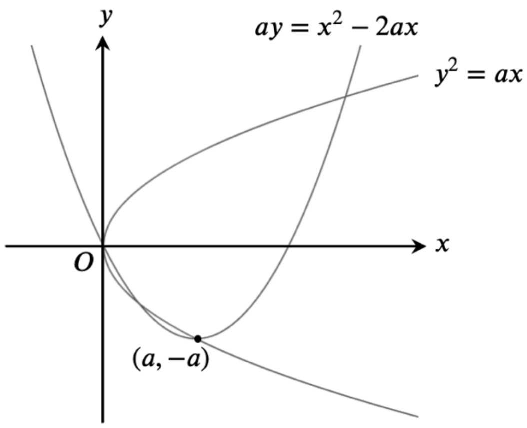
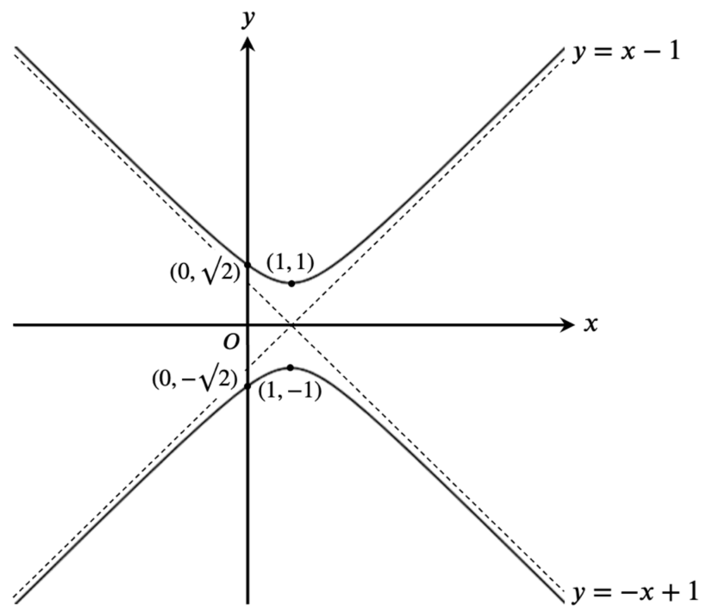
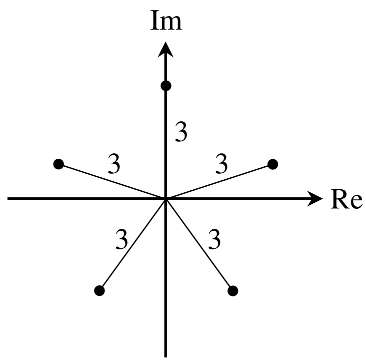
MATHEMATICS

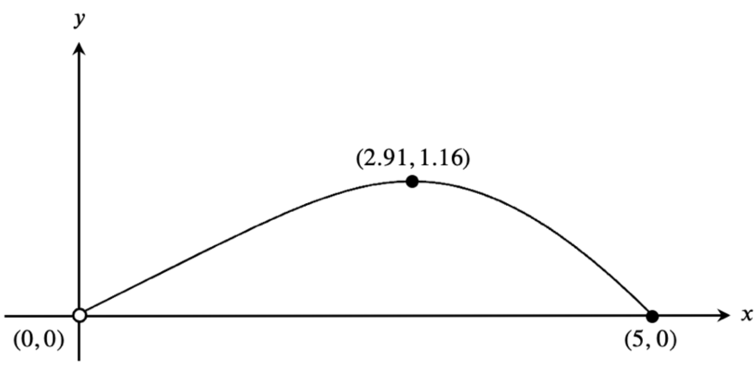
Paper 9758/01

Set III – Paper 1

Topic identification and short answers

Qn	Topic(s)	Part	Answers
1	Differentiation; Sequences and series (Geometric sequence)		$r = -\mathrm{e}^{\pi}$
2	Sequences and series (Arithmetic sequence)	(a)	[shown]
		(b)	$d = 2$
3	Graphs (sketching); Inequalities	(a)	
		(b)	$x = 0$ or $\frac{3-\sqrt{5}}{2} \le x \le 1$ or $x \ge \frac{3+\sqrt{5}}{2}$
4	Differentiation (maxima and minima)	(a)	[shown]
		(b)	$x = -1$ and $x = 2$
5	Integration techniques (by parts); Definite integrals	(a)	$\int \{\mathrm{f}(x) + \mathrm{g}(x)\}\,\mathrm{d}x = x\ln(\ln x) + C$
		(b)	$\mathrm{e}^2 \ln 2 - \frac{\mathrm{e}^2}{2} + \mathrm{e}$

6	Differentiation (implicit); Integrals; Graphs (sketching, conics)	**(a)**	$k = 1$ $\mathrm{f}(x) = x + \frac{2}{x} - 2$
		(b)	
7	Recurrence relations; Complex numbers (four operations)	**(a)**	[shown] Base cases: $u_1 = 1$, $u_2 = 5$
		(b)	$r^2 - 2r + 5 = 0$
		(c)	[shown] $m = 4$ $\alpha^4 + \beta^4 = -6$
8	Complex numbers (conjugate, argument, modulus, Argand diagram, geometrical effects)	**(a)**	[shown]
		(b)	$z_3 = 1$ [shown]
		(c)	$\lvert z_4 - z_3 \rvert = \sqrt{\left(\frac{5-\sqrt{5}}{2}\right)}$
		(d)	Required area $= \frac{5}{8}\sqrt{(10 + 2\sqrt{5})}$ units^2
		(e)	

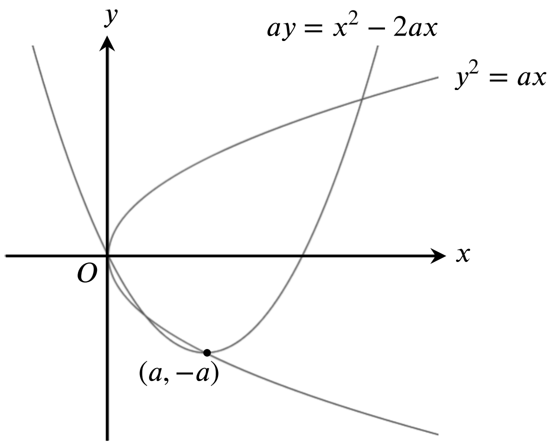
9	Differentiation (parametric equations, tangents); Graphs (parametric, conics)	**(a)**	$\frac{dy}{dx}=\frac{\cos\theta-1}{\sin\theta}$
		(b)	[shown]
		(c)	$q=p+\pi$ [shown] $y=\sin 2\theta$
		(d)	Circle with radius 1 unit and centre $(0,0)$ Clockwise direction
10	Differential equation; Differentiation; Graphs (sketching)	**(a)**	$\frac{d^2x}{dt^2}+7\frac{dx}{dt}+6x=0$
		(b)	$x=4e^{-t}+e^{-6t}$ $y=2e^{-t}-2e^{-6t}$
		(c)	
11	Functions (inverse, composite)	**(a) (i)**	[shown]
		(a) (ii)	$R_{h^{-1}}=[0,1]$ $D_f=[0,1]$ [shown]
		(a) (iii)	$D_{h^{-1}}=[-2,2]$ $R_{fh^{-1}}=[0,1]$ [shown]
		(b)	[shown]
		(c)	$Q(x)=1-\left\|1-\|4x-2\|\right\|,\ 0\le x\le 1$

12	Vectors (three dimensions, normal, angle between two vectors, distances, relationships); Inequalities; Differentiation (implicit, maxima-minima)	**(a)**	$\begin{pmatrix} \sqrt{3} \\ 0 \\ -1 \end{pmatrix}$
		(b)	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ The orbits are not coplanar.
		(c)	[shown]
		(d)	$\frac{1}{4}\pi \le t \le \frac{1}{2}\pi$ or $\frac{5}{4}\pi \le t \le \frac{3}{2}\pi$
		(e)	$t = \frac{3}{8}\pi, \frac{11}{8}\pi$ Distance ≈ 2.31 units
		(f)	The perfect condition is not feasible. The statement claims that the distance between the satellite and the starspot is the shortest possible when they are collinear with the centre of the star. However, mathematically the three points cannot be collinear.

