# 5 PROJECTILE MOTION

H2 Physics 9478

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## Learning Outcomes

Candidates should be able to:

(a) describe and use the concept of weight as the force experienced by a mass in a gravitational field.

(b) describe and explain motion due to a uniform velocity in one direction and a uniform acceleration in a perpendicular direction.

(c) derive, from the definition of work done by a force, the equation $\Delta E_P = mg\Delta h$ for gravitational potential energy changes in a uniform gravitational field (e.g. near the Earth's surface).

(d) recall and use the equation $\Delta E_P = mg\Delta h$ to solve problems.

(e) describe qualitatively, with reference to forces and energy, the motion of bodies falling in a uniform gravitational field with air resistance, including the phenomenon of terminal velocity.

## 5.1 Mass vs Weight

**Mass**

The mass $m$ of a body is the intrinsic property of a body which resists change in motion.

i.e. it is a measure of the inertia of a body. The same force, when applied to a body of larger mass will cause a smaller change in motion.

**Weight**

The weight $W$ of a body is the force experienced by a mass in a gravitational field.

Weight or the gravitational force on a body is an extrinsic property as it depends not just on the mass of the body, but also the gravitational field strength $g$ at the point where the body is.

Mathematically

$$\boxed{W = mg}$$

It is important not to confuse weight with mass.

## 5.2 Free Fall

### ❖ Galileo's Free Fall Experiment

Galileo's free fall experiment is one of his most famous contributions to physics. It involved the study of bodies falling freely under the influence of gravity.

The purpose of the experiment was to challenge the prevailing Aristotelian belief that heavier bodies fall faster than lighter bodies. Galileo hypothesized that in the absence of air resistance, all bodies would fall at the same rate, *regardless of their mass*.

## ❖ Acceleration

Since the gravitational force is the only force acting on a falling body,

$$F_{net} = mg$$

By Newton's 2nd Law,

$$mg = ma$$
$$a = g$$

Thus, the acceleration of the body is dependent on the gravitational field strength, i.e. $a = g$.

The acceleration of a free-falling object near the surface of the Earth has an average value of $9.81\ \text{m s}^{-2}$. This is known as the acceleration of free fall.

Professor Brian Cox visits NASA's Space Power Facility in Ohio to see what happens when a bowling ball and a feather are dropped together in the world's biggest vacuum.

## ❖ Free Falling Bodies in a Uniform Gravitational Field

An object moving freely near the surface of the Earth in the absence of air resistance is said to be undergoing free fall.

In such a case, whether the body is moving upwards or downwards, it experiences a **constant acceleration directed downwards** (towards centre of the Earth) with magnitude of $g = 9.81\ \text{m s}^{-2}$.

For illustration, let us consider the motion of a ball that is projected vertically upwards and falling back to its original position and beyond. Upwards is defined as the positive direction and the reference point where displacement is zero is set at point A.

Fig. 5.1 shows the sign of the displacement, velocity and acceleration vectors of the ball at the various points along its motion.

| | **displacement** | **velocity** | **acceleration** |
|---|---|---|---|
| **A** | zero | + | – |
| **B** | + | + | – |
| **C** | + | zero | – |
| **D** | + | – | – |
| **E** | zero | – | – |
| **F** | – | – | – |

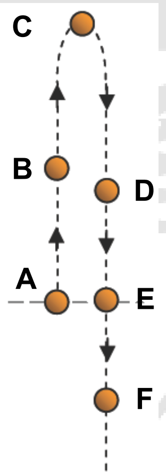



**Fig. 5.1**

## 5.3 One-Dimensional Motion

The four kinematic equations that were introduced in Chapter 3 are as follows:

- $v = u + at$
- $s = \frac{1}{2}(u + v)t$
- $s = ut + \frac{1}{2}at^2$
- $v^2 = u^2 + 2as$

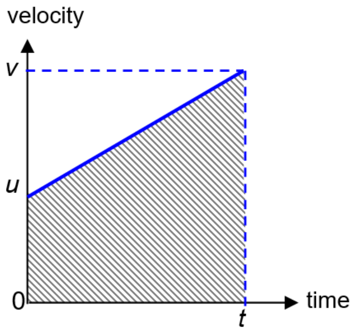


**Fig 5.2**

Note that these four equations apply only to motion in a *straight line* with *constant acceleration* as shown in Fig. 5.2.

