow{PF}$ and thus perpendicular distance $PF = |\overrightarrow{PF}|$.

Method 3: Finding length of projection

If the question first asks for length of projection of AP on the line L (i.e. AF), then we can use Pythagoras' Theorem on $\triangle APF$:

Step 1: Find AF , the length of projection of AP on the line L (Recall Chapter 6A Section 5.4 Projection of a Vector in a Given Direction)

Step 2: Find PF using Pythagoras' Theorem on $\triangle APF$.

Example 4

Find the shortest distance from $P(1, 3, 2)$ to the line with equation $\mathbf{r} = (1-t)\mathbf{i} + (2-3t)\mathbf{j} + 2\mathbf{k}$.

Solution

Shortest distance from $P(1, 3, 2)$ to the line

■

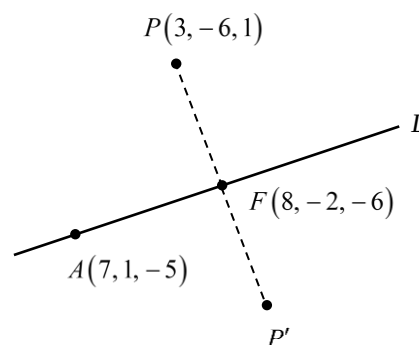
Example 5

Using Example 3, find the position vector of the reflection of point P on line L .

Solution

From Example 3, the foot of perpendicular F is $(8, -2, -6)$

Since F is midpoint of PP' ,



Alternatively, F is midpoint of PP' .

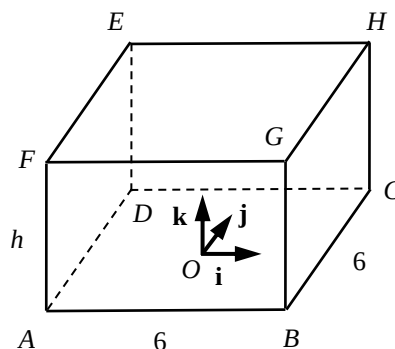
Using ratio theorem, $\overrightarrow{OF} = \frac{\overrightarrow{OP} + \overrightarrow{OP'}}{2}$.

$$\Rightarrow \overrightarrow{OP'} = 2\overrightarrow{OF} - \overrightarrow{OP} = 2 \begin{pmatrix} 8 \\ -2 \\ -6 \end{pmatrix} - \begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix} = \begin{pmatrix} 13 \\ 2 \\ -13 \end{pmatrix}.$$

■

Example 6

$ABCDEFGH$ is a wooden block with a square base $ABCD$ of side 6 units. The height of the block is h units. The point O at the centre of the base, is the origin of position vectors. The vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors in the direction AB , BC and AF respectively.



The vector $\overrightarrow{OH}$ makes an angle of 45° with the horizontal.

- (i) Find, h , the height of the block.
- (ii) Write down the vector equation of the line FB and hence find the position vector of the foot of the perpendicular from H to FB .

[NJC/2003]

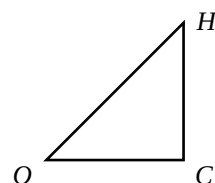
Solution

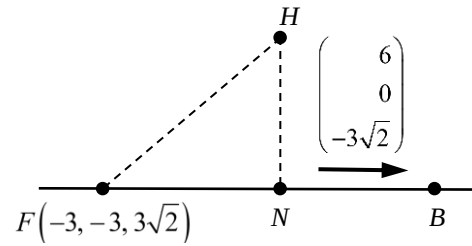
(i)

(ii) A vector parallel to line FB is

$$\overrightarrow{FB} = \overrightarrow{OB} - \overrightarrow{OF} = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} - \begin{pmatrix} -3 \\ -3 \\ 3\sqrt{2} \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ -3\sqrt{2} \end{pmatrix}$$

Vector equation of line FB is





Let N be the foot of perpendicular from H to the line FB .

Since point N lies on line FB ,

$$\overrightarrow{ON} = \begin{pmatrix} 3+6\lambda \\ -3 \\ -3\sqrt{2}\lambda \end{pmatrix} \text{ for some } \lambda \in \mathbb{R}$$

$$HN \perp FB \Rightarrow \overrightarrow{HN} \cdot \begin{pmatrix} 6 \\ 0 \\ -3\sqrt{2} \end{pmatrix} = 0$$

$$\Rightarrow (\overrightarrow{ON} - \overrightarrow{OH}) \cdot \begin{pmatrix} 6 \\ 0 \\ -3\sqrt{2} \end{pmatrix} = 0$$

$$\Rightarrow \left(\begin{pmatrix} 3+6\lambda \\ -3 \\ -3\sqrt{2}\lambda \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 3\sqrt{2} \end{pmatrix} \right) \cdot \begin{pmatrix} 6 \\ 0 \\ -3\sqrt{2} \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 6\lambda \\ -6 \\ -3\sqrt{2}\lambda - 3\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 0 \\ -3\sqrt{2} \end{pmatrix} = 0$$

$$\Rightarrow 36\lambda + 18\lambda + 18 = 0$$

$$\Rightarrow \lambda = -\frac{1}{3}$$

$$\therefore \overrightarrow{ON} = \begin{pmatrix} 3+6(-\frac{1}{3}) \\ -3 \\ -3\sqrt{2}(-\frac{1}{3}) \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ \sqrt{2} \end{pmatrix}$$

■

3 RELATIONSHIPS BETWEEN TWO LINES

3.1 Parallel, Intersecting or Skew Lines

Two distinct lines in three-dimensional space may be located in one of the following configurations:

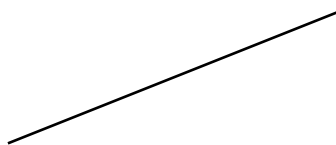
- (a) the lines are parallel,
- (b) the lines are not parallel and intersect,
- (c) the lines are not parallel and do not intersect. These are called **skew** lines.

It is also possible that, given two equations, they may be of the same line even though they look different.

To determine the relationship between two given lines from their equations, we first consider the directions of the lines.

Direction vectors are parallel

Same line

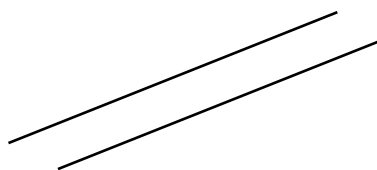


Equate the **i**, **j**, and **k** components and solve for λ and μ :

Solutions exist.

(See Example 7(iii))

Parallel Lines



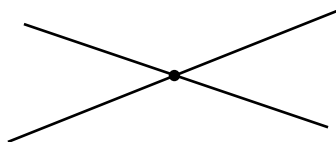
Equate the **i**, **j**, and **k** components and solve for λ and μ :

Solutions **do not** exist.

(See Example 7(iv))

Direction vectors are not parallel

Intersecting Lines

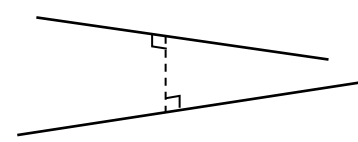


Equate the **i**, **j**, and **k** components and solve for λ and μ :

Solution exists.

(See Example 7(i))

Skew Lines



Equate the **i**, **j**, and **k** components and solve for λ and μ :

Solutions **do not** exist.

(See Example 7(ii))

Example 7

Determine the geometrical relationship between the following pairs of lines. If the lines intersect, find the position vector of the point of intersection.

(i) $l_1 : \mathbf{r} = \mathbf{i} - \mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k})$, $l_2 : \mathbf{r} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} + 3\mathbf{k})$,

where $\lambda, \mu \in \mathbb{R}$,

(ii) $l_1 : x-1 = \frac{y}{3} = \frac{z-1}{4}$, $l_2 : \frac{x-2}{4} = 3-y = z$,

(iii) $l_1 : \frac{x-4}{3} = \frac{y-6}{4} = \frac{z+9}{-12}$, $l_2 : \frac{x-1}{-6} = \frac{y-2}{-8} = \frac{z-3}{24}$,

(iv) $l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$, $l_2 : \mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -6 \\ 12 \end{pmatrix}$, where $\lambda, \mu \in \mathbb{R}$.

Solution

(i) $l_1 : \mathbf{r} = \mathbf{i} - \mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k}) = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+\lambda \\ -1-\lambda \\ 3+\lambda \end{pmatrix}$

$$l_2 : \mathbf{r} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2+2\mu \\ 4+\mu \\ 6+3\mu \end{pmatrix}$$

Since $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \neq \alpha \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ for any scalar α , l_1 and l_2 are not parallel.

For intersection of two lines, we equate the two expressions to solve for λ and μ .

$$\text{Let } \begin{pmatrix} 1+\lambda \\ -1-\lambda \\ 3+\lambda \end{pmatrix} = \begin{pmatrix} 2+2\mu \\ 4+\mu \\ 6+3\mu \end{pmatrix}, \text{ i.e., } \begin{cases} \lambda - 2\mu = 1 \\ -\lambda - \mu = 5 \\ \lambda - 3\mu = 3 \end{cases}$$

In the algebraic method, we choose two equations to solve for λ and μ . Then the third equation is used to check if the system yields any solution for λ and μ .

GC Use [APPS][PlySmlt2]

The first screenshot shows the calculator in the 'SIMULT EQN SOLVER MODE'. The 'EQUATIONS' field is set to 3, and the 'UNKNOWN' field is set to 3. The 'AUTO DEC' menu is open, showing 'NORMAL' selected. The 'FLOAT' menu is open, showing '0' selected. The 'RADIAN' menu is open, showing 'DEGREE' selected. The 'MAIN' button is highlighted.

The second screenshot shows the 'SYSTEM MATRIX (3 x 3)' screen. The matrix is displayed as:

$$\begin{bmatrix} 1 & -2 & 1 \\ -1 & -1 & 5 \\ 1 & -3 & 3 \end{bmatrix}$$

The 'SYSM[3,3]=3' status is shown. The 'MAIN' button is highlighted.

The third screenshot shows the 'SOLUTION' screen. The solutions are displayed as:

$$\begin{aligned} x_1 &= -3 \\ x_2 &= -2 \end{aligned}$$

The 'MAIN' button is highlighted.