Suggested solutions and post-mortem

Qn	Suggested Solutions	Post-mortem
1 **[3]**	For stationary points, $\rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x \cos x - \mathrm{e}^x \sin x = \mathrm{e}^x(\cos x - \sin x) = 0$ $\rightarrow \cos x = \sin x \ (\because \mathrm{e}^x > 0)$ $\rightarrow \tan x = 1$ $\therefore x = \frac{1}{4}\pi + n\pi$, where $n \in \mathbb{Z}$ Substituting this result into the curve equation, $\rightarrow y = \mathrm{e}^{\left(\frac{1}{4}\pi + n\pi\right)} \cos\left(\frac{1}{4}\pi + n\pi\right)$ $\rightarrow y = \mathrm{e}^{\left(\frac{1}{4}\pi + n\pi\right)} \left[\cos\frac{1}{4}\pi \cos n\pi - \sin\frac{1}{4}\pi \sin n\pi\right]$ $\rightarrow y = \mathrm{e}^{\left(\frac{1}{4}\pi + n\pi\right)} \left[\frac{\sqrt{2}}{2}\cos n\pi - \frac{\sqrt{2}}{2}(0)\right]$ $\rightarrow y = \mathrm{e}^{\left(\frac{1}{4}\pi + n\pi\right)} \left[\frac{\sqrt{2}}{2}(-1)^{n+1}\right]$ $\therefore y = -\frac{\sqrt{2}}{2}\mathrm{e}^{\frac{1}{4}\pi}(-\mathrm{e}^{\pi})^n$ Letting $n = k + 1$ and $n = k$ to find ratio of the y-coordinates of two consecutive turning points, $\rightarrow \dfrac{-\dfrac{\sqrt{2}}{2}\mathrm{e}^{\frac{1}{4}\pi}(-\mathrm{e}^{\pi})^{k+1}}{-\dfrac{\sqrt{2}}{2}\mathrm{e}^{\frac{1}{4}\pi}(-\mathrm{e}^{\pi})^{k}} = -\mathrm{e}^{\pi}$, which is constant. $\therefore$ The y-coordinates of the turning points follow a geometric progression. Required common ratio $= -\mathrm{e}^{\pi}$	This question assesses on differentiation and geometric series. The keyword "turning points" should sufficiently hint at the use of calculus, which will reveal that $\tan x = 1$ at all turning points. Finding the required progression afterwards requires candidates to appreciate that there are infinitely many solutions to $\tan x = 1$, not just $x = \frac{1}{4}\pi$. With the general formula for the x-coordinates of the turning points, candidates can substitute it into the curve equation and reorganise the terms to obtain the required ratio. Note the negative sign in the ratio, which may elude the unwary. This could easily be prevented by cross-checking using GC which will confirm the curve's oscillation about the x-axis.

2 (a) [2]	Given that $(u_{n+1}+u_{n+2}+\cdots+u_{2n})$ is a constant multiple of $(u_1+u_2+\cdots+u_n)$, $\rightarrow u_{n+1}+u_{n+2}+\cdots+u_{2n}=k(u_1+u_2+\cdots+u_n)$, where $k\in\mathbb{R}$. $\rightarrow \frac{n}{2}[(1+nd)+(1+(2n-1)d)]=k\cdot\frac{n}{2}[(2(1)+(n-1)d)]$ $\rightarrow 2+(3n-1)d=k(2+(n-1)d)$ $\rightarrow \frac{2+(3n-1)d}{2+(n-1)d}=k$, which is constant.	This question mainly tests on arithmetic series, in particular the expression for the arithmetic sum.
2 (b) [2]	Let $\frac{2+(3n-1)d}{2+(n-1)d}=k$ $\rightarrow 2+(3n-1)d=2k+(n-1)kd$ $\rightarrow 2+3nd-d=2k+knd-kd$ $\rightarrow 3nd-knd=2k-kd-2+d$ $\rightarrow nd(3-k)=k(2-d)-(2-d)$ $\rightarrow nd(3-k)=(k-1)(2-d)$ d and k are constants $\rightarrow (k-1)(2-d)$ is constant $\rightarrow nd(3-k)$ is constant for all integers $n\geq 1$ $\rightarrow nd(3-k)=0$ $\therefore k=3$ or $d=0$ (rej. $\because$ sequence is non-constant) Given $nd(3-k)=0$ and $k=3$, $\rightarrow 0=(2-d)(3-1)$ $\rightarrow d=2$	This question may prove challenging. The keyword "hence" hints the use of the expression in **(a)**, from which candidates are expected to gain further insight. This suggested solution yields a factorised form for a convenient deduction. Most importantly, this question calls for candidates to appreciate that **an expression containing constants and variables can only be constant when the expression itself evaluates to 0**. Upon recognising this, along with the fact that that the sequence is non-constant, candidates should be able to deduce d.

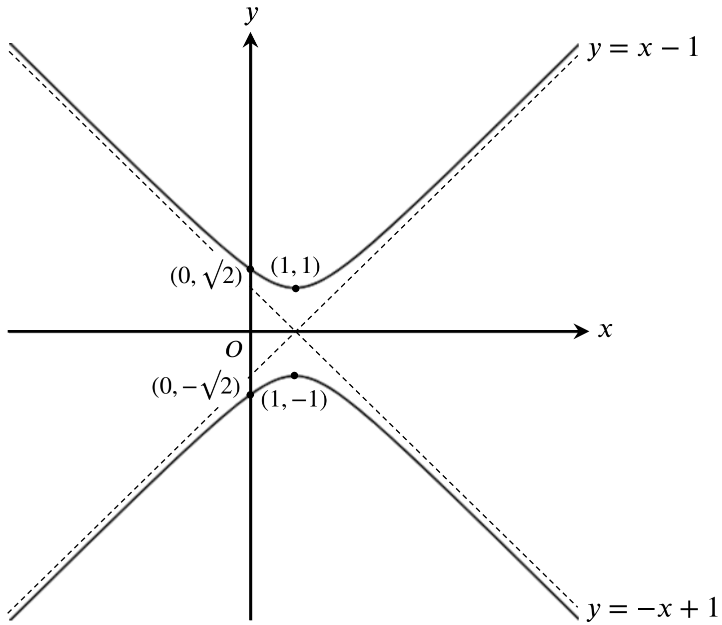
3 (a) [2]	 Substituting $y = -a$ to $y^2 = ax$, $\to (-a)^2 = ax$ $\to a^2 = ax$ $\to x = a$ Substituting $y = -a$ to $ay = x^2 - 2ax$, $\to a(-a) = x^2 - 2ax$ $\to -a^2 = x^2 - 2ax$ $\to 0 = x^2 - 2ax - a^2 = (x - a)^2$ $\to x = a$ Both curves have a point in common at $(a, -a)$.	This question assesses candidates on graph sketching, and tangentially on graph transformation. Candidates may encounter difficulties visualising the effects of the constant a to each graph simultaneously. Substituting a with a real value to obtain a sketch using GC would be most helpful, but graph transformations may also prove to be a handy tool here: • Consider $y^2 = x$ with $y^2 = ax$. Observe that the two graphs are obtainable by scaling the other graph parallel to the x-axis. • Consider $y = x^2$ with $ay = x^2 - 2ax$. Observe that the second quadratic equation can be expressed as $y = \frac{1}{a}(x - a)^2 - a$. This expression gives insight to not only the transformations between these two graphs, but also the coordinates of the turning point. Marks would be awarded each for the correct shape of both curves, and the coordinates of the turning point of $ay = x^2 - 2ax$, which must lie on the curve $y^2 = ax$. This part also aims to confirm two of the four x-coordinates at which the two curves intersect ($x = 0$ and $x = a$). These, along with other information deduced in this question, are relevant in the next part, which may prove much more challenging.