In the absence of air resistance, bodies moving in air (i.e., free falling) experience a <u>constant vertical acceleration</u> ($a_y = g = 9.81 \text{ m s}^{-2}$) <u>directed downwards</u>, and no horizontal acceleration ($a_x = 0$).

## 5.4 Two-Dimensional Motion

❖ **Combining Horizontal & Vertical Motion**

A body that is projected near the surface of the Earth, such as a kicked football or a batted baseball, describes a curved path in a vertical plane. This kind of motion is called **projectile motion**.

Projectile motion involves motion in the horizontal and vertical directions simultaneously. These perpendicular components of motion are independent yet coordinated, with the vertical motion being similar to an object in free fall.

❖ **Important Terms**

When describing a projectile, the following terms are often used:

- **Trajectory** – the path described by a body, which for this chapter is parabolic.
- **Range** – the horizontal displacement between the point of projection and the point of impact.
- **Angle of projection** – the angle between the direction of projection and the horizontal plane through the point of projection.
- **Time of flight** – time taken from the point of projection to the point of impact

## ❖ Key Characteristics of Projectile Motion

1) The trajectory of the projectile is symmetrical about the vertical axis through the highest point.
2) The time the body takes to go from $s_y = 0$ to the highest point equals to the time the body takes to go from the highest point to $s_y = 0$ because acceleration is constant at $g$ throughout.
3) The path of a projectile is always a parabola.

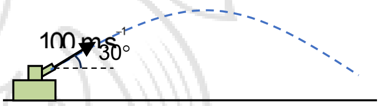
## ❖ General Analysis of the Trajectory of a Projectile

Projectile motion can be analysed in the following steps:

1) Resolving the ***displacement***, ***velocity***, and ***acceleration*** vectors into their horizontal and vertical components (or two perpendicular components).
2) Applying the kinematic equations along the horizontal and vertical directions separately.
3) The resultant velocity at any point can be determined by doing a vector addition of the horizontal and vertical components of the velocity at that point.

## ❖ Applying Kinematic Equations

Consider a cannon ball shot out of a tank at an angle of 30° to the horizontal at a velocity of $100 \text{ m s}^{-1}$, where air resistance is negligible. The path of the cannon ball would be a parabola shown in Fig. 5.5.




### Horizontal Motion

If we are looking down from above, we would observe the cannon ball to be moving in a straight line. Neglecting air resistance, the cannon ball <u>moves in a straight line with constant velocity</u> as the acceleration in the horizontal direction is zero.

**Kinematics Equations**

- $a_x = 0$
- $u_x = v_x$
- $s_x = u_x t$

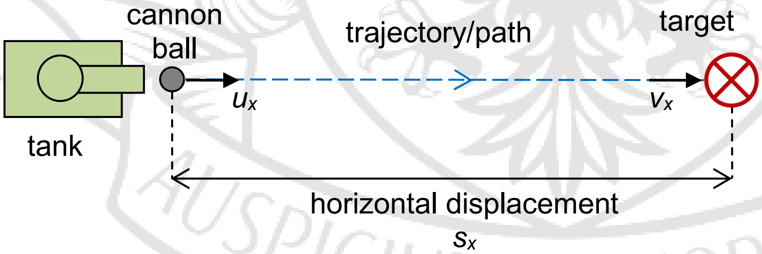



**Fig 5.3**

**Vertical Motion**

If we are looking from the front view of the tank, we would observe the cannon ball slowing down while moving upwards until it stops momentarily in the vertical direction, then speeding up while moving downwards. The magnitude of its acceleration is the acceleration of free fall $g$ where $g = 9.81\ \text{m s}^{-2}$ and is directed downwards.