Algebraic

$$\lambda - 2\mu = 1 \quad \text{---(1)}$$

$$-\lambda - \mu = 5 \quad \text{---(2)}$$

$$\lambda - 3\mu = 3 \quad \text{---(3)}$$

$$(1)+(2): -3\mu = 6 \Rightarrow \mu = -2$$

Substitute into (2),

$$-\lambda + 2 = 5 \Rightarrow \lambda = -3$$

Sub. $\lambda = -3, \mu = -2$ into (3),

$$\text{LHS} = \lambda - 3\mu = 3$$

$$\text{RHS} = 3$$

Since $\text{LHS} = \text{RHS}$, we get consistent values for λ and μ .

$\therefore$ The lines l_1 and l_2 intersect.

Substitute $\lambda = -3$ in the equation of l_1 to find the position vector of

$$\text{the intersection point, i.e. } \begin{pmatrix} 1+(-3) \\ -1-(-3) \\ 3+(-3) \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}.$$

$$(ii) \quad l_1: x-1 = \frac{y}{3} = \frac{z-1}{4} = \lambda, \text{ i.e. } \mathbf{r} = \begin{pmatrix} 1+\lambda \\ 3\lambda \\ 1+4\lambda \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

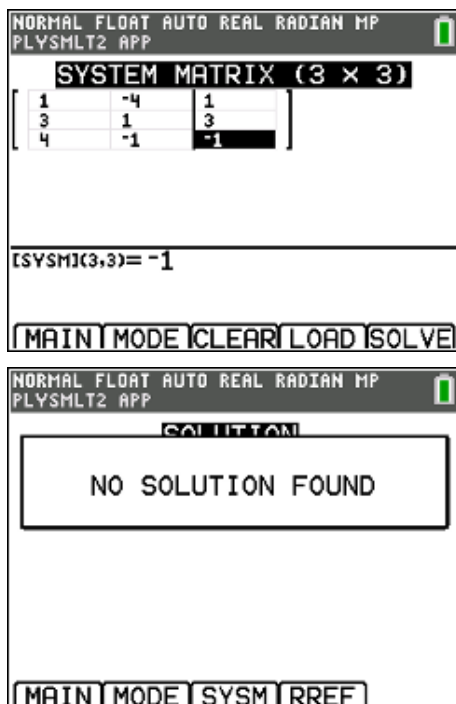
$$l_2: \frac{x-2}{4} = 3-y = z = \mu, \text{ i.e., } \mathbf{r} = \begin{pmatrix} 2+4\mu \\ 3-\mu \\ \mu \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$$

Since $\begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \neq \alpha \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$ for any scalar α , l_1 and l_2 are not parallel.

$$\text{Let } \begin{pmatrix} 1+\lambda \\ 3\lambda \\ 1+4\lambda \end{pmatrix} = \begin{pmatrix} 2+4\mu \\ 3-\mu \\ \mu \end{pmatrix}, \text{ i.e., } \begin{cases} \lambda - 4\mu = 1 \\ 3\lambda + \mu = 3 \\ 4\lambda - \mu = -1 \end{cases}$$

GC

Solving using GC as in (i), there are no solutions for λ and μ . Therefore the lines l_1 and l_2 do not intersect. Hence they are skew lines.



Algebraic

$$\lambda - 4\mu = 1 \quad \text{---(1)}$$

$$3\lambda + \mu = 3 \quad \text{---(2)}$$

$$4\lambda - \mu = -1 \quad \text{---(3)}$$

$$(2) \times 4: 12\lambda + 4\mu = 12 \quad \text{---(2a)}$$

$$(1) + (2a): 13\lambda = 13 \Rightarrow \lambda = 1$$

Substitute into (2):

$$3 + \mu = 3 \Rightarrow \mu = 0$$

Substitute $\lambda = 1, \mu = 0$ into (3):

$$\text{LHS} = 1 + 4\lambda = 5$$

$$\text{RHS} = \mu = 0$$

Since $\text{LHS} \neq \text{RHS}$, we do not get consistent values for λ and μ , which implies there are no intersection points between l_1 and l_2 .

$\therefore$ The lines l_1 and l_2 are skew.

$$(iii) \quad l_1: \frac{x-4}{3} = \frac{y-6}{4} = \frac{z+9}{-12}, \text{ i.e., } \mathbf{r} = \begin{pmatrix} 4+3\lambda \\ 6+4\lambda \\ -9-12\lambda \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ -9 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ -12 \end{pmatrix}$$

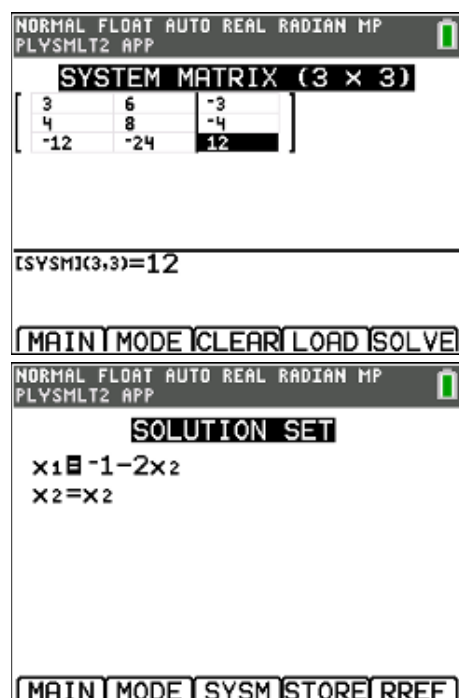
$$l_2: \frac{x-1}{-6} = \frac{y-2}{-8} = \frac{z-3}{24}, \text{ i.e., } \mathbf{r} = \begin{pmatrix} 1-6\mu \\ 2-8\mu \\ 3+24\mu \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -6 \\ -8 \\ 24 \end{pmatrix}$$

Since $-2 \begin{pmatrix} 3 \\ 4 \\ -12 \end{pmatrix} = \begin{pmatrix} -6 \\ -8 \\ 24 \end{pmatrix}$, l_1 and l_2 are either parallel or the same line.

GC

$$\text{Let } \begin{pmatrix} 4+3\lambda \\ 6+4\lambda \\ -9-12\lambda \end{pmatrix} = \begin{pmatrix} 1-6\mu \\ 2-8\mu \\ 3+24\mu \end{pmatrix}, \text{ i.e., } \begin{cases} 3\lambda + 6\mu = -3 \\ 4\lambda + 8\mu = -4 \\ -12\lambda - 24\mu = 12 \end{cases}$$

Solving using GC, we get
 $\lambda = -1 - 2\mu$ (l_1 and l_2 have infinitely many points in common).
 The equations are consistent.
 $\therefore$ The lines l_1 and l_2 are the same line.



Algebraic

The point $(1, 2, 3)$ lies on l_2 .

To check if it lies on l_1 :

$$\begin{pmatrix} 4+3\lambda \\ 6+4\lambda \\ -9-12\lambda \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \left| \begin{array}{l} \text{If the point } (1, 2, 3) \text{ lies} \\ \text{on } l_1 \text{ as well then } l_1 \text{ and} \\ l_2 \text{ are the same line.} \end{array} \right.$$

Observe that $\lambda = -1$ satisfies all three components, thus $(1, 2, 3)$ lies on l_1 .

$\therefore$ The lines l_1 and l_2 are the same line.

$$(iv) \quad l_1: \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1+\lambda \\ 2-2\lambda \\ 3+4\lambda \end{pmatrix}$$

$$l_2: \mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -6 \\ 12 \end{pmatrix} = \begin{pmatrix} 2+3\mu \\ 4-6\mu \\ -5+12\mu \end{pmatrix}$$

Since $3 \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ 12 \end{pmatrix}$, l_1 and l_2 are either parallel or the same line.

GC

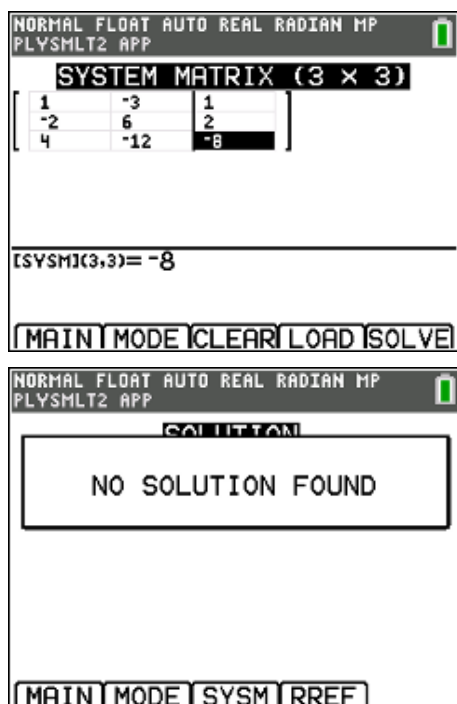
$$\text{Let } \begin{pmatrix} 1+\lambda \\ 2-2\lambda \\ 3+4\lambda \end{pmatrix} = \begin{pmatrix} 2+3\mu \\ 4-6\mu \\ -5+12\mu \end{pmatrix}, \text{ i.e.,}$$

$$\begin{cases} \lambda - 3\mu = 1 \\ -2\lambda + 6\mu = 2 \\ 4\lambda - 12\mu = -8 \end{cases}$$

Solving using GC, we do not get consistent values for λ and μ .

$\Rightarrow$ The lines l_1 and l_2 are not the same line (they do not have any points in common).

$\therefore$ The lines l_1 and l_2 are parallel.



Algebraic

The point $(2, 4, -5)$ lies on l_2 .

To check if it lies on l_1 :

$$\begin{pmatrix} 1+\lambda \\ 2-2\lambda \\ 3+4\lambda \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

$$(1): 1 + \lambda = 2 \Rightarrow \lambda = 1$$

$$(2): 2 - 2\lambda = 4 \Rightarrow \lambda = -1$$

Since the system of equations does not have a consistent solution, $(2, 4, -5)$ does not lie on l_1 .

Hence the lines l_1 and l_2 are parallel.

LECTURE 3

Lesson Outline

- Relationship between two lines in space
- Angle between two lines

Example 8

The position vectors of A , B referred to an origin O are $\mathbf{a}$, $\mathbf{b}$ respectively. The point P is the midpoint of AB and Q is the midpoint of OP . The line AQ produced cuts OB at R . Find, in terms of $\mathbf{a}$, $\mathbf{b}$,

- (a) the position vector of Q referred to O ,
 (b) the vector $\overrightarrow{AQ}$.

Prove that $\overrightarrow{OR} = \frac{1}{3}\mathbf{b}$.