3 (b) [3]	Letting $a = 1$ yields $y^2 = x$ and $y = x^2 - 2x$. Using the sketch in **(a)**, the following are deduced: • $y = \sqrt{x}$ is the modulus transformation of $y^2 = x$. • Hence, $y = \sqrt{x}$ and $y = \lvert x^2 - 2x \rvert$ meet at the same x-coordinate as do $y^2 = x$ and $y = x^2 - 2x$. • Hence, $y = \sqrt{x}$ and $y = \lvert x^2 - 2x \rvert$ meet at $x = 0$ and $x = 1$. Finding the remaining principal values for the inequality using the graph in **(a)** as proxy, $\rightarrow x = (x^2 - 2x)^2$ $\rightarrow 0 = x^4 - 4x^3 + 4x^2 - x$ $\rightarrow 0 = x(x^3 - 4x^2 + 4x - 1)$ $\rightarrow 0 = x(x-1)(x^2 - 3x + 1)$, by considering root $x = 1$ (from sketch) and comparing coefficient $\rightarrow x = 0 \text{ or } x = 1 \text{ or } x = \dfrac{3 \pm \sqrt{5}}{2}$ $\therefore x = 0 \text{ or } \dfrac{3-\sqrt{5}}{2} \leq x \leq 1 \text{ or } x \geq \dfrac{3+\sqrt{5}}{2}$	This question may prove challenging. Success in this part is highly dependent on the correctness of the answers in part **(a)**. With $a = 1$, we have the two curves $y^2 = x$ and $y = x^2 - 2x$ pertinent to the given inequality. This question requires candidates to appreciate that **modulus transformation preserves *x*-coordinates.** Note that $y^2 = x$ can also be written as $y = \pm\sqrt{x}$, which upon modulus transformation $y = \lvert \pm\sqrt{x} \rvert$ will simply yield $y = \sqrt{x}$. Transforming $y = x^2 - 2x$ similarly into $y = \lvert x^2 - 2x \rvert$ will lead to the important deduction that **the graphs in part (a) would intersect at the same *x*-values after undergoing modulus transformation**. (Imagine drawing the sketch in part (a) on a transparency sheet and folding it along the x-axis.) Note from **(a)** that out of the four points at which the two graphs intersect, two can be deduced from observation, yielding principal values $x = 0$ and $x = a = 1$. To find the remaining principal values, note that the four principal values make up roots of a quartic function. **Having deduced two of these roots, the remaining two roots would thus form a quadratic factor** that can be solved in exact to yield the remaining principal values. Upon presenting the final inequality, a calculator may help confirm where the surd principal values lie with respect to $x = 0$ and $x = 1$. Beware of dismissing $x = 0$ in the final answer, as equality also holds here.

4 (a) [2]	$\mathrm{f}(x)=\mathrm{f}(1-x)$ $\rightarrow \mathrm{f}'(x)=-\mathrm{f}'(1-x)$ $\rightarrow \mathrm{f}'(x)+\mathrm{f}'(1-x)=0$ When $x=1-x$, $x=\frac{1}{2}$ and $2\mathrm{f}'\left(\frac{1}{2}\right)=0$ $\therefore$ The curve has a turning point at $x=\frac{1}{2}$.	This question tests on differentiation, particularly the use of chain rule and the condition $\mathrm{f}'(a)=0$ for a stationary point to exist at $x=a$. Upon recognising that the result in this part entails equating $x=1-x$, candidates may expect similar working and manipulation in the next part.
4 (b) [4]	$\mathrm{f}(x)=\mathrm{f}\left(\frac{1}{x}\right)$ $\rightarrow \mathrm{f}'(x)=-\frac{1}{x^2}\mathrm{f}'\left(\frac{1}{x}\right)$ $\rightarrow \mathrm{f}'(x)+\frac{1}{x^2}\mathrm{f}'\left(\frac{1}{x}\right)=0$ When $x=\frac{1}{x}$, $x=\pm1$ and $2\mathrm{f}'(\pm1)=0$. Since $x\neq1$, the curve has a turning point at $x=-1$. $\mathrm{f}\left(\frac{1}{x}\right)=\mathrm{f}(1-x)$ $\rightarrow -\frac{1}{x^2}\mathrm{f}'\left(\frac{1}{x}\right)=-\mathrm{f}'(1-x)$ $\rightarrow \frac{1}{x^2}\mathrm{f}'\left(\frac{1}{x}\right)=\mathrm{f}'(1-x)$ When $x=-1$, $\mathrm{f}'(-1)=\mathrm{f}'(2)$. Since $\mathrm{f}'(-1)=0$, the curve has a turning point at $x=2$.	Candidates who find success in part **(a)** should be able to proceed similarly in this part and find the x-coordinates of the remaining stationary points accordingly, though unaware candidates may not see that $\mathrm{f}\left(\frac{1}{x}\right)=\mathrm{f}(1-x)$ for one of the stationary points. Note the need to engage with the asymptotes in this part, which comes in handy when rejecting spurious solutions.