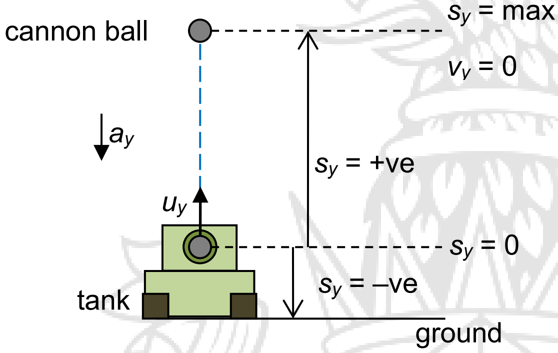



**Fig 5.4**

**Kinematics Equations**

**(taking up as positive)**

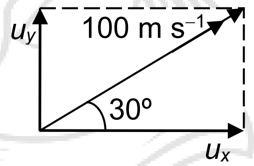
- $a_y = -g = -9.81\ \text{m s}^{-2}$
- $v_y = u_y + a_y t$
  $= u_y + (-9.81)t$
- $s_y = u_y t + \frac{1}{2}a_y t^2$
  $= u_y t + \frac{1}{2}(-9.81)t^2$
- $v_y^2 = u_y^2 + 2a_y s_y$
  $= u_y^2 + 2(-9.81)s_y$

* note that $t = \frac{s_x}{u_x}$

**Combining Horizontal & Vertical Motions**

The components of the initial velocity $u$ of the cannon ball are given by

$$u_x = u\cos\theta = 100\cos 30° = 86.6\ \text{m s}^{-1}$$
$$u_y = u\sin\theta = 100\sin 30° = 50.0\ \text{m s}^{-1}$$




The velocity $v$ of the cannon ball at any time in the motion can be described as

- having a magnitude of $v = \sqrt{v_x^2 + v_y^2}$

  where $v_x = v\cos\theta$ and $v_y = v\sin\theta$, and
- travelling at an angle $\theta$ to the horizontal where $\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$

The combination of two linear paths result in a parabola.

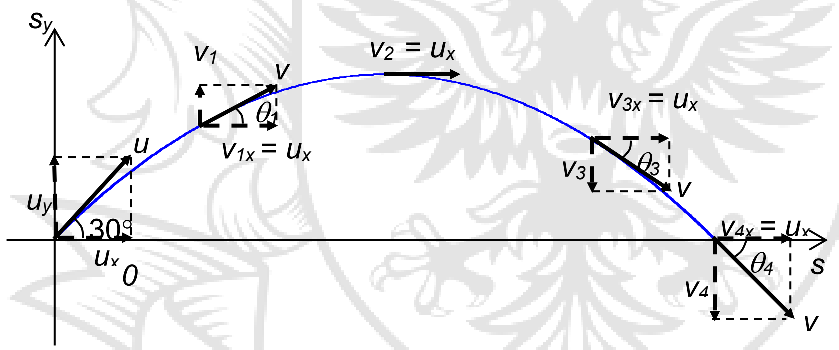
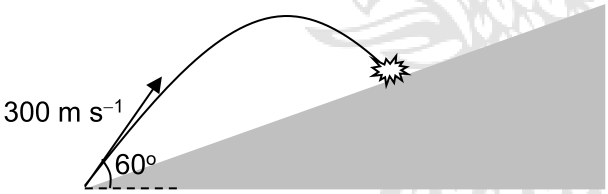



**Fig 5.5**

**Example 1**

**(Resolving Vertical & Horizontal Components)**

An artillery shell at the base of a mountain is fired with an initial velocity of $300 \text{ m s}^{-1}$ at an angle of $60^\circ$ from the horizontal, as shown in the diagram.