[N72/II/5(b)]

Solution

- (a) P is the midpoint of AB i.e. $AP : PB = 1 : 1$

By ratio theorem, $\overrightarrow{OP} =$

Q is the midpoint of OP , hence $\overrightarrow{OQ} =$

- (b) $\overrightarrow{AQ} = \overrightarrow{OQ} - \overrightarrow{OA} =$

Method 1

Since the two lines meet at R , to find intersection, we equate the equations of the two lines.

$$l_{AQ} : \mathbf{r} = \mathbf{a} + \lambda \left(-\frac{3}{4}\mathbf{a} + \frac{1}{4}\mathbf{b} \right)$$

$$l_{OB} : \mathbf{r} = \mu\mathbf{b}$$

For intersection,

$$\mathbf{a} + \lambda \left(-\frac{3}{4}\mathbf{a} + \frac{1}{4}\mathbf{b} \right) = \mu\mathbf{b}$$

Method 2

$$\overrightarrow{AR} = \lambda \overrightarrow{AQ} = \lambda \left(-\frac{3}{4}\mathbf{a} + \frac{1}{4}\mathbf{b} \right)$$

$$\therefore \overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR}$$

$$= \mathbf{a} + \lambda \left(-\frac{3}{4}\mathbf{a} + \frac{1}{4}\mathbf{b} \right)$$

$$= \left(1 - \frac{3}{4}\lambda \right) \mathbf{a} + \frac{\lambda}{4}\mathbf{b}$$

On the other hand, R is on OB ,

$$\overrightarrow{OR} = \mu\mathbf{b}$$

$$\text{Therefore } \left(1 - \frac{3}{4}\lambda \right) \mathbf{a} + \frac{\lambda}{4}\mathbf{b} = \mu\mathbf{b}.$$

Since $\mathbf{a}$ is not parallel to $\mathbf{b}$

$$\mathbf{a} : 1 - \frac{3\lambda}{4} = 0 \quad \mathbf{b} : \frac{\lambda}{4} = \mu$$

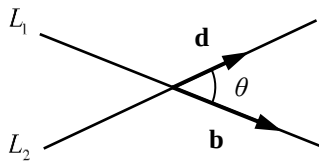
$$\Rightarrow \lambda = 43, \mu = \frac{1}{3}$$

$$\text{Hence } \overrightarrow{OR} = \frac{1}{3}\mathbf{b}.$$

■

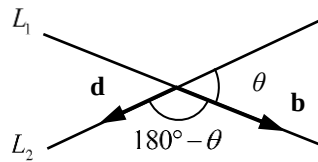
3.2 Angle between Two Lines

Given two lines L_1 and L_2 with equations $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and $\mathbf{r} = \mathbf{c} + \mu\mathbf{d}$ respectively,



$$\begin{aligned} \mathbf{b} \cdot \mathbf{d} &= |\mathbf{b}| |\mathbf{d}| \cos \theta \\ \cos \theta &= \frac{\mathbf{b} \cdot \mathbf{d}}{|\mathbf{b}| |\mathbf{d}|} \\ &= \frac{|\mathbf{b} \cdot \mathbf{d}|}{|\mathbf{b}| |\mathbf{d}|} \end{aligned}$$

because $\mathbf{b} \cdot \mathbf{d}$ is a positive number.



$$\begin{aligned} \mathbf{b} \cdot \mathbf{d} &= |\mathbf{b}| |\mathbf{d}| \cos (180^\circ - \theta) \\ &= -|\mathbf{b}| |\mathbf{d}| \cos \theta \\ \cos \theta &= \frac{-\mathbf{b} \cdot \mathbf{d}}{|\mathbf{b}| |\mathbf{d}|} \\ &= \frac{|\mathbf{b} \cdot \mathbf{d}|}{|\mathbf{b}| |\mathbf{d}|} \end{aligned}$$

because $\mathbf{b} \cdot \mathbf{d}$ is a negative number.

Let θ be the **acute** angle between the two lines, then

$$\cos \theta = \frac{|\mathbf{b} \cdot \mathbf{d}|}{|\mathbf{b}| |\mathbf{d}|}.$$

This formula can also be used to find the angle between skew lines.

Example 9

- (i) Find the cosine of the acute angle between the lines L_1 and L_2 , where L_1 passes through point $(3, -4, 3)$ and is parallel to vector

$$2\mathbf{i} - \mathbf{j} + 2\mathbf{k} \text{ and } L_2 \text{ has equation } \mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \text{ where } \mu \in \mathbb{R}.$$

- (ii) Find the acute angle between L_2 and the z-axis.

Solution

(i) $L_1 : \mathbf{r} =$

$$L_2 : \mathbf{r} =$$

Let θ be the acute angle between L_1 and L_2

$$\cos \theta =$$

- (ii) Let θ be the acute angle between L_2 and the z-axis. The z-axis is parallel to the vector $\mathbf{k}$.

■

LECTURE 4

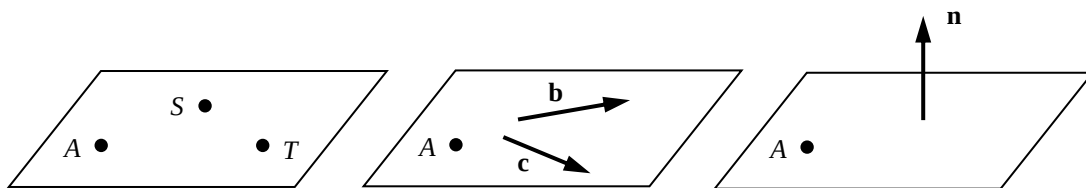
Lesson Outline

- Vector equation of a plane in parametric form & scalar product form
- Converting between parametric form and scalar product form

4 PLANES

A particular plane can be specified in a number of ways, such as the following:

- Given three non-collinear points A , S and T , exactly one plane contains all three points A , S and T .
- Given one point A and two non-parallel vectors $\mathbf{b}$ and $\mathbf{c}$, exactly one plane contains the point A and is parallel to both $\mathbf{b}$ and $\mathbf{c}$.
- Given one point A and a vector $\mathbf{n}$, exactly one plane contains the point A and is perpendicular to $\mathbf{n}$. In this case $\mathbf{n}$ is called the **normal** vector to the plane.



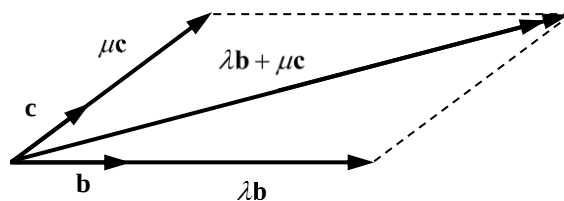
There are other ways to specify a plane but these are the more useful ones.

For example, if the two lines intersect or are parallel, then they lie on the same plane. We say that the lines are coplanar.

4.1 Vector Equation of a Plane in Parametric Form

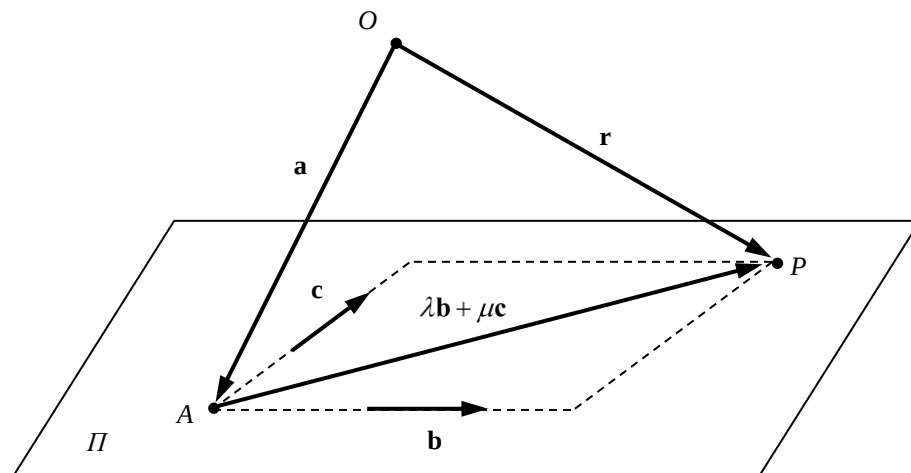
Suppose we want to find the equation of a plane Π containing point A and parallel to the two vectors $\mathbf{b}$ and $\mathbf{c}$ (where $\mathbf{b}$ is not parallel to $\mathbf{c}$).

Note that for any $\lambda, \mu \in \mathbb{R}$, the vector $\lambda\mathbf{b} + \mu\mathbf{c}$ is also parallel to Π .



Conversely, any vector coplanar with $\mathbf{b}$ and $\mathbf{c}$ can be expressed as $\lambda\mathbf{b} + \mu\mathbf{c}$ for some $\lambda, \mu \in \mathbb{R}$. (Recall *Chapter 6A Section 3.3 Linear Combinations of Vectors*)

Therefore, if a point P with position vector $\mathbf{r}$ lies on the plane Π , then $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$.



Here, $\mathbf{r}$ is the position vector of any point on the plane π .

Hence, a vector equation of the plane in **parametric form** is given by

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}, \text{ where } \lambda, \mu \in \mathbb{R}.$$

Example 10

(a) Find the vector equation of the plane in parametric form that passes

through $(0, 1, 2)$ and parallel to $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$.

(b) Find the vector equation of the plane in parametric form that passes through the points with position vectors $3\mathbf{i} - 2\mathbf{j}$, $2\mathbf{i} + 3\mathbf{k}$ and $\mathbf{i} - \mathbf{j} + \mathbf{k}$.

Solution

Note that the answers for vector equations of planes in parametric form are not unique.

(a) Equation of plane is

(b) Two vectors parallel to the plane:

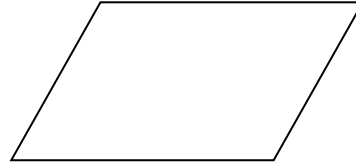
■

Example 11

Find the vector equation of the plane Π in parametric form that contains the line $l: \mathbf{r} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} + t(4\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ and passing through the point P with position vector $3\mathbf{i} - \mathbf{j} + 5\mathbf{k}$.

Solution

Two vectors parallel to the plane:



■

4.2 Vector Equation of a Plane in Scalar Product Form

Suppose we want to find an equation of a plane Π containing a given point A (with position vector $\overrightarrow{OA} = \mathbf{a}$) and perpendicular to the given vector $\mathbf{n}$.