5 (a) [2]	Integrating $\mathrm{f}(x)$ by parts, $\int \ln(\ln x)\,\mathrm{d}x = x\ln(\ln x) - \int x\left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right)\mathrm{d}x + C$ $\int \ln(\ln x)\,\mathrm{d}x = x\ln(\ln x) - \int \frac{1}{\ln x}\,\mathrm{d}x + C$ $\int \ln(\ln x)\,\mathrm{d}x + \int \frac{1}{\ln x}\,\mathrm{d}x = x\ln(\ln x) + C$ $\therefore \int \{\mathrm{f}(x) + \mathrm{g}(x)\}\,\mathrm{d}x = x\ln(\ln x) + C$	This question may prove challenging. At the academic point where candidates are preparing for A–Levels, candidates can already expect logarithmic integrals to have a certain trick to them. Candidates who realise that $\mathrm{f}'(x) = \frac{1}{x\ln x}$, may decide to integrate $\mathrm{f}(x)$ by parts after noticing that $\int x\mathrm{f}'(x)\,\mathrm{d}x = \int \mathrm{g}(x)\,\mathrm{d}x$. Beware of leaving out the integration constant $(+C)$ that is required for indefinite integrals.
5 (b) [4]	Integrating $\mathrm{g}(x)$ by parts from $x = \mathrm{e}$ to $x = \mathrm{e}^2$, $\int_{\mathrm{e}}^{\mathrm{e}^2} \frac{1}{\ln x}\mathrm{d}x = \left[\frac{x}{\ln x}\right]_{\mathrm{e}}^{\mathrm{e}^2} - \int_{\mathrm{e}}^{\mathrm{e}^2} x\left\{\frac{-1}{(\ln x)^2}\right\}\left(\frac{1}{x}\right)\mathrm{d}x$ $\int_{\mathrm{e}}^{\mathrm{e}^2} \frac{1}{\ln x}\mathrm{d}x = \left[\frac{x}{\ln x}\right]_{\mathrm{e}}^{\mathrm{e}^2} + \int_{\mathrm{e}}^{\mathrm{e}^2} \frac{1}{(\ln x)^2}\mathrm{d}x$ Adding $\int_{\mathrm{e}}^{\mathrm{e}^2} \ln(\ln x)\,\mathrm{d}x$ to both sides, $\int_{\mathrm{e}}^{\mathrm{e}^2} \left\{\ln(\ln x) + \frac{1}{\ln x}\right\}\mathrm{d}x = \left[\frac{x}{\ln x}\right]_{\mathrm{e}}^{\mathrm{e}^2} + \int_{\mathrm{e}}^{\mathrm{e}^2} \left\{\ln(\ln x) + \frac{1}{(\ln x)^2}\right\}\mathrm{d}x$ $\int_{\mathrm{e}}^{\mathrm{e}^2} \{\mathrm{f}(x) + \mathrm{g}(x)\}\,\mathrm{d}x = \left[\frac{x}{\ln x}\right]_{\mathrm{e}}^{\mathrm{e}^2} + \int_{\mathrm{e}}^{\mathrm{e}^2} \left\{\mathrm{f}(x) + \left(\mathrm{g}(x)\right)^2\right\}\mathrm{d}x$ $\therefore \int_{\mathrm{e}}^{\mathrm{e}^2} \left\{\mathrm{f}(x) + \left(\mathrm{g}(x)\right)^2\right\}\mathrm{d}x$ $= [x\ln(\ln x)]_{\mathrm{e}}^{\mathrm{e}^2} - \left[\frac{x}{\ln x}\right]_{\mathrm{e}}^{\mathrm{e}^2}$ $= \mathrm{e}^2\ln 2 - \frac{\mathrm{e}^2}{2} + \mathrm{e}$	This question may also prove challenging. Candidates may be quick to begin by writing $\int_{\mathrm{e}}^{\mathrm{e}^2} \left\{\mathrm{f}(x) + \left(\mathrm{g}(x)\right)^2\right\} = \int_{\mathrm{e}}^{\mathrm{e}^2} \left\{\ln(\ln x) + \frac{1}{(\ln x)^2}\right\}\mathrm{d}x,$ which would soon lead to a dead-end. It would be best for candidates to presume relevance of earlier results. This could help conserve precious time by avoiding unfruitful initial workings. Candidates who are successful in part **(a)** may be spurred to consider similarly doing integration by parts with the other function $\mathrm{g}(x)$. Upon eventually coming across $\int_{\mathrm{e}}^{\mathrm{e}^2} \frac{1}{\ln x}\mathrm{d}x$ and $\int_{\mathrm{e}}^{\mathrm{e}^2} \frac{1}{(\ln x)^2}\mathrm{d}x$ in either side of the same equation, it would be intuitive to add $\int_{\mathrm{e}}^{\mathrm{e}^2} \ln(\ln x)\,\mathrm{d}x$ to both sides to summon the result in **(a)** and the required definite integral respectively.

6 (a) [4]	Given that $y^2 = x^k\mathrm{f}(x)$, $\rightarrow 2y\dfrac{\mathrm{d}y}{\mathrm{d}x} = kx^{k-1}\mathrm{f}(x) + x^k\mathrm{f}'(x)$ $\rightarrow 2xy\dfrac{\mathrm{d}y}{\mathrm{d}x} = kx^k\mathrm{f}(x) + x^{k+1}\mathrm{f}'(x)$ $\therefore 2xy\dfrac{\mathrm{d}y}{\mathrm{d}x} = ky^2 + x^{k+1}\mathrm{f}'(x)$ Substituting this into the curve's differential equation gives: $ky^2 + x^{k+1}\mathrm{f}'(x) = y^2 + x^2 - 2$ Comparing the term in terms of y, $\rightarrow ky^2 = y^2$ $\therefore k = 1$ Comparing the term in terms of x, $\rightarrow x^{k+1}\mathrm{f}'(x) = x^2 - 2$ $\rightarrow \mathrm{f}'(x) = \dfrac{x^2-2}{x^{k+1}} = \dfrac{x^2-2}{x^2} = 1 - \dfrac{2}{x^2}$ $\therefore \mathrm{f}(x) = x + \dfrac{2}{x} + D,\quad$ for some $D \in \mathbb{R}$ Given $y^2 = x^k\mathrm{f}(x)$ and $k = 1$, substitution gives: $\dfrac{y^2}{x} = x + \dfrac{2}{x} + D$ Substituting the point $(1, 1)$ to find D, $\rightarrow 1 = 1 + 2 + D$ $\therefore D = -2$ $\therefore k = 1$ and $\mathrm{f}(x) = x + \dfrac{2}{x} - 2$	Despite the given differential equation, this question is geared more towards implicit differentiation – the part that is even remotely pertinent to solving differential equation merely involves rudimentary integration. The keyword "calculus" and the term $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the question should sufficiently hint that differentiation is expected. With further manipulation, it can be deduced that $ky^2 + x^{k+1}\mathrm{f}'(x) = y^2 + x^2 - 2$. Upon comparison, candidates may quickly obtain k, and $\mathrm{f}'(x)$ follows. Finding a particular solution for $\mathrm{f}(x)$ is conveniently done by integration and engaging with the given info that C passes through $(1, 1)$.

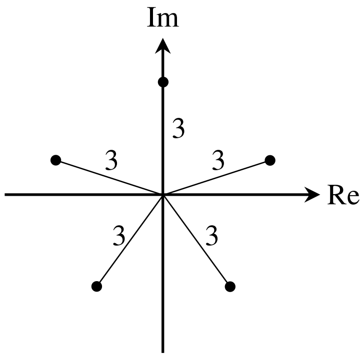
6 (b) [3]	Reconciling the results in **(a)**, $\rightarrow y^2 = x\left(x + \frac{2}{x} - 2\right) = x^2 + 2 - 2x$ $\rightarrow y^2 = (x-1)^2 + 1$ $\rightarrow y^2 - (x-1)^2 = 1$ $\therefore C$ is a hyperbola Finding asymptotes of C, $\rightarrow y = \pm(x-1)$ $\rightarrow y = x - 1$ and $y = -x + 1$ Finding y-intercepts of C, $\rightarrow y^2 - (0-1)^2 = 1$ $\rightarrow y^2 = 2$ $\rightarrow y = \pm\sqrt{2}$  	This question is purely on sketching conic graphs. Candidates who successfully find the required results in part **(a)** can proceed to manipulate the resulting equation $y^2 = x^k\mathrm{f}(x)$ to find the equation of a hyperbola, and from there start sketching accordingly. With the correct shape of curve, correct coordinates for all points of interest, and correct asymptote equations and positions, candidates are awarded all marks attributed to this question.