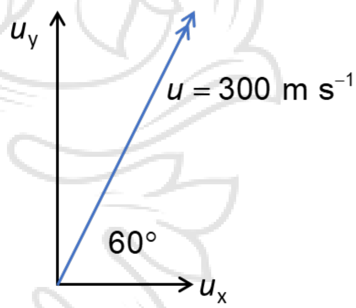



The shell explodes on the side of the mountain 40 s after firing.

**(a)** Calculate the magnitude of the horizontal and vertical components of the initial velocity.

**(b)** Find the horizontal and vertical distance of the point where the shell explodes relative to the firing point.

Remember to use sign convention!



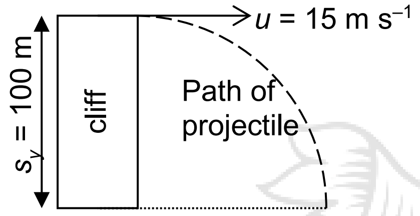
## Example 2

**(Projected horizontally i.e.** $u_y = 0$ & $u_x = u$**)**

A stone is projected horizontally with a speed of $15\ \mathrm{m\ s^{-1}}$ from the top of a cliff 100 m high. Calculate the

**(a)** speed of the stone just before it hits the ground,
**(b)** horizontal distance from the base of the cliff to where the stone lands.

Taking downwards as positive



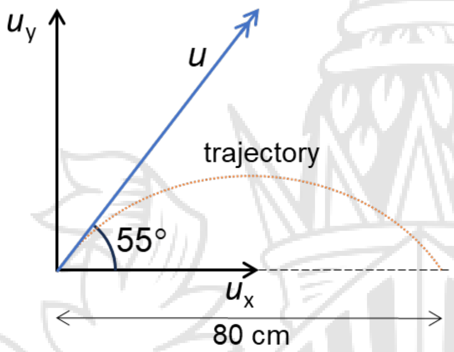

It makes more sense to take downwards as positive, as both movement and acceleration are downwards.

## Example 3

### (Horizontal Range)

Locusts have been observed to jump distances up to 80 cm on a level floor. Photographs of their jump show that they usually take off at an angle of about $55^\circ$ from the horizontal.

Calculate the initial velocity of locusts jumping 80 cm horizontally with a take-off angle of $55^\circ$.



## 5.5 Work Done & GPE of a Projectile (without air resistance)

The concept of energy is one of the most fundamental in science and is discussed in the context of Newtonian mechanics. Energy is present in various forms, with endless conversion from one form to another. Energy is transferred or converted when work is done.

**Work Done by a Force**

Work done by a constant force is the product of the force and the displacement in the direction of the force.

Recall from Chapter 4

Consider a body being raised upwards, at a constant velocity, from a height of $h_1$ to a height of $h_2$ near the Earth's surface. Since the velocity at which the body moves is constant, the force $F$ required to lift the body must be equal to its weight $mg$.

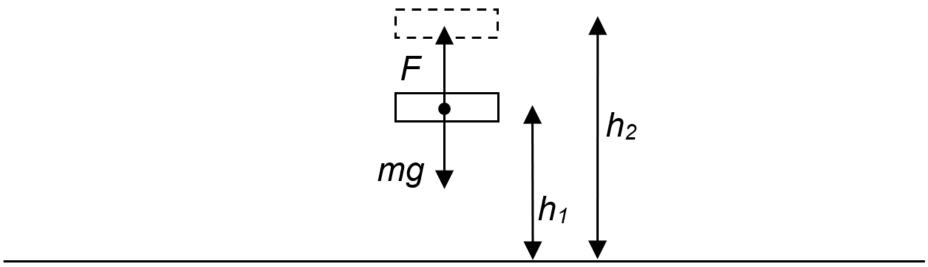



**Fig. 5.7**

Work done by $F$ in displacing the centre of mass of body vertically upward,

$$W = Fs$$
$$= mg(h_2 - h_1)$$
$$= mg\Delta h \text{ where } \Delta h = h_2 - h_1$$

In raising the body, the force $F$ does work on it which results in a gain in gravitational potential energy $E_P$ of the body.