A vector equation of Π in scalar product form is

$$\mathbf{r} \cdot \mathbf{n} = d \quad \text{where } d = \mathbf{a} \cdot \mathbf{n}$$

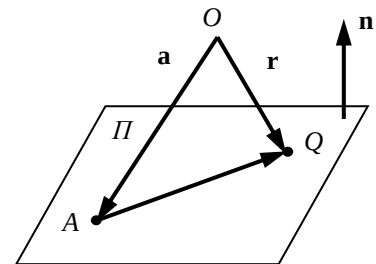
Note that $d \in \mathbb{R}$.

Proof 1

Let Q be an arbitrary point in Π with position vector $\mathbf{r}$.

Since AQ lies in plane Π , it is perpendicular to $\mathbf{n}$. Hence,

$$\begin{aligned} \overrightarrow{AQ} \cdot \mathbf{n} &= 0 \\ (\overrightarrow{OQ} - \overrightarrow{OA}) \cdot \mathbf{n} &= 0 \\ \overrightarrow{OQ} \cdot \mathbf{n} &= \overrightarrow{OA} \cdot \mathbf{n} \\ \mathbf{r} \cdot \mathbf{n} &= \mathbf{a} \cdot \mathbf{n} \end{aligned}$$



Since $\mathbf{a}$ and $\mathbf{n}$ are given, we can take $\mathbf{a} \cdot \mathbf{n}$ to be a constant scalar p . Hence we have $\mathbf{r} \cdot \mathbf{n} = d$.

Proof 2

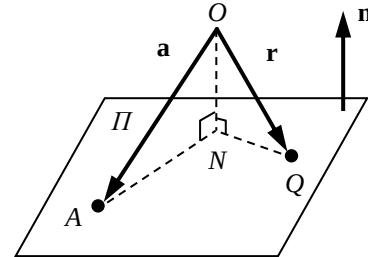
Let Q be an arbitrary point in Π with position vector $\mathbf{r}$.

Let N be the foot of the perpendicular from O to Π , so that $\overrightarrow{ON}$ is parallel to $\mathbf{n}$.

Then AN and NQ are perpendicular to ON .

Therefore the projections of $\overrightarrow{OA}$ onto $\overrightarrow{ON}$ and of $\overrightarrow{OQ}$ onto $\overrightarrow{ON}$ are equal. We have

$$\begin{aligned}\mathbf{r} \cdot \hat{\mathbf{n}} &= \mathbf{a} \cdot \hat{\mathbf{n}} \\ \mathbf{r} \cdot \left(\frac{\mathbf{n}}{|\mathbf{n}|} \right) &= \mathbf{a} \cdot \mathbf{n} \\ \mathbf{r} \cdot \mathbf{n} &= \mathbf{a} \cdot \mathbf{n}\end{aligned}$$



Since $\mathbf{a}$ and $\mathbf{n}$ are given, we can take $\mathbf{a} \cdot \mathbf{n}$ to be a constant scalar p . Hence we have $\mathbf{r} \cdot \mathbf{n} = d$.

(Actually we have only illustrated the case where $\overrightarrow{ON}$ is in the same direction as $\mathbf{n}$. Try and do the case where $\overrightarrow{ON}$ and $\mathbf{n}$ are in opposite directions on your own.)

Example 12

Find the vector equation of the plane Π in scalar product form if the plane Π is perpendicular to vector $\mathbf{i} + \mathbf{j} - \mathbf{k}$ and contains the point A with coordinates $(4, 2, 5)$.

Solution

Write down the coordinates of another point on the plane Π .

What is special about planes whose equations are of the form $\mathbf{r} \cdot \mathbf{n} = 0$?

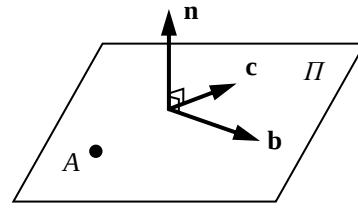
■

Conversion between parametric form and scalar product form

Suppose a vector equation of Π in parametric form is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$. To convert it to scalar product form, we need to find a vector $\mathbf{n}$ perpendicular to the plane.

One possible choice is

$$\mathbf{n} = \mathbf{b} \times \mathbf{c}.$$



Then a vector equation of Π in scalar product form is $\mathbf{r} \cdot \mathbf{n} = d$, where $d = \mathbf{a} \cdot \mathbf{n}$.

Converting vector equations of planes from scalar product form to parametric form is less common.

Suppose a vector equation of plane Π in scalar product form is $\mathbf{r} \cdot \mathbf{n} = d$.

To convert it to parametric form, we need to find a point A on the plane as well as two vectors parallel to the plane (but not parallel to each other).

Example 13

(i) Given that the vector equation of plane Π_1 in parametric form is $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + m(2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) + t(\mathbf{i} + 3\mathbf{j} - 3\mathbf{k})$, find a vector perpendicular to Π_1 and hence write down the vector equation of Π_1 in scalar product form.

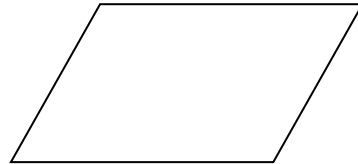
(ii) Find the vector equation of the plane Π_2 in scalar product form, given that it passes through the point $(1, 0, 1)$ and is perpendicular to

$$\text{the line } l : \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + k \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

(iii) Given that the vector equation of the plane Π_3 is $\mathbf{r} \cdot (\mathbf{i} + \mathbf{j} - \mathbf{k}) = 1$, find a vector equation of Π_3 in parametric form.

*Solution***(i)**

How do you check
that you have a
correct normal
vector?

(ii) $\mathbf{n} =$ **(iii)** $\Pi_3 : \mathbf{r}.$

By observation,
are points on Π_3 .

Two vectors parallel to the plane:

■

LECTURE 5

Lesson Outline

- Cartesian equation of a plane
 - Special planes
 - Determining if a point lies on a plane
 - Foot of the perpendicular from a point to a plane
 - Perpendicular distance from a point to a plane
-

4.3 Cartesian Equation of a Plane

Suppose the equation of a plane Π is $\mathbf{r} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = d$, where a, b, c and d are real numbers.

The Cartesian equation of the plane can be obtained by letting $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

and evaluating the scalar product, i.e. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = d$ gives

$$ax + by + cz = d$$

The form $ax + by + cz = d$ is the **Cartesian equation** of the plane.

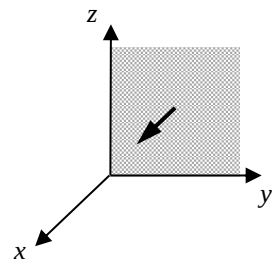
Example

The plane Π has a vector equation $\mathbf{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} = 4$. The Cartesian equation of

Π is $2x - 3y + 5z = 4$.

4.4 Special Planes

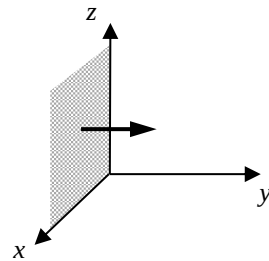
These are planes perpendicular to each of the axes and contains the origin.



y-z plane

Cartesian equation: $x = 0$

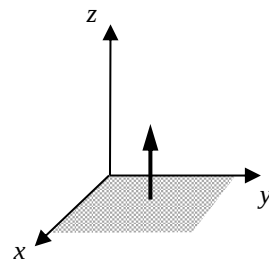
Scalar product form: $\mathbf{r} \cdot \mathbf{i} = 0$ or $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$



x-z plane

Cartesian equation: $y = 0$

Scalar product form: $\mathbf{r} \cdot \mathbf{j} = 0$ or $\mathbf{r} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$



x-y plane

Cartesian equation: $z = 0$

Scalar product form: $\mathbf{r} \cdot \mathbf{k} = 0$ or $\mathbf{r} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$

Example 14

Find the vector equations of the following planes in the form $\mathbf{r} \cdot \mathbf{n} = d$.

(i) $-x + 2y + 5 = 0$

(ii) $-2x + 3y + z - 2 + m(x - 2y + z - 1) = 0$, where m is a constant

Solution

(i) $-x + 2y + 5 = 0 \Rightarrow$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot$$

$$\Rightarrow \mathbf{r} \cdot$$

(ii) $-2x + 3y + z - 2 + m(x - 2y + z - 1) = 0$

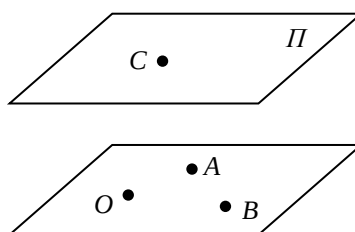
■

Example 15

Points A , B and C have position vectors, relative to the origin O , given by $\overrightarrow{OA} = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\overrightarrow{OB} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ and $\overrightarrow{OC} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Find a Cartesian equation of the plane through C parallel to the plane OAB .

Solution

$$\overrightarrow{OA} \times \overrightarrow{OB} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$$



$\therefore$ The Cartesian equation of plane through C parallel to plane OAB is $x + 2y - z = 2$. ■

The different forms of the vector equation of the plane above are:

Parametric Form	$\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}, \text{ where } \lambda, \mu \in \mathbb{R}$
Scalar Product Form	$\mathbf{r} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 2$
Cartesian Equation	$x + 2y - z = 2$

In general, the different forms of the vector equation of a plane are as follows:

Parametric Form	$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$ where $\lambda, \mu \in \mathbb{R}$
Scalar Product Form	$\mathbf{r} \cdot \mathbf{n} = d$ where $\mathbf{n} = \mathbf{b} \times \mathbf{c}$ and $d = \mathbf{a} \cdot \mathbf{n}$
Cartesian Equation	$ax + by + cz = d$ where $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Extra Practice

Annex A: Practice Question 4, 5, 6

5 RELATIONSHIPS BETWEEN POINT AND PLANE

5.1 Point on a Plane

Substituting the coordinates or position vector of a point into the equation of a plane allows us to determine if the point lies on the plane.

Example 16

Determine if the point $(3, -1, 0)$ lies on the following planes.

(i) $\mathbf{r} \cdot (3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}) = 13$

(ii) $3x - 4y + 5z = 13$

(iii) $\mathbf{r} = 4\mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) + \mu(-2\mathbf{i} + \mathbf{j} + 2\mathbf{k}), \lambda, \mu \in \mathbb{R}$

Solution

(i) Substitute $\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$ into $\mathbf{r}$,

LHS =

Therefore the point $(3, -1, 0)$ on the plane

$\mathbf{r} \cdot (3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}) = 13$.

(ii) Substitute $x = 3, y = -1, z = 0$

LHS =

Therefore the point $(3, -1, 0)$ on the plane
 $3x - 4y + 5z = 13$.