7 (a) [4]	Starting from the RHS, $2u_{n+1} - 5u_n$ $= 2\left[\frac{5}{2}(1+2\mathrm{i})^{n+1} + \frac{5}{2}(1-2\mathrm{i})^{n+1}\right] - 5\left[\frac{5}{2}(1+2\mathrm{i})^{n} + \frac{5}{2}(1-2\mathrm{i})^{n}\right]$ $= 5(1+2\mathrm{i})^{n+1} + 5(1-2\mathrm{i})^{n+1} - \frac{25}{2}(1+2\mathrm{i})^{n} - \frac{25}{2}(1-2\mathrm{i})^{n}$ $= (1+2\mathrm{i})^{n+2}\left[\frac{5}{1+2\mathrm{i}} - \frac{25}{2(1+2\mathrm{i})^2}\right] + (1-2\mathrm{i})^{n+2}\left[\frac{5}{1-2\mathrm{i}} - \frac{25}{2(1-2\mathrm{i})^2}\right]$ $= \frac{5}{2}(1+2\mathrm{i})^{n+2} + \frac{5}{2}(1-2\mathrm{i})^{n+2}$ $= u_{n+2}$, which equals the LHS. Using calculator to find the two base cases, $u_1 = 1, u_2 = 5$. By observation, → Given that u_1 and u_2 are real, u_3 is real. → Given that u_2 and u_3 are real, u_4 is real. → In general, given that u_n and u_{n+1} are real, then u_{n+2} is real. $\therefore u_n$ is real for all n	Despite the appearance of complex numbers, this question mainly tests more on recurrence relations. For the show part, it is arguably more intuitive to start from the right-hand side for greater ease. Note that the use of calculator is not barred by the question, and therefore can be exploited to evaluate complex operations in the intermediary steps. To find the base cases, the use of calculator is again allowed. Note that the question demands two base cases, and thus the correct values for both u_1 and u_2 are required for the mark. When proving u_n is real for all n, note the keyword "hence" which expects the use of the given recurrence relation and the found base cases. The proof should be intuitive after discovering that u_1 and u_2 are real.

7 (b) [2]	Using $u_{n+2} = 2u_{n+1} - 5u_n$, $\rightarrow \dfrac{u_{n+2}}{u_{n+1}} = 2 - 5\dfrac{u_n}{u_{n+1}}$ As $n \to \infty, \dfrac{u_{n+2}}{u_{n+1}} \to r$ and $\dfrac{u_{n+1}}{u_n} \to r$ $\rightarrow r = 2 - \dfrac{5}{r}$ $\rightarrow r^2 = 2r - 5$ $\rightarrow r^2 - 2r + 5 = 0$ $\therefore a = -2, b = 5$	This question tests on candidates' ability to utilise the property of a recurrence relation for infinite iterations, in particular recognising that when such property applies to $\frac{u_{n+1}}{u_n}$, the same also applies to $\frac{u_{n+2}}{u_{n+1}}$. The manipulation necessary in this question requires the recognition that $\frac{u_n}{u_{n+1}} = \frac{1}{r}$. (Without explicit mention, it should be clear that $r \neq 0$.)
7 (c) [2]	Finding the roots of $r^2 - 2r + 5 = 0$, $\rightarrow r = \dfrac{2 \pm \sqrt{4-20}}{2} = \dfrac{2 \pm \sqrt{-16}}{2} = \dfrac{2 \pm 4\mathrm{i}}{2} = 1 \pm 2\mathrm{i}$ $\rightarrow \alpha, \beta = 1 \pm 2\mathrm{i}$ $\therefore \alpha^m + \beta^m = (1+2\mathrm{i})^m + (1-2\mathrm{i})^m = \dfrac{2}{5}u_m$ Using the recursion, $u_3 = 2(5) - 5(1) = 5 \rightarrow \alpha^3 + \beta^3 = 2$ $u_4 = 2(5) - 5(5) = -15 \rightarrow \alpha^4 + \beta^4 = -6$ $\therefore$ Required minimum $m = 4$, and required value $\alpha^4 + \beta^4 = -6$	Candidates who are successful in finding the correct constants in part **(b)** should easily find the resemblance between $\alpha^m + \beta^m$ and u_m and show the required result accordingly. Again, the use of calculator is allowed in this part. Candidates may choose to key in the given general formula or the recurrence relation in **(a)**, or even proceed manually, to eventually find that the first m which yields $\alpha^m + \beta^m < 0$ is $m = 4$. As an aside, this 3-part recurrence relation question teases the concept of *second order homogenous linear recurrence relation* ($u_{n+2} = 2u_{n+1} - 5u_n$) and the use of its *characteristic equation* ($r^2 - 2r + 5 = 0$) to find its *general formula* ($u_n = \frac{5}{2}(1+2\mathrm{i})^n + \frac{5}{2}(1-2\mathrm{i})^n$). All of these are topics under the H2 Further Mathematics syllabus that can be extended from H2 Mathematics.

8 (a) [1]	The equation $z^5 - 1 = 0$ has all real coefficients $\rightarrow z_1, z_2, z_3, z_4$ and z_5 come in conjugates $\rightarrow$ Any point in $\{Z_1, Z_2, Z_3, Z_4, Z_5\}$ can be reflected about the real axis to give a point also in that set. $\rightarrow$ The regular polygon is symmetrical about the real axis.	This question tests on the conjugate root theorem. While subtle, it must be recognised that $z^5 = 1$ is an equation with all real coefficients; it is in fact the only piece of information that implies conjugacy and naturally explains the reflective symmetry.
8 (b) [2]	The root z_3 is real, and $z_3 = 1$ Since $Z_1Z_2Z_3Z_4Z_5$ is a regular polygon, $\rightarrow$ Each complex root differs in argument from the next by $\frac{2\pi}{n}$, where n is the number of sides ($n = 5$) $\rightarrow$ It is also given that $\arg(z_3) = 0$ and $n = 5$ $\therefore \arg(z_4) = \frac{2\pi}{5}$	Since it is established that z_1, z_2, z_3, z_4 and z_5 are ordered in increasing arguments, and that from part **(a)** these complex roots also come in conjugates, it should follow that the real root is exactly in the middle of the sequence, which is z^3. Only then can the remaining roots "wing out" from z^3 in an Argand diagram for each to have an assignable conjugate. Having discovered $z_3 = 1$, the argument of z_4 can be easily visualised and explained by means of geometry.
8 (c) [2]	**<u>Method 1: Using cosine rule</u>** $\lvert z_4 - z_3\rvert^2$ $= \lvert z_3\rvert^2 + \lvert z_4\rvert^2 - 2\lvert z_3\rvert\lvert z_4\rvert \cos\frac{2\pi}{5}$ $= 1^2 + 1^2 - 2(1)(1)\left[\frac{\left(\frac{-1+\sqrt{5}}{4}\right)}{1}\right]$ $= 2 - \left(\frac{-1+\sqrt{5}}{2}\right)$ $= \frac{5-\sqrt{5}}{2}$ $\therefore \lvert z_4 - z_3\rvert = \sqrt{\left(\frac{5-\sqrt{5}}{2}\right)}$	This question assesses on the concept of modulus as applied on complex numbers. The first suggested method is recommended for candidates who are more visually inclined. It makes use of the cosine rule, the angle between OZ_3 and OZ_4 as can be deduced from **(b)**, and the fact that the modulus of all roots of $z^5 = 1$ is 1. The correct exact value for k should eventually be found with careful working.