Hence, the change in $E_P$ is

$$\Delta E_P = mg\Delta h$$

where $\Delta h = h_2 - h_1$ or the change in height.

The absolute value of gravitational potential energy, where $E_P = mgh$, is arbitrary. The point where the gravitational potential energy is taken to be zero can be changed by simply choosing a different point of reference for $h$, and this does not affect the physics involved. This is because it is the *change* in the gravitational potential energy $\Delta E_P$, arising from a change in position of the body, that is the more important quantity.

When an object moves in projectile motion, the body's speed is non-zero (i.e., it is moving) at every position along its trajectory.

This implies that the body possesses kinetic energy at every position along its trajectory.

As it moves along its trajectory, its height changes with respect to its starting position and hence its gravitational potential energy will change.

This means that at every position along the trajectory, the body will possess both

1) **kinetic energy** (due to its speed) as well as
2) **gravitational potential energy** (due to the body's position in the gravitational field).

Since there is no work done against air resistance, by the principle of conservation of energy, any increase in gravitational potential energy must be accompanied by a decrease in kinetic energy and vice versa.

**Example 4**

A 2.0 kg box is released from the top of a frictionless slope. It slides a distance of 3.0 m down the slope, which is inclined at an angle of $30°$ to the horizontal.

Determine

**(a)** the work done by gravitational force on the box.
**(b)** the speed of the box after sliding 3.0 m down the slope.

✏ **Try solving Example 2(a) using Conservation of Energy!**

## 5.6 Effects of Air Resistance

**Air resistance** is a resistive force that opposes motion. It is an example of drag force.

It depends on the:

- shape of the body,
- speed of the body, and
- viscosity of the fluid.

At low speeds, this resistive force is proportional to the speed of the body.

### ❖ Horizontal Motion

Consider an object moving in a horizontal direction in which air resistance is not negligible. Horizontally, air resistance is the net force acting on the body.

$$F_{net} = F_{air}$$

By Newton's 2nd Law,

$$F_{air} = ma$$

For a body of mass $m$ moving horizontally with an initial velocity $v_0$, the velocity of the body changes with time is shown in Fig. 5.8. The corresponding acceleration-time graph is shown in Fig. 5.9.

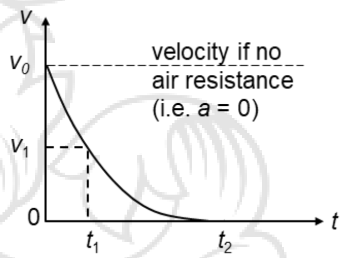



Fig. 5.8

Note: the area under the $v - t$ graph is now less. This implies that displacement will be less than when there is no air resistance

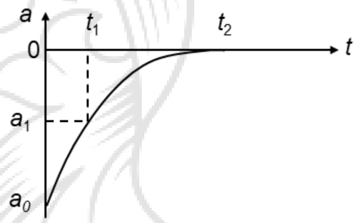



Fig. 5.9

At $t = 0$, air resistance is maximum as the body is moving at its maximum speed. Since air resistance opposes motion (i.e., acts in the opposite direction to $v_0$), acceleration is negative and as a result, the speed of the body decreases rapidly.

This in turn results in a smaller opposing force and hence a negative acceleration of a smaller magnitude as can be seen at $t = t_1$.

The speed of the body continues to decrease at a decreasing rate until it stops moving as shown at $t = t_2$.

Once it stops moving, its speed is zero and air resistance stops acting on the body.

## ❖ Vertical Motion

The forces acting on a body moving vertically are weight and air resistance. Weight always acts downwards, whereas air resistance opposes motion and will act in the opposite direction to the body's velocity. Hence, the net force acting on a body moving upwards will be different from that of a body moving downwards.