(iii) Substitute $\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$ into $\mathbf{r}$,

Using GC, $\lambda = -1, \mu = 0$.

Since there are unique values for λ and μ , the point satisfies the equation of the plane. Therefore the point $(3, -1, 0)$ on the plane $\mathbf{r} = 4\mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) + \mu(-2\mathbf{i} + \mathbf{j} + 2\mathbf{k}), \lambda, \mu \in \mathbb{R}$. ■

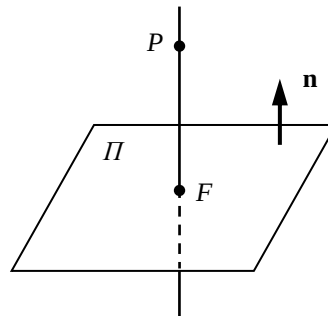
Extra Practice

Annex A: Practice Question 7

5.2 Foot of Perpendicular from a Point to a Plane

Suppose we have a plane Π with equation $\mathbf{r} \cdot \mathbf{n} = d$, and a point P not on the plane. Let F be the foot of perpendicular from P to Π .

We wish to find the coordinates of F as well as the length of PF , the perpendicular distance from P to Π .

Method 1: Intersection of plane and perpendicular line

Step 1: Find equation of line through P perpendicular to Π , i.e.

$$\mathbf{r} = \overrightarrow{OP} + \lambda \mathbf{n}$$

Step 2: Substitute into equation of plane and solve for λ , i.e.

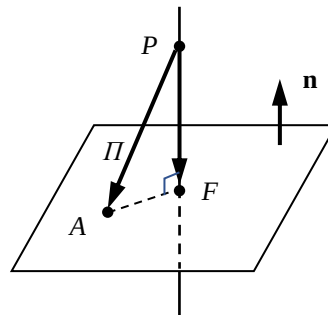
$$(\overrightarrow{OP} + \lambda \mathbf{n}) \cdot \mathbf{n} = d$$

Step 3: Substitute λ into equation of line to find $\overrightarrow{OF}$.

Note that $\overrightarrow{PF} = \lambda \mathbf{n}$.
Therefore the
perpendicular
distance from P to
the plane is
 $|\overrightarrow{PF}| = |\lambda| |\mathbf{n}|$.

Method 2: Projection

(Recall Chapter 6A Section 5.4 Projection of a Vector in a Given Direction)



Note the
perpendicular
distance from P to
the plane is
 $|\overrightarrow{PF}| = |\overrightarrow{PA} \cdot \hat{\mathbf{n}}|$.

Step 1: Find/choose a point A in Π .

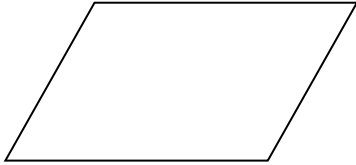
Step 2: $\overrightarrow{PF}$ is the vector component of $\overrightarrow{PA}$ parallel to $\mathbf{n}$,

$$\overrightarrow{PF} = (\overrightarrow{PA} \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}}$$

Step 3: Find $\overrightarrow{OF}$ using $\overrightarrow{OF} = \overrightarrow{OP} + \overrightarrow{PF}$.

Example 17

The plane Π has the equation $\mathbf{r} \cdot (\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = 1$. The point B has coordinates $(7, 0, 3)$ and C is the foot of the perpendicular from B to Π . Find the coordinates of C and the shortest distance from B to Π .

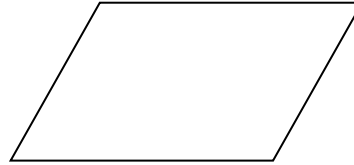
*Solution**Method 1*

Since point C passes through the line which passes through B and is perpendicular to Π ,

$$\overrightarrow{OC} =$$

Since point C is on the plane Π ,

Coordinates of C are $(5, -2, -1)$.

Method 2

The point $(1, 0, 0)$ lies on Π (since

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 1). \text{ Let this point be } A.$$

Using the formula,

$$\overrightarrow{BC} = (\overrightarrow{BA} \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}}$$

$$\begin{aligned} \overrightarrow{BC} &= \left(\begin{pmatrix} -6 \\ 0 \\ -3 \end{pmatrix} \cdot \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right) \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \\ &= \frac{-6-6}{6} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \end{aligned}$$

$$\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC}$$

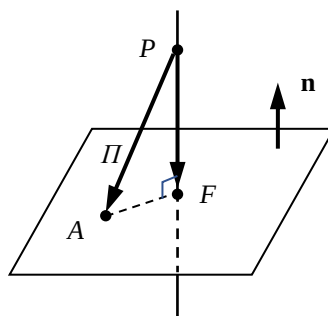
$$= \begin{pmatrix} 7 \\ 0 \\ 3 \end{pmatrix} + (-2) \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix}$$

Coordinates of C are $(5, -2, -1)$.

Shortest distance from B to plane Π

$$= |\overrightarrow{BC}| = \left| \begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix} - \begin{pmatrix} 7 \\ 0 \\ 3 \end{pmatrix} \right| = \left| \begin{pmatrix} -2 \\ -2 \\ -4 \end{pmatrix} \right| = 2 \left| \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right| = 2\sqrt{6}.$$

5.3 Perpendicular Distance from a Point to a Plane



Method 1: Projection

If the question only asks for the perpendicular distance from P to plane Π , then we use the fact that perpendicular distance from P to plane Π is the length of projection of $\overrightarrow{PA}$ on $\mathbf{n}$, i.e.

$$\text{Perpendicular distance from } P \text{ to } \Pi = \left| \overrightarrow{PA} \cdot \hat{\mathbf{n}} \right|$$

Method 2: Finding foot of perpendicular

If the question first asks for position vector of the foot of perpendicular from point P to plane Π , then we can use this position vector to deduce the perpendicular distance PF :

Step 1: Find $\overrightarrow{OF}$, where F is the foot of perpendicular from point P to plane Π . (Recall *Chapter 6B Section 5.2 Foot of Perpendicular from a Point to a Plane*)

Step 2: Find $\overrightarrow{PF}$ and thus perpendicular distance $PF = \left| \overrightarrow{PF} \right|$.

Example 18

The plane Π has the equation $2x + 2y - z = 4$. The point P has coordinates $(6, 1, 3)$. Find the shortest distance from P to Π .

Solution

■

LECTURE 6

Lesson Outline

- Distance of the origin O from a plane
 - Reflection of a point in a plane
 - Possible configurations of a line and a plane
 - Intersection of a line and a plane
 - Angle between a line and a plane
-

5.4 Distance of the Origin O from a Plane

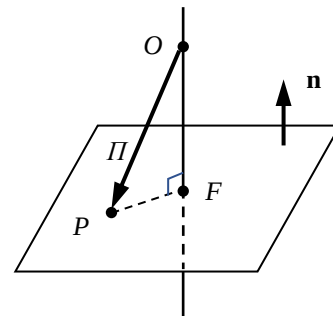
Given a plane Π with equation $\mathbf{r} \cdot \mathbf{n} = d$, i.e. $\mathbf{r} \cdot \hat{\mathbf{n}} = \frac{d}{|\mathbf{n}|}$,

Distance of the origin O from $\Pi = \frac{|d|}{|\mathbf{n}|}$.

Proof

Let P be any point on the plane Π . Then $\overrightarrow{OP} \cdot \mathbf{n} = d$.

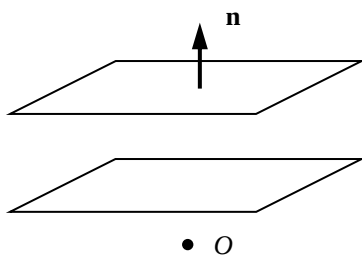
$$\text{Distance from origin } O \text{ to } \Pi = |\overrightarrow{OP} \cdot \hat{\mathbf{n}}| = \frac{|\overrightarrow{OP} \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{|d|}{|\mathbf{n}|}.$$



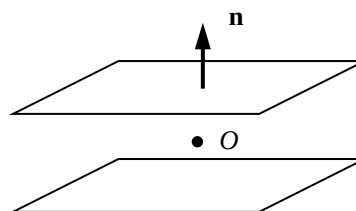
Note

Given two planes $\Pi_1 : \mathbf{r} \cdot \mathbf{n} = d_1$ and $\Pi_2 : \mathbf{r} \cdot \mathbf{n} = d_2$ with the **same** normal vector $\mathbf{n}$,

- if d_1 and d_2 have the **same sign**, then the planes are located on the **same side** of the origin,
- if d_1 and d_2 have **opposite signs**, then the planes are located on **opposite sides** of the origin, i.e. origin is located between the planes.



(a)



(b)

Example 19

- (a) Find the distance from the origin to the plane p with equation $2x - y + z = 6$. Write down the equation of another plane parallel to p that is the same distance from the origin.
- (b) Find the cartesian equations of the planes that are perpendicular to the vector $2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and 4 units from the origin.

*Solution***(a)****(b)**

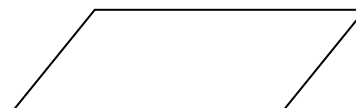
■

5.5 Reflection of a Point in a Plane**Example 20**

Using Example 17, find the position vector of B' , the image of B when reflected in the plane Π .

Solution

$$\therefore \overrightarrow{OB'} = 2 \begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix} - \begin{pmatrix} 7 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ -5 \end{pmatrix}$$



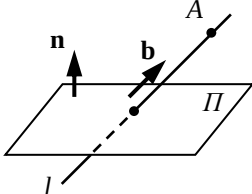
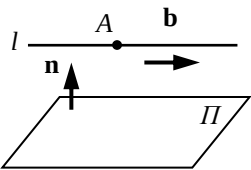
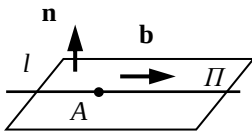
Alternatively, C is the midpoint of B and B' .