	[Continued] **<u>Method 2: Using modulus</u>** $\lvert z_4 - z_3\rvert$ $= \left\lvert\frac{1}{4}(-1+\sqrt{5}) + \frac{1}{4}\sqrt{(10+2\sqrt{5})}\mathrm{i} - 1\right\rvert$ $= \frac{1}{4}\left\lvert(-5+\sqrt{5}) + \sqrt{(10+2\sqrt{5})}\mathrm{i}\right\rvert$ $= \frac{1}{4}\sqrt{\left\{(-5+\sqrt{5})^2 + \left(\sqrt{(10+2\sqrt{5})}\right)^2\right\}}$ $= \frac{1}{4}\sqrt{(30-10\sqrt{5}+10+2\sqrt{5})}$ $= \frac{1}{4}\sqrt{(40-8\sqrt{5})}$ $= \sqrt{\left(\frac{40-8\sqrt{5}}{16}\right)}$ $= \sqrt{\left(\frac{5-\sqrt{5}}{2}\right)}$	The second suggested method directly evaluates the given modulus, having confirmed in **(b)** that $z_3 = 1$. While candidates stronger on arithmetic may opt to use this method, still this may take longer than the first suggested method as in here candidates require greater attention to correctly evaluate all resulting surd forms.
8 **(d)** **[2]**	Required area $= 5\left[\frac{1}{2}(1)(1)\sin\frac{2\pi}{5}\right]$ $= \frac{5}{2}\left[\frac{\frac{1}{4}\sqrt{(10+2\sqrt{5})}}{1}\right]$ $= \frac{5}{8}\sqrt{(10+2\sqrt{5})}$ units^2	This question summons the use of $\arg(z_4) = \frac{2}{5}\pi$ and the sine formula for the area of a triangle. Candidates are further required to deduce $\sin\frac{2\pi}{5}$ exactly, which should be a quick endeavour for those who opt for the first method in part **(c)**, having established the required lengths and angle.

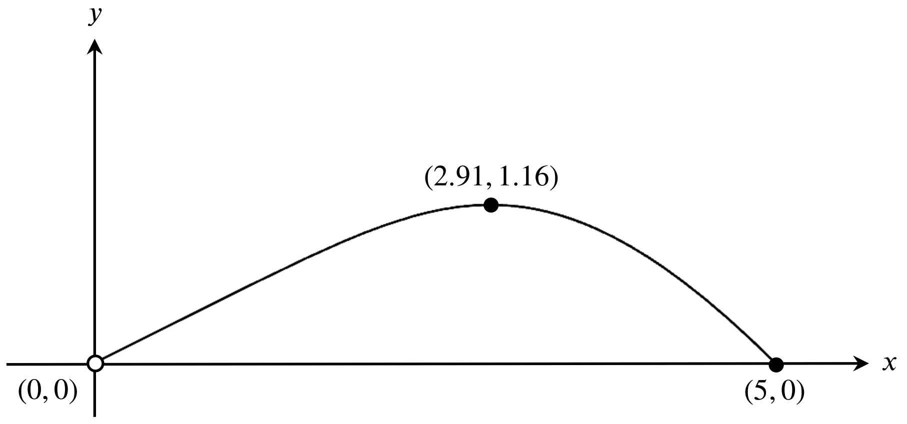
8 (e) [2]		This question tests on geometrical effects of operations to complex numbers on an Argand diagram. Upon discovering that $z^5 = -243\mathrm{i}$ can be written as $\left(\frac{\mathrm{i}z}{3}\right)^5 = 1$, candidates should be able to produce a diagram showing the transformed points. As a side, this 5-part question teases the concept of *n*th *roots of unity*, which are roots of the equation $z^n = 1$. This topic is not explicitly within the syllabus of H2 Mathematics or (surprisingly) H2 Further Mathematics. Candidates may wish to check out similar questions that appeared in past H2 Mathematics papers: • 2015/P1/Q9(b)

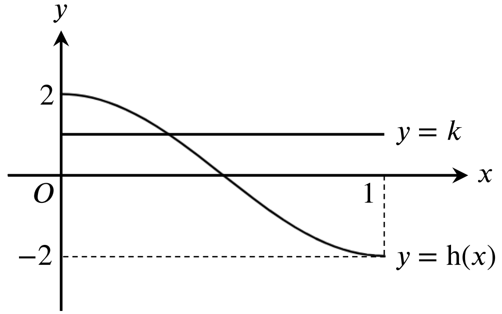
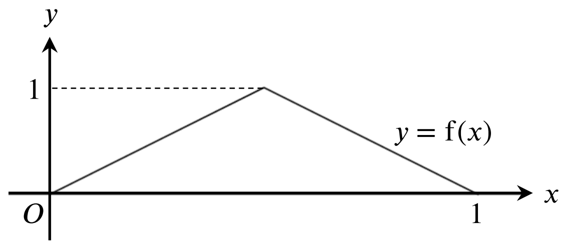
9 (a) [3]	$\dfrac{\mathrm{d}y}{\mathrm{d}\theta} = 2\cos\theta - 2\cos 2\theta$ $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = -2\sin\theta - 2\sin 2\theta$ $\therefore \dfrac{\mathrm{d}y}{\mathrm{d}x}$ $= \dfrac{2\cos\theta - 2\cos 2\theta}{-2\sin\theta - 2\sin 2\theta}$ $= \dfrac{2\cos\theta - 2\left(2\cos^2\theta - 1\right)}{-2\sin\theta - 2(2\sin\theta\cos\theta)}$ $= \dfrac{\cos\theta - 2\cos^2\theta + 1}{-\sin\theta - 2\sin\theta\cos\theta}$ $= \dfrac{(\cos\theta - 1)(-2\cos\theta - 1)}{\sin\theta\,(-2\cos\theta - 1)}$ $= \dfrac{\cos\theta - 1}{\sin\theta}$	This question assesses on differentiation as applied to parametric curves. The differentiation to be executed should prove manageable, and any outstanding challenge in this question, if any, would lie in the trigonometric manipulation to achieve the desired form. With careful working, candidates eventually obtain that $a = -1$.
9 (b) [2]	For a point with parameter θ, the tangent equation is given by: $y - (2\sin\theta - \sin 2\theta) = \dfrac{\cos\theta - 1}{\sin\theta}\left(x - (2\cos\theta + \cos 2\theta)\right)$ $\rightarrow y\sin\theta - 2\sin^2\theta + \sin 2\theta\sin\theta = (\cos\theta - 1)(x - 2\cos\theta - \cos 2\theta)$ $\rightarrow y\sin\theta - 2\sin^2\theta + \sin 2\theta\sin\theta = (\cos\theta - 1)x - 2\cos^2\theta + 2\cos\theta - \cos 2\theta\cos\theta + \cos 2\theta$ $\rightarrow y\sin\theta + 2(\cos^2\theta - \sin^2\theta) + \cos 2\theta\cos\theta + \sin 2\theta\sin\theta = (\cos\theta - 1)x + 2\cos\theta + \cos 2\theta$ $\rightarrow y\sin\theta + 2\cos 2\theta + \cos(2\theta - \theta) = (\cos\theta - 1)x + 2\cos\theta + \cos 2\theta$ $\rightarrow y\sin\theta + \cos 2\theta = (\cos\theta - 1)x + \cos\theta$	This question mainly tests on tangent equations of a parametric curve. It also however calls for even greater rigour in trigonometric manipulation than part **(a)** and thus requires due attentiveness. A mark is awarded each for writing the correct initial tangent equation and showing **<u>all</u>** necessary trigonometric manipulation to obtain the final equation. Candidates may wish to check out similar questions that appeared in past H2 Mathematics papers: • 2010/P1/Q11(i)