Visualisation of vertical motion under the effect of air resistance

### Moving Upwards

For a small body of mass $m$ moving vertically upwards with an initial velocity $v_0$, the free body diagram of the body is shown in Fig. 5.10. Since it is moving upwards, air resistance acts downwards.

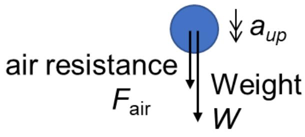



**Fig. 5.10**

$$F_{net} = F_{air} + W$$

By Newton's 2nd Law,

$$F_{air} + W = ma_{up}$$

$$a_{up} = \frac{F_{air}}{m} + g$$

As can be seen from both Fig. 5.10 and the equation above, *as the ball moves up*, net acceleration

- acts downwards, and
- has a magnitude that is ***larger*** than $9.81 \text{ m s}^{-2}$.

The speed of the body decreases. This in turn leads to a decrease in the magnitude of air resistance.

As the body continues to move upwards, its speed decreases at a decreasing rate, causing air resistance and hence the magnitude of the net acceleration to also decrease.

At maximum height, the body is momentarily at rest and there is no air resistance. The net downwards acceleration acting on the body is now equal to $9.81 \text{ m s}^{-2}$ as shown in Fig. 5.11.

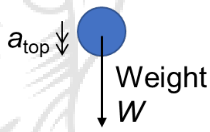



**Fig. 5.11**

The velocity time graph for the body's upwards motion is shown in Fig. 5.12, with upwards taken as the positive direction.

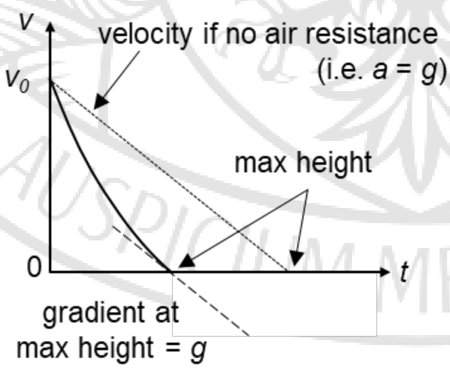



**Fig. 5.12**

Note:

- at maximum height, net acceleration = $g$
- air resistance reduces the area under the graph.
  This implies that the *maximum height achieved will be less* than when there is no air resistance

**Moving downwards**

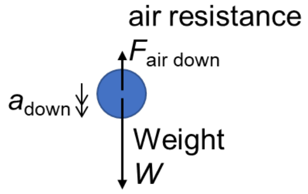



**Fig. 5.13**

As the body begins to descend from its maximum height, the free body diagram is shown in Fig. 5.13. Since it is moving downwards, air resistance acts upwards.

$$F_{\text{net}} = W - F_{\text{air}}$$

By Newton's 2nd Law,

$$W - F_{\text{air}} = ma_{\text{down}}$$

$$a_{\text{down}} = g - \frac{F_{\text{air}}}{m}$$

As can be seen from both Fig. 5.13 and the equation above, *as the ball moves down*, net acceleration

- acts downwards, and
- has a magnitude that is ***smaller*** than $9.81\ \text{m s}^{-2}$.

The speed of the body increases. This in turn leads to an increase in the magnitude of air resistance.

As the body continues to move downwards, its speed increases at a decreasing rate, causing air resistance to increase and the magnitude of the net acceleration to decrease.

**Terminal Velocity**

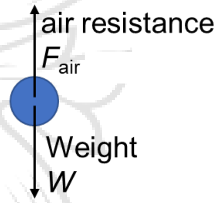



**Fig. 5.14**

If the body is allowed to speed up such that the magnitude of the *air resistance* is now equal to the magnitude of its *weight*, the net force and hence net acceleration acting on it will now be zero. The free body diagram is shown in Fig. 5.14.

$$F_{\text{net}} = W - F_{\text{air}} = 0$$

By Newton's 2nd Law,

$$W - F_{\text{air}} = 0$$

$$a_{\text{down}} = 0$$

Since there is no acceleration, the body's velocity will remain constant. This velocity is known as its terminal velocity.