By ratio theorem, we have $\overrightarrow{OC} = \frac{\overrightarrow{OB} + \overrightarrow{OB'}}{2} \Rightarrow \overrightarrow{OB'} = 2\overrightarrow{OC} - \overrightarrow{OB}$. ■

6 RELATIONSHIPS BETWEEN LINE AND PLANE

6.1 Possible Configurations of a Line and a Plane

Given a line $l : \mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$ and a plane $\Pi : \mathbf{r} \cdot \mathbf{n} = d$, there are three possible configurations of l and Π with respect to each other.

l intersects Π (at a point)	l is parallel to Π	l is contained in Π
 $\mathbf{b} \cdot \mathbf{n} \neq 0$ We may be asked to find the coordinates of the point of intersection, and the angle between the line and the plane.	 $\mathbf{b} \cdot \mathbf{n} = 0$ and $\mathbf{a} \cdot \mathbf{n} \neq d$ We may be asked to find the distance between the line and the plane.	 $\mathbf{b} \cdot \mathbf{n} = 0$ and $\mathbf{a} \cdot \mathbf{n} = d$

Example 21

The line l is defined by $l : \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$ where $t \in \mathbb{R}$. The planes Π_1

and Π_2 are defined by $\Pi_1 : \mathbf{r} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 1$ and $\Pi_2 : \mathbf{r} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = c$.

- (a) Show that l is parallel to Π_1 .
 (b) For which real value of c does the l lie in Π_2 ?

Solution

(a)

(b) From above, for l to lie in Π_2 ,

$$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = c \Rightarrow c = -3. \quad \blacksquare$$

6.2 Intersection of a Line and a Plane

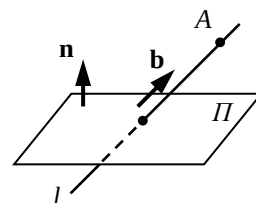
When does a line intersect a plane at one point?

Given a line $l: \mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$ and a plane $\Pi: \mathbf{r} \cdot \mathbf{n} = d$, to find the point of intersection,

Step 1: Substitute $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$ into $\mathbf{r} \cdot \mathbf{n} = d$ to solve for λ , i.e.

$$(\mathbf{a} + \lambda \mathbf{b}) \cdot \mathbf{n} = d$$

Step 2: Substitute λ into l to find the point of intersection.



Example 22

The position vectors of the points A and B are given by $\overrightarrow{OA} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\overrightarrow{OB} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$. Find a vector equation of the line passing through A and B . Find the position vector of the point where this line meets the plane $z = 0$.

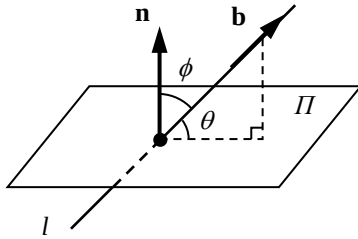
Solution

Extra Practice

Annex A: Practice Question 8

6.3 Angle between a Line and a Plane

Suppose a line $l: \mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$ and a plane $\Pi: \mathbf{r} \cdot \mathbf{n} = d$ intersect at a point. We wish to find θ , the **acute angle** between line l and plane Π .

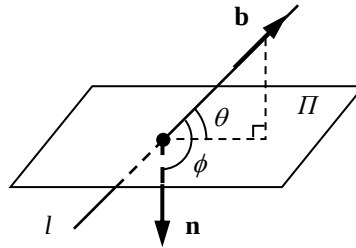


$$\cos \phi = \frac{\mathbf{n} \cdot \mathbf{b}}{|\mathbf{n}| |\mathbf{b}|}$$

Since $\phi = 90^\circ - \theta$, we have $\cos \phi = \sin \theta$. Therefore,

$$\sin \theta = \frac{\mathbf{n} \cdot \mathbf{b}}{|\mathbf{n}| |\mathbf{b}|} = \hat{\mathbf{n}} \cdot \hat{\mathbf{b}} = |\hat{\mathbf{n}} \cdot \hat{\mathbf{b}}| \text{ since}$$

$\hat{\mathbf{n}} \cdot \hat{\mathbf{b}}$ is positive (ϕ is acute).



$$\cos \phi = \frac{\mathbf{n} \cdot \mathbf{b}}{|\mathbf{n}| |\mathbf{b}|}$$

Since $\phi = 90^\circ + \theta$, we have $\cos \phi = -\sin \theta$. Therefore,

$$\sin \theta = -\frac{\mathbf{n} \cdot \mathbf{b}}{|\mathbf{n}| |\mathbf{b}|} = -\hat{\mathbf{n}} \cdot \hat{\mathbf{b}} = |\hat{\mathbf{n}} \cdot \hat{\mathbf{b}}| \text{ since}$$

$\hat{\mathbf{n}} \cdot \hat{\mathbf{b}}$ is negative (ϕ is obtuse).

Therefore, if θ is the acute angle between a line $l: \mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$ and a plane $\Pi: \mathbf{r} \cdot \mathbf{n} = d$, then

$$\sin \theta = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}| |\mathbf{b}|} = |\hat{\mathbf{n}} \cdot \hat{\mathbf{b}}|.$$

Example 23

Referred to the origin O , the points A and B have position vectors given respectively by $\overrightarrow{OA} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\overrightarrow{OB} = 4\mathbf{j} + 7\mathbf{k}$. The plane p has equation $2x + 4y + 3z = 1$.

- (i) Show that the point A lies in the plane p .
- (ii) Find the acute angle between the line AB and plane p .
- (iii) Find a vector equation of the plane in parametric form which contains the line AB and is perpendicular to the plane p .

Solution

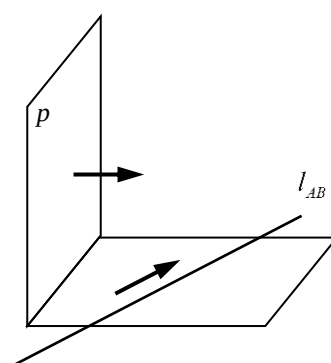
- (i) Substitute $A(3, -2, 1)$ into $2x + 4y + 3z = 1$.

$$\text{LHS} = 2(3) + 4(-2) + 3(1) = 6 - 8 + 3 = 1$$

Since it satisfies the equation, A is on p .

(ii)

(iii)



■

LECTURE 7

Lesson Outline

- Reflection of a line in a plane
- Two parallel planes
- Intersection of two planes
- Angle between two planes

6.4 Reflection of a Line in a Plane

Example 24

Using Example 22, find the vector equation of the reflection of the line AB in the plane $z = 0$.

Solution

From Example 22, $l_{AB} : \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and position vector of point of

intersection with $z = 0$ is $\frac{1}{3} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$.

Let F be the foot of perpendicular of A to the plane $z = 0$.

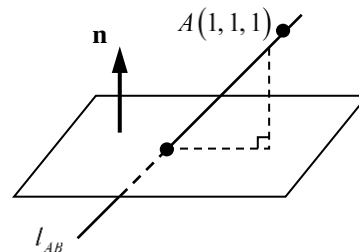
$$l_{AF} : \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } z = 0 \Rightarrow \mathbf{r} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \Rightarrow 1 + \lambda = 0 \Rightarrow \lambda = -1$$

$$\therefore \overrightarrow{OF} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (-1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\overrightarrow{AF} = \overrightarrow{FA'} \Rightarrow \overrightarrow{OF} - \overrightarrow{OA} = \overrightarrow{OA'} - \overrightarrow{OF}$$

$$\overrightarrow{OA'} = 2\overrightarrow{OF} - \overrightarrow{OA} = 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$



Alternatively, since $z = 0$ is the x - y plane, the reflection of A in the plane $z = 0$ is $(1, 1, -1)$.

Hence direction of the line of reflection is

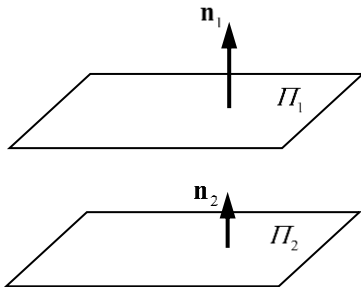
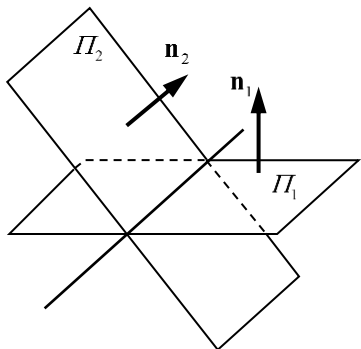
Alternatively, since $z=0$ is the x - y plane, we can also see that the direction of the line of reflection is $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

Hence, equation of reflected line is

■

7 RELATIONSHIPS BETWEEN TWO PLANES

Given two distinct planes $\Pi_1 : \mathbf{r} \cdot \mathbf{n}_1 = d_1$ and $\Pi_2 : \mathbf{r} \cdot \mathbf{n}_2 = d_2$, there are two possible configurations of Π_1 and Π_2 with respect to each other.

Π_1 and Π_2 are parallel	Π_1 and Π_2 intersect (at a line)
 $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel We may be asked to find the distance between the planes.	 $\mathbf{n}_1$ and $\mathbf{n}_2$ are not parallel We may be asked to find the equation of the line of intersection, and the angle between the two planes.

7.1 Two Parallel Planes

Example 25

The plane π_1 has equation $-7x - 2y + 3z = 4$. The plane π_2 has equation $7x + 2y - 3z = 7$. Find the distance between π_1 and π_2 .

*Solution**Method 1: Distance from point to plane*

(Recall Chapter 6B Section 5.3 Perpendicular Distance from a Point to a Plane)

Planes π_1 and π_2 are parallel since their normal

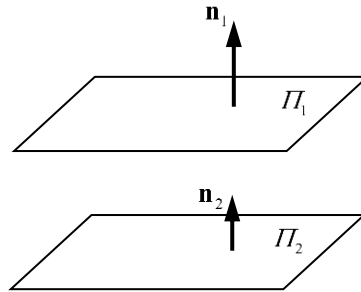
vectors are parallel, $\begin{pmatrix} -7 \\ -2 \\ 3 \end{pmatrix} = -\begin{pmatrix} 7 \\ 2 \\ -3 \end{pmatrix}$.

$A(0, -2, 0)$ is a point on π_1 since
 $-7(0) - 2(-2) + 3(0) = 4$.

$B(1, 0, 0)$ is a point on π_2 since
 $7(1) + 2(0) - 3(0) = 7$.