9 (c) [4]	Given that $q = p + \pi$, $\left(\frac{\cos p - 1}{\sin p}\right)\left(\frac{\cos q - 1}{\sin q}\right)$ $= \left(\frac{\cos p - 1}{\sin p}\right)\left(\frac{\cos(p+\pi) - 1}{\sin(p+\pi)}\right)$ $= \left(\frac{\cos p - 1}{\sin p}\right)\left(\frac{-\cos p - 1}{-\sin p}\right)$ $= \frac{-\cos^2 p + 1}{-\sin^2 p} = \frac{\sin^2 p}{-\sin^2 p} = -1$ $\therefore$ The tangent at P and the tangent at Q are indeed perpendicular to one another. Tangent at P: $\rightarrow y\sin p + \cos 2p = (\cos p - 1)x + \cos p$ Tangent at Q: $\rightarrow y\sin q + \cos 2q = (\cos q - 1)x + \cos q$ $\rightarrow y\sin(p+\pi) + \cos(2p + 2\pi) = (\cos(p+\pi) - 1)x + \cos(p+\pi)$ $\rightarrow -y\sin p + \cos 2p = (-\cos p - 1)x - \cos p$ Eliminating $y\sin p$ by adding the two tangent equations, $\rightarrow 2\cos 2p = -2x$ $\therefore x = -\cos 2p$ Substituting $x = \cos 2p$ into the tangent equation of P, $\rightarrow y\sin p + \cos 2p = (\cos p - 1)(-\cos 2p) + \cos p$ $\rightarrow y\sin p + \cos 2p = -\cos 2p\cos p + \cos 2p + \cos p$ $\rightarrow y\sin p + \cos 2p = -\cos p\left(1 - 2\sin^2 p\right) + \cos 2p + \cos p$ $\rightarrow y\sin p + \cos 2p = -\cos p + 2\sin^2 p\cos p + \cos 2p + \cos p$ $\rightarrow y\sin p = 2\sin^2 p\cos p$ $\therefore y = 2\sin p\cos p = \sin 2p$	Despite the many demands, this question mainly tests on tangents of a parametric curve and the intersection of two tangent lines. Firstly, the keyword "verify" commands candidates to use the given $q = p + \pi$ to obtain a result implying the perpendicular property – not to be misunderstood as going the opposite direction, which is instead demanded by keywords "show" or "proof". Candidates must also recognise that two lines are perpendicular if and only if their gradients multiply to -1; this knowledge is implicitly expected within H2 Mathematics. Some trigonometric manipulations are due in the verification. The subsequent part on finding the x-coordinate of R can be deduced by elimination after substituting $q = p + \pi$. The last part on finding the y-coordinate of R requires further trigonometric manipulation to obtain a similar simplified form. Candidates may wish to check out similar questions that appeared in past H2 Mathematics papers: • 2011/P1/Q3 • 2013/P1/Q11(i)-(ii) • 2014/P2/Q1(ii) • 2017/P2/Q1(ii)
9 (d) [2]	$x^2 + y^2 = \cos^2 2p + \sin^2 2p = 1$ R traces a circle with radius 1 unit and centre $(0, 0)$. R traces this circle in a clockwise direction.	With the correct coordinates in part **(c)**, the locus of R is easily defined. The unusual question on direction of trace can be deduced using GC, which draws parametric curves always from minimum to maximum parameter.

10 (a) [3]	Differentiating $\frac{\mathrm{d}x}{\mathrm{d}t} = 2y - 2x$ with respect to t, $\rightarrow \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 2\frac{\mathrm{d}y}{\mathrm{d}t} - 2\frac{\mathrm{d}x}{\mathrm{d}t}$ Given that $\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{\mathrm{d}x}{\mathrm{d}t} - 3y$, $\rightarrow \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 2\left(-\frac{\mathrm{d}x}{\mathrm{d}t} - 3y\right) - 2\frac{\mathrm{d}x}{\mathrm{d}t}$ $\rightarrow \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -4\frac{\mathrm{d}x}{\mathrm{d}t} - 6y$ $\rightarrow \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} = -6y$ Rearranging $\frac{\mathrm{d}x}{\mathrm{d}t} = 2y - 2x$ gives $2y = \frac{\mathrm{d}x}{\mathrm{d}t} + 2x$ and therefore $\rightarrow \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} = -3\left(\frac{\mathrm{d}x}{\mathrm{d}t} + 2x\right)$ $\rightarrow \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} = -3\frac{\mathrm{d}x}{\mathrm{d}t} - 6x$ $\rightarrow \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 7\frac{\mathrm{d}x}{\mathrm{d}t} + 6x = 0$ $\therefore a = 7, b = 6.$	This question assesses candidates' aptitude in manipulating differential equations to obtain results through elimination, substitution, and further differentiation. Like several A–Level questions, success in the first part is compulsory before candidates could even proceed any further to the latter parts, and the lack thereof will unfortunately inhibit progress in total. Admittedly, the second-order differential equation to be shown may appear daunting with exposure to only H2 Mathematics syllabus (less so with exposure to H2 Further Mathematics syllabus). Still, this should not hinder candidates at large from properly utilising mathematical tools within the H2 Mathematics syllabus to obtain full marks in this part.

10 (b) [6]	$x = Ae^{kt} + Be^{mt}$ $\rightarrow \frac{dx}{dt} = kAe^{kt} + mBe^{mt}$ $\rightarrow \frac{d^2x}{dt^2} = k^2Ae^{kt} + m^2Be^{mt}$ Substituting into the result in **(i)**, $\rightarrow k^2Ae^{kt} + m^2Be^{mt} + 7\left(kAe^{kt} + mBe^{mt}\right) + 6\left(Ae^{kt} + Be^{mt}\right) = 0$ $\rightarrow \left(k^2 + 7k + 6\right)Ae^{kt} + \left(m^2 + 7m + 6\right)Be^{mt} = 0$ $\rightarrow k^2 + 7k + 6 = 0$ and $m^2 + 7m + 6 = 0$ $\rightarrow (k+1)(k+6) = 0$ and $(m+1)(m+6) = 0$ $\rightarrow k = -1$ or $k = -6$ and $m = -1$ or $m = -6$ Since $k \neq m$, let $k = -1$ and $m = -6$, $\therefore x = Ae^{-t} + Be^{-6t}$ Finding y, $\rightarrow \frac{dx}{dt} = 2y - 2x$ $\rightarrow 2y = -Ae^{-t} - 6Be^{-6t} + 2Ae^{-t} + 2Be^{-6t}$ $\rightarrow y = \frac{1}{2}Ae^{-t} - 2Be^{-6t}$ When $t = 0$, $x = 5$ and $y = 0$ $\rightarrow 5 = A + B$ and $0 = \frac{1}{2}A - 2B$ $\rightarrow$ Using calculator to solve simultaneously, $A = 4$ and $B = 1$ $\therefore x = 4e^{-t} + e^{-6t}$ and $y = 2e^{-t} - 2e^{-6t}$	This question revolves around differential equation, but in truth puts greater emphasis on differentiation. Upon further differentiation of the given general expression for x in t followed by substitution into the result obtained and shown in **(a)**, two separate quadratic equation in k and m respectively can be obtained to give the same roots. Without loss of generality, candidates may assign either value to k and m so long as $k \neq m$ – in fact, the value of A and B follows suit with the respective value of k and m. With the decided values for k and m, candidates will end up with expressions for x and y in terms of A, B and t. Substituting initial conditions will yield a system of linear equations, which can be solved simultaneously either using a calculator or manually by hand, both of which are equally quick alternatives. Candidates may wish to check out similar questions that appeared in past H2 Mathematics papers: • 2013/P1/Q10(i)-(iii) • 2018/P1/Q10