Continuing from Fig. 5.12, the complete velocity time graph of the body for its entire journey back to the starting point is shown in Fig. 5.15, along with its acceleration time graph.

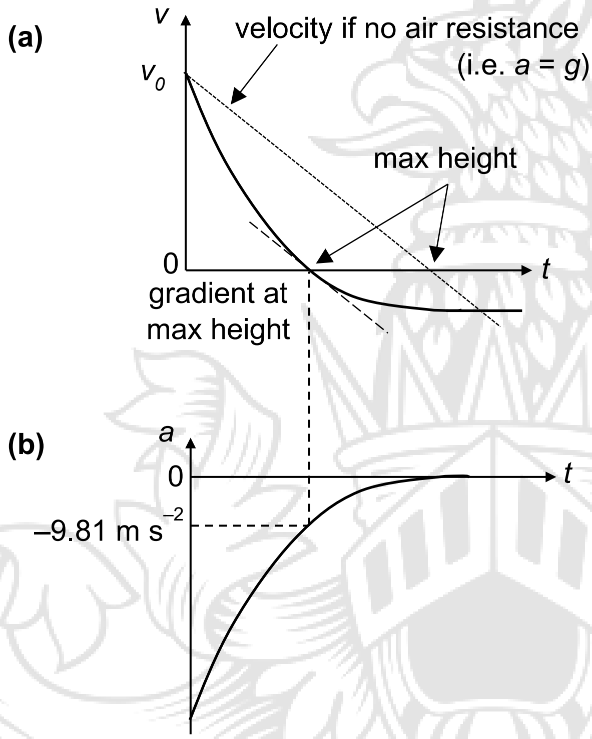



**Fig. 5.15**

From the ***velocity-time graph*** in Fig. 5.15 (a), the average speed on the way up, is more than the average speed on the way down.

Since the distance travelled upwards to the highest point must equal the distance travelled back to its starting position, the time taken to travel to maximum height (upwards) is shorter than the time taken to travel from maximum height to its starting position (downwards).

## ❖ Trajectory of Projectile Motion With Air Resistance

Since projectile motion involves motion in the horizontal and vertical directions simultaneously, both the horizontal and vertical motions of the projectile will be affected by the presence of air resistance.

Fig. 5.16 shows the trajectory of the cannon ball both with and without air resistance. The variation of both horizontal and vertical velocities with time are also shown.

(described in Fig. 5.3 and 5.4 on page 5)