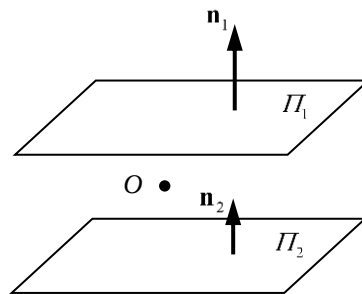
$$\overrightarrow{BA} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix}$$

Hence distance between the two planes is the distance from A to π_2 ,

*Method 2: Using distance from origin to plane*

(Recall Chapter 6B Section 5.4 Distance of the Origin O from a Plane)

$$\pi_1 : \mathbf{r} \cdot \begin{pmatrix} -7 \\ -2 \\ 3 \end{pmatrix} = 4$$



Divide throughout by $\left| \begin{pmatrix} -7 \\ -2 \\ 3 \end{pmatrix} \right| = \sqrt{62}$, we get $\mathbf{r} \cdot \frac{1}{\sqrt{62}} \begin{pmatrix} -7 \\ -2 \\ 3 \end{pmatrix} = \frac{4}{\sqrt{62}}$.

$$\pi_2 : \mathbf{r} \cdot \begin{pmatrix} 7 \\ 2 \\ -3 \end{pmatrix} = 7 \Leftrightarrow \mathbf{r} \cdot \begin{pmatrix} -7 \\ -2 \\ 3 \end{pmatrix} = -7$$

Divide throughout by $\left| \begin{pmatrix} -7 \\ -2 \\ 3 \end{pmatrix} \right| = \sqrt{62}$, we get $\mathbf{r} \cdot \frac{1}{\sqrt{62}} \begin{pmatrix} -7 \\ -2 \\ 3 \end{pmatrix} = -\frac{7}{\sqrt{62}}$

Hence the two planes are parallel planes on opposite sides of the origin, $\frac{4}{\sqrt{62}}$ and $\frac{7}{\sqrt{62}}$ units away from the origin respectively.

$\therefore$ Distance between planes = . ■

In general, given two parallel planes $\Pi_1 : \mathbf{r} \cdot \mathbf{n}_1 = d_1$ and $\Pi_2 : \mathbf{r} \cdot \mathbf{n}_2 = d_2$, with the **same** normal vector $\mathbf{n}$,

$$\text{Distance between } \Pi_1 \text{ and } \Pi_2 = \frac{|d_1 - d_2|}{|\mathbf{n}|}.$$

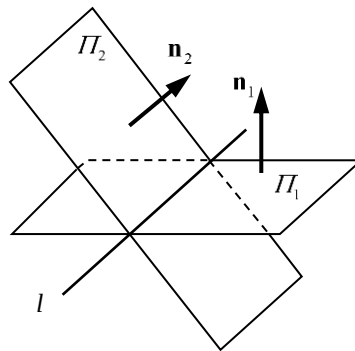
Extra Practice

Annex A: Practice Question 9

7.2 Intersection of Two Planes

The intersection of two non-parallel planes is a line.

Consider two planes $\Pi_1 : \mathbf{r} \cdot \mathbf{n}_1 = d_1$ and $\Pi_2 : \mathbf{r} \cdot \mathbf{n}_2 = d_2$.



Since the line of intersection l lies in both planes, $l \perp \mathbf{n}_1$ and $l \perp \mathbf{n}_2$, hence

$$l // \mathbf{n}_1 \times \mathbf{n}_2,$$

i.e., the line of intersection l of the planes is parallel to $\mathbf{n}_1 \times \mathbf{n}_2$.

There are three methods of finding the equation of the line of intersection of two planes.

Method 1: Solve simultaneous equation using GC

- Step 1: Express equations of both planes in Cartesian form.
- Step 2: Use GC to solve the simultaneous equations (2 equations in 3 unknowns)
- Step 3: Express two of the variables in terms of the third, e.g. x and y respectively in terms of z .

If the equations of two planes are in any other form, then convert the equations to cartesian form and solve the system of equations.

Method 2: Find point and direction of line

Given the equations of two planes in scalar product form

$$\Pi_1 : \mathbf{r} \cdot \mathbf{n}_1 = d_1 \text{ and } \Pi_2 : \mathbf{r} \cdot \mathbf{n}_2 = d_2,$$

then the equation of line of intersection, l , can be found by finding the direction vector and a point on the line.

- Step 1: Since $l \parallel \mathbf{n}_1 \times \mathbf{n}_2$, find

$$\mathbf{n}_1 \times \mathbf{n}_2.$$

- Step 2: For a point (x, y, z) on l , set one of the coordinates to 0, e.g. $z = 0$, and solve the Cartesian equations of Π_1 and Π_2 involving x and y only.

Why can you set one of the coordinates to 0?

This method is usually used only if there are unknowns in the equations of the planes.

Method 3: Solving parametric and scalar product form

If the equations of two planes are in parametric and scalar product form respectively, the line of intersection can be solved by substituting the parametric form into the scalar product form, e.g.

$$\Pi_1 : \mathbf{r} = \mathbf{a} + \lambda \mathbf{b} + \mu \mathbf{c} \text{ and } \Pi_2 : \mathbf{r} \cdot \mathbf{n} = d$$

- Step 1: Substitute $\Pi_1 : \mathbf{r} = \mathbf{a} + \lambda \mathbf{b} + \mu \mathbf{c}$ into $\Pi_2 : \mathbf{r} \cdot \mathbf{n} = d$ i.e.

$$(\mathbf{a} + \lambda \mathbf{b} + \mu \mathbf{c}) \cdot \mathbf{n} = d$$

to obtain λ in terms of μ .

- Step 2: Substitute λ in terms of μ into the parametric equation and re-express $\mathbf{r}$ in terms of μ only.

Note

- As the equation of a line is not unique, you may obtain a different expression from others depending on the method chosen or the choices you made in the algebraic working.

Example 26

- (a) The planes p and q have equations $3x - y - z = 2$ and $x + 5y + z = 14$ respectively. Find the equation of the line of intersection of p and q .

- (b) The equations of two planes are

$$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix}, \lambda, \mu \in \mathbb{R} \text{ and } \mathbf{r} \cdot \begin{pmatrix} -7 \\ -2 \\ 3 \end{pmatrix} = -4.$$

Without the use of the graphing calculator, find the equation of the line of intersection of the two planes.

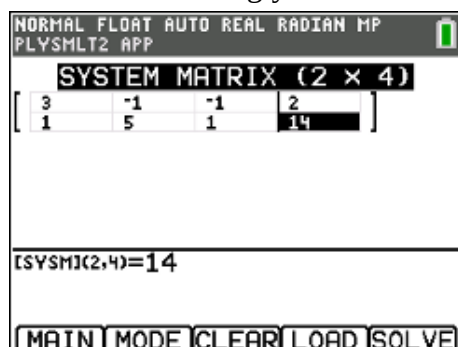
Solution

- (a) $p: 3x - y - z = 2$ and $q: x + 5y + z = 14$

Method 1

- 1) Press **[apps]** **[4:PlySmlt2]** **[2:SIMULTANEOUS EQN SOLVER]**. Select 2 for **EQUATIONS** and 3 for **UNKNOWN**s. Select **[NEXT]** by pressing **[graph]**.

- 2) Enter the coefficients and constants accordingly.



- 3) Select **[SOLVE]** by pressing **[graph]**.



Method 2

$$p: \mathbf{r} \cdot \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = 2 \quad q: \mathbf{r} \cdot \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} = 14$$

Let the line of intersection of the two planes be l .

A vector parallel to l is

To find a point (x, y, z) , let $z = 0$ in the Cartesian equations of the two planes to get

You may also let z be any other value and find the corresponding x and y values.

Solving we get,

$$x = \frac{3}{2}, y = \frac{5}{2}.$$

Thus $\left(\frac{3}{2}, \frac{5}{2}, 0\right)$ lies on l .

Hence equation of l is

Solving using the GC, we get

$$\mathbf{r} = \begin{pmatrix} \frac{3}{2} \\ \frac{5}{2} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}, \lambda \in \mathbb{R}.$$

If you substituted $y=0$ instead, the equation of l would become

$$\mathbf{r} = \begin{pmatrix} 4 \\ 0 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}.$$

How do you check if this is the same line as that obtained in the solution?

$\therefore$ Equation of line of intersection is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{5}{2} \\ 0 \end{pmatrix} + \frac{1}{4}z \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$$

Let $\lambda = \frac{1}{4}z$, we get

$$\mathbf{r} = \begin{pmatrix} \frac{3}{2} \\ \frac{5}{2} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}, \lambda \in \mathbb{R}.$$

(b) Method 3

Let the two planes be p and q ,

$$p: \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2-2\mu \\ 1+4\mu \\ -1+5\lambda+3\mu \end{pmatrix}$$

$$q: \mathbf{r} \cdot \begin{pmatrix} -7 \\ -2 \\ 3 \end{pmatrix} = -4$$

For line of intersection of p and q ,

$$\begin{pmatrix} 2-2\mu \\ 1+4\mu \\ -1+5\lambda+3\mu \end{pmatrix} \cdot \begin{pmatrix} -7 \\ -2 \\ 3 \end{pmatrix} = -4$$

$$\Rightarrow (2-2\mu)(-7) + (1+4\mu)(-2) + (-1+5\lambda+3\mu)(3) = -4$$

$$\Rightarrow -19 + 15\lambda + 15\mu = -4$$

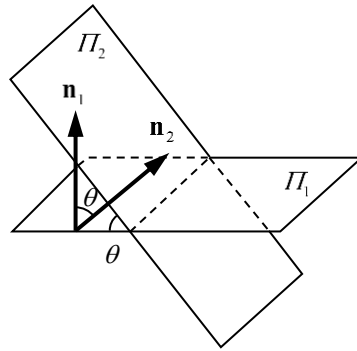
$$\Rightarrow \lambda = 1 - \mu$$

$\therefore$ Equation of the line of intersection is

$$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + (1-\mu) \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix}, \text{ i.e. } \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix}, \mu \in \mathbb{R}. \quad \blacksquare$$

7.3 Angle between Two Planes

Suppose $\Pi_1 : \mathbf{r} \cdot \mathbf{n}_1 = d_1$ and $\Pi_2 : \mathbf{r} \cdot \mathbf{n}_2 = d_2$ are two planes that intersect. We wish to find θ , the acute angle between the two planes.

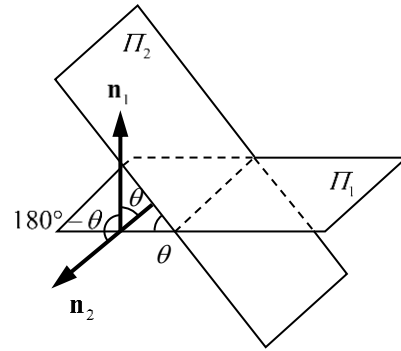


Angle between $\mathbf{n}_1$ and $\mathbf{n}_2$ is θ .

Hence

$$\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2 = |\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2|$$

since $\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2$ is positive (θ is acute).