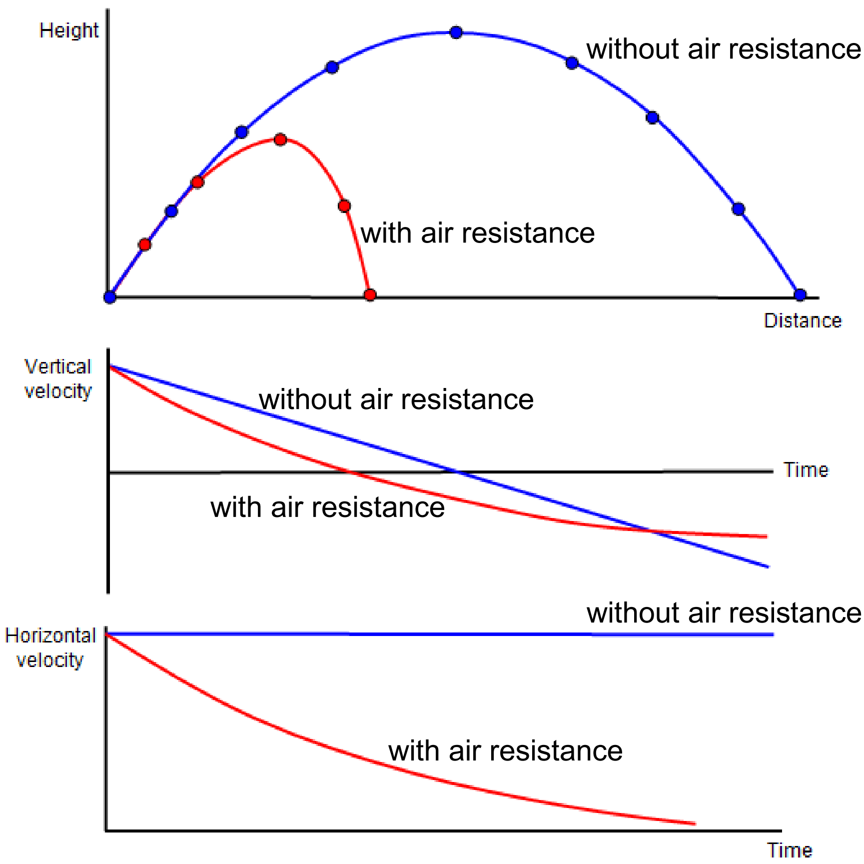



**Fig. 5.16**

Visualisation of projectile motion graphs with and without air resistance.

Without air resistance, the trajectory is symmetrical.

With air resistance,

- Horizontal velocity is no longer constant (it decreases)
  Range (horizontal displacement) is now smaller.

- On the way up, vertical acceleration is more than $g$
  $$a_{up} = \frac{F_{air}}{m} + g \Rightarrow a_{up} > g$$
  Vertical velocity decreases to zero in a shorter time
  Maximum height is now smaller.

Recall:

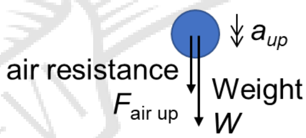



**Fig. 5.10**

As a result, the trajectory reaches a shorter height and ends with a shorter range. It is also asymmetrical.

## ❖ Energy Considerations

With the introduction of air resistance, it is now necessary to consider work done against air resistance alongside kinetic energy as well as gravitational potential energy.

### Moving Upwards

As the cannon ball moves upwards through the air, any decrease in kinetic energy will result in the same increase in gravitational potential energy and thermal energy. This thermal energy is due to the work done against air resistance and is dissipated to the surroundings.

### Moving Downwards

As the cannon ball moves downwards through the air, any decrease in gravitational energy will result in the same increase in kinetic energy and thermal energy. This thermal energy is due to the work done against air resistance and is dissipated to the surroundings.

### Terminal Velocity

At terminal velocity, kinetic energy remains constant as the body continues to fall. The decrease in the gravitational potential energy is totally converted to thermal energy that is dissipated to the surroundings.

## 5.7 Appendix

❖ **Proof of Parabolic Path of a Projectile**

Consider a particle P with velocity u projected at an angle $\theta$ from the ground as shown in Fig. 5.17.

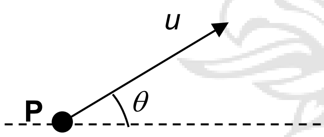



**Fig. 5.17**

Assume air resistance is negligible.

Applying the equations of motion in both directions,

$$s_X = (u\cos\theta)t \qquad ..................(1)$$

$$s_Y = (u\sin\theta)t - \frac{1}{2}gt^2 \qquad ..................(2)$$

Rearranging (1), we get :

$$t = \frac{s_X}{u\cos\theta} \qquad ................(3)$$

Substitute (3) in to (2):

$$s_Y = (u\sin\theta)\left(\frac{s_X}{u\cos\theta}\right) - \frac{1}{2}g\left(\frac{s_X}{u\cos\theta}\right)^2$$

$$s_Y = (u\tan\theta)s_X - \left(\frac{g}{2u^2\cos^2\theta}\right)s_X{}^2$$

The above expression shows that the relationship between the vertical displacement $s_y$ and the horizontal displacement $s_x$ is a quadratic one.

Hence, the trajectory of a projectile travelling through negligible air resistance is a parabola.