Angle between $\mathbf{n}_1$ and $\mathbf{n}_2$ is $180^\circ - \theta$.

$$\cos(180^\circ - \theta) = -\cos \theta$$

Hence

$$\cos \theta = -\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = -\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2 = |\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2|$$

since $\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2$ is negative ($180^\circ - \theta$ is obtuse).

Therefore the acute angle, θ , between two intersecting planes $\Pi_1 : \mathbf{r} \cdot \mathbf{n}_1 = d_1$ and $\Pi_2 : \mathbf{r} \cdot \mathbf{n}_2 = d_2$ is given by

$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|} = |\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2|$$

Example 27

Planes π_1 and π_2 have equations $x - 2y + 3z = 0$ and $3x + y + 2z = 0$ respectively. Show that the acute angle between π_1 and π_2 is 60° .

Solution

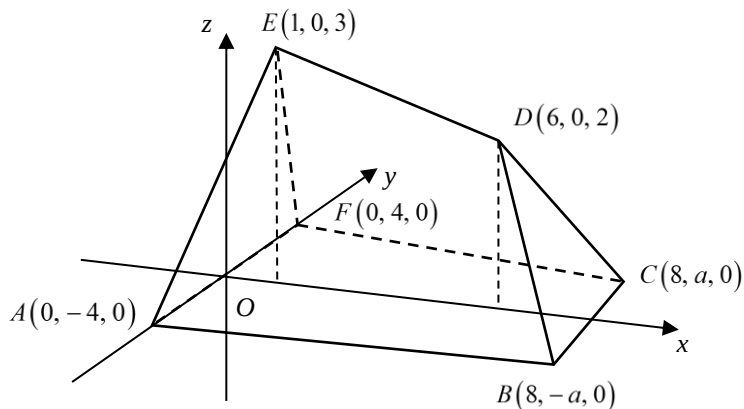
■

Extra Practice

Annex A: Practice Question 10

Example 28

A tent has vertices $ABCDEF$ with coordinates as shown. Lengths are in metres. The xy -plane is horizontal.



- (i) Find the length of the ridge of the tent DE , and the angle this makes with the horizontal.
- (ii) Find the equation of the plane ADE in scalar product form. Given that B also lies in this plane, find a .
- (iii) Find the acute angle between the planes $ABDE$ and BCD .

Solution

$$(i) \quad \overrightarrow{DE} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 6 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} \Rightarrow |\overrightarrow{DE}| = \sqrt{(-5)^2 + 1^2} = \sqrt{26}$$

Let the angle DE makes with the horizontal be θ .

$$\sin \theta = \frac{\left| \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|}{\sqrt{(-5)^2 + 1^2} \sqrt{1^2}} = \frac{1}{\sqrt{26}} \Rightarrow \theta = 11.3^\circ$$

$$(ii) \quad \overrightarrow{AD} = \begin{pmatrix} 6 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}, \quad \overrightarrow{AE} = \begin{pmatrix} 6 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix}$$

$$\text{Normal} = \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix} \times \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ 16 \\ -20 \end{pmatrix} = -4 \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix}$$

Equation of plane ADE :

$$\mathbf{r} \cdot \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix} = \overrightarrow{OA} \cdot \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix} = 16. \text{ Hence } \mathbf{r} \cdot \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix} = 16.$$

$$\text{Since } B \text{ lies on plane } ADE, \begin{pmatrix} 8 \\ -a \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix} = 16 \Rightarrow 8 + 4a = 16 \Rightarrow a = 2$$

$$(iii) \quad \overrightarrow{BC} = \begin{pmatrix} 8 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 8 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}, \quad \overrightarrow{BD} = \begin{pmatrix} 6 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 8 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}.$$

$$\text{Normal} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} \times \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 8 \end{pmatrix} = 8 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Angle between planes

$$= \cos^{-1} \frac{\left| \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix} \right|}{\sqrt{1^2 + 1^2} \sqrt{1^2 + (-4)^2 + 5^2}} = \cos^{-1} \frac{6}{\sqrt{2} \sqrt{42}} = 49.1^\circ. \quad \blacksquare$$

ANNEX A

Practice Questions on Lines and Planes

1 Find the vector equations of the lines joining the following pairs of points.

(a) $(2, -1, 2), (3, -1, 4)$

(b) $(1, 2, 2), (2, -2, 2)$

(c) $(3, 1, 4), (-1, 2, 3)$

2 Which of these equations represent the same straight line?

(a) $\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$

(b) $\mathbf{r} = \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$

(c) $\mathbf{r} = 5\mathbf{i} + 2\mathbf{j} + 6\mathbf{k} + \lambda(-2\mathbf{i} + \mathbf{j} - 2\mathbf{k}), \lambda \in \mathbb{R}$

(d) $\mathbf{r} = \mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + \lambda(-2\mathbf{i} + \mathbf{j} - 2\mathbf{k}), \lambda \in \mathbb{R}$

(e) $\mathbf{r} = -\mathbf{i} + 5\mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j} + 2\mathbf{k}), \lambda \in \mathbb{R}$

(f) $\mathbf{r} = -\mathbf{i} + 5\mathbf{j} + \lambda(-2\mathbf{i} + \mathbf{j} - 2\mathbf{k}), \lambda \in \mathbb{R}$

3 Determine if the point $(-3, 1, 5)$ lies on each of the following lines.

(a) $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$

(b) $\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \lambda \in \mathbb{R}$

(c) $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ -3 \\ -1 \end{pmatrix}, \lambda \in \mathbb{R}$

4 Find the equation of the following planes in (i) parametric, (ii) scalar product and (iii) cartesian forms.

(a) Plane Π_1 passes through the points $(1, 2, 3), (2, -1, 2)$ and $(3, 1, -1)$.

(b) Plane Π_2 is parallel to both $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $3\mathbf{i} - \mathbf{j} + \mathbf{k}$ and passes through the point $(2, 0, -3)$.

5 Find the equation of the following planes in (i) scalar product and (ii) cartesian form.

(a) Plane Π_3 passes through $(2, -1, 1)$ and is normal to the vector $2\mathbf{i} - \mathbf{j} - \mathbf{k}$.

(b) Plane Π_4 passes through $(1, 2, -1)$ and is parallel to $5x + y + 7z = 20$.

6 Find a vector equation of the line passing through $(4, 2, -1)$ and perpendicular to the plane $3x + 4y - z = 1$.

- 7 Determine if the point $(-3, 1, 5)$ lies on each of the following planes.

(a) $\mathbf{r} \cdot \begin{pmatrix} 4 \\ -3 \\ 3 \end{pmatrix} = 0$

(b) $3x - 4y + 5z = 13$

(c) $\mathbf{r} = 4\mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) + \mu(-2\mathbf{i} + \mathbf{j} + 2\mathbf{k}), \lambda, \mu \in \mathbb{R}$

- 8 Given the following planes and lines, determine whether the line is parallel to the plane, intersects the plane or is contained in the plane. Find the coordinates of the point of intersection if the line intersects the plane.

(a) $5x - 4y - 7z = 49$ and $\frac{x-3}{2} = \frac{y+1}{5} = \frac{z-2}{-3}$

(b) $\mathbf{r} \cdot (\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) = 4$ and $\mathbf{r} = (2\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}) + \lambda(\mathbf{i} - \mathbf{j} - \mathbf{k})$

(c) $\mathbf{r} \cdot (3\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) = 3$ and $\mathbf{r} = (\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}) + \lambda(\mathbf{i} - 3\mathbf{j} - 3\mathbf{k})$

- 9 Find the exact distance between these pairs of parallel objects.

(a) The lines $\mathbf{r} = (\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) + \lambda(2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = (3\mathbf{i} + 4\mathbf{k}) + \mu(2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

(b) The line $\mathbf{r} = (\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}) + \lambda(\mathbf{i} - 3\mathbf{j} - 3\mathbf{k})$ and the plane $\mathbf{r} \cdot (3\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) = 3$

(c) The planes $\mathbf{r} \cdot (3\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) = 3$ and $\mathbf{r} \cdot (3\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) = 37$

- 10 Find the acute angle between the following pairs of objects.

(a) The lines $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix}$

(b) The line $\mathbf{r} = \begin{pmatrix} 1 \\ 7 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$ and the plane $\mathbf{r} \cdot \begin{pmatrix} -3 \\ 11 \\ 8 \end{pmatrix} = 4$

(c) The planes $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1$ and $\mathbf{r} \cdot \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} = 6$

Answers

$$1 \quad \text{(a)} \mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R} \quad \text{(b)} \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 0 \end{pmatrix}, \lambda \in \mathbb{R} \quad \text{(c)} \mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 1 \\ -1 \end{pmatrix}, \lambda \in \mathbb{R}$$

2 They all represent the same straight line.

3 (a) Yes (b) No (c) No

$$4 \quad \text{(a)(i)} \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}, \lambda, \mu \in \mathbb{R} \quad \text{(ii)} \mathbf{r} \cdot \begin{pmatrix} 11 \\ 2 \\ 5 \end{pmatrix} = 30 \quad \text{(iii)} 11x + 2y + 5z = 30$$

$$\text{(b)(i)} \mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}, \lambda, \mu \in \mathbb{R} \quad \text{(ii)} \mathbf{r} \cdot \begin{pmatrix} 1 \\ -4 \\ -7 \end{pmatrix} = 23 \quad \text{(iii)} x - 4y - 7z = 23$$

$$5 \quad \text{(a)(i)} \mathbf{r} \cdot \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = 4 \quad \text{(ii)} 2x - y - z = 4 \quad \text{(b)(i)} \mathbf{r} \cdot \begin{pmatrix} 5 \\ 1 \\ 7 \end{pmatrix} = 0 \quad \text{(ii)} 5x + y + 7z = 0$$

$$6 \quad \mathbf{r} = 4\mathbf{i} + 2\mathbf{j} - \mathbf{k} + \lambda(3\mathbf{i} + 4\mathbf{j} - \mathbf{k}), \lambda \in \mathbb{R}$$

7 (a) Yes (b) No (c) No

8 (a) intersect at $(11, 19, -10)$ (b) contained (c) parallel

$$9 \quad \text{(a)} 2\sqrt{2} \quad \text{(b)} \frac{29}{\sqrt{34}} \quad \text{(c)} \sqrt{34}$$

$$10 \quad \text{(a)} 55.9^\circ \quad \text{(b)} 24.7^\circ \quad \text{(c)} 82.9^\circ$$