10 (c) [4]	Sketching a curve with parametric equations $x = 4e^{-t} + e^{-6t}$ and $y = 2e^{-t} - 2e^{-6t}$:  At $t = 0$, $x = 5$ and $y = 0$. $\therefore$ The curve has an **inclusive** endpoint at $(5, 0)$. As $t \to \infty$, $x \to 0$ and $y \to 0$. $\therefore$ The curve has an **exclusive** endpoint at $(0, 0)$. Turning point occurs when $\frac{dy}{dx} = 0$ $\to \frac{dy}{dt} = 0$ $\to -2e^{-t} + 12e^{-6t} = 0$ $\to e^{5t} = 6$ $\to t = \frac{1}{5}\ln 6$ $\therefore$ The turning point has coordinates $(2.91, 1.16)$	This question assesses on graph sketching, in particular sketching a curve defined parametrically and the awareness of inclusive/exclusive endpoints. Points are awarded each for the following: • correct curve shape • correct indicators and coordinates for each endpoint • correct coordinates for the turning point To obtain the correct sketch of the curve, candidates must first obtain the correct parametric equation for x and y in terms of t – should candidates fail to achieve these from the previous part, there is unfortunately little chance of scoring well here. This question also requires candidates to appreciate the implicit fact that **graphs in terms of time t are only defined within the range $t \geq 0$**. Such implicit understanding is called for in many H2 Mathematics assessments, particularly in differential equation questions that asks candidates to sketch a related curve with respect to time. (See Specimen Paper/P1/Q11.) Candidates are also reminded that coordinates must be explicitly written as so wherever necessary. **Particularly, axial intercepts should not be indicated with only their values – so too should the origin O.** While it is commonly understood that in a graph O is synonymous with $(0, 0)$, it is better to be safe than sorry. Evaluating the turning point should prove routine for well-prepared candidates. Note that candidates are allowed to leave all coordinates in three significant figures, as no exact values are required by the question.

11 (a) (i) [1]	 From the sketch above, the line $y = k$, where $k \in \mathbb{R}$, intersects the graph of $y = \mathrm{h}(x)$ at most once. $\rightarrow$ h is a one-one function $\therefore$ h^{-1} exists.	This question tests on the condition for the existence of an inverse function. The graph of $y = \mathrm{h}(x)$ and the required (regurgitative) explanation are both required for one whole mark in this part.
11 (a) (ii) [1]	$\mathrm{R}_{\mathrm{h}^{-1}} = \mathrm{D_h} = [0, 1]$ and $\mathrm{D_f} = [0, 1]$ $\rightarrow \mathrm{R}_{\mathrm{h}^{-1}} \subseteq \mathrm{D_f}$ $\therefore$ fh^{-1} exists	This question tests on the condition for the existence of a composite function. All relevant intervals can be easily deduced and must be correctly stated for the mark. Note that $\mathrm{fh}^{-1}(x)$ is not equivalent to $(\mathrm{fh})^{-1}(x)$.
11 (a) (iii) [2]	 Considering the mapping of domains, $\mathrm{R}_{\mathrm{h}^{-1}} = \mathrm{D_h} = [0, 1] \xrightarrow{\mathrm{f}} [0, 1]$ $\rightarrow \mathrm{R}_{\mathrm{fh}^{-1}} = [0, 1]$ $\rightarrow$ Since $\mathrm{D_h} = [0, 1]$, $\mathrm{R}_{\mathrm{fh}^{-1}} \subseteq \mathrm{D_h}$ $\therefore$ hfh^{-1} exists	This question also tests on the condition for the existence of a composite function. All relevant intervals should be correctly stated for one mark in this part. Another mark is awarded for sketching $y = \mathrm{f}(x)$ as compulsory evidence of deducing $\mathrm{R}_{\mathrm{fh}^{-1}} = [0, 1]$. Note that $\mathrm{hfh}^{-1}(x)$ is not equivalent to $(\mathrm{hfh})^{-1}(x)$.

11 (b) [5]	When $0 \le x \le \frac{1}{2}$, $\mathrm{f}(x) = 1 - 1 + 2x = 2x$ When $\frac{1}{2} < x \le 1$, $\mathrm{f}(x) = 1 - 2x + 1 = 2 - 2x$ Let $y = \mathrm{h}(x)$ $\rightarrow y = 2\cos(\pi x)$ $\rightarrow \cos^{-1}\left(\frac{y}{2}\right) = \pi x$ $\rightarrow x = \frac{1}{\pi}\cos^{-1}\left(\frac{y}{2}\right)$ $\therefore \mathrm{h}^{-1}(x) = \frac{1}{\pi}\cos^{-1}\left(\frac{x}{2}\right)$ When $0 \le \mathrm{h}^{-1}(x) \le \frac{1}{2}$ $\therefore \mathrm{hfh}^{-1}(x)$ $= 2\cos\left(2\cos^{-1}\left(\frac{x}{2}\right)\right)$ $= 2\left(2\cos^2\left(\cos^{-1}\left(\frac{x}{2}\right)\right) - 1\right)$ $= 4\left(\frac{x}{2}\right)^2 - 2$ $= x^2 - 2$ When $\frac{1}{2} < \mathrm{h}^{-1}(x) \le 1$ $\therefore \mathrm{hfh}^{-1}(x)$ $= 2\cos\left(2\pi - 2\cos^{-1}\left(\frac{x}{2}\right)\right)$ $= 2\cos\left(2\cos^{-1}\left(\frac{x}{2}\right)\right)$ $= x^2 - 2$, as before. $\therefore \mathrm{g}(x) = x^2 - 2$ $\mathrm{D_g} = \mathrm{D_{hfh^{-1}}} = \mathrm{D_{h^{-1}}} = \mathrm{R_h} = [-2, 2]$, as seen in part **(a)(i)**.	This part may prove challenging. To begin, candidates must first find an expression for $\mathrm{h}^{-1}(x)$, which appears in the composite function $\mathrm{g}(x) = \mathrm{hfh}^{-1}(x)$. The part on writing $\mathrm{f}(x)$ as two separate non-modulus functions in the given two domains requires the appreciation that the modulus function can be omitted in for each case so long as the proper signs are applied. Using relevant trigonometric identities, both functions from the respective domains can eventually be simplified to yield the same result $\mathrm{g}(x) = x^2 - 2$. Explaining the domain of g can be easily done by deducing the domain of the composite function it equals to, which turns out to equal the range of h. Should candidates have sketched the graph of $y = \mathrm{h}(x)$ in part **(a)(i)**, it should be straightforward to verify that the intervals defining $\mathrm{D_g}$ and $\mathrm{R_h}$ respectively are in fact one and the same.

11 **(c)** **[4]**	$\mathrm{P}(x)$ $= x^4 - 4x^2 + 2$ $= x^4 - 4x^2 + 4 - 2$ $= (x^2 - 2)^2 - 2$ $= (\mathrm{g}(x))^2 - 2$ $= \mathrm{gg}(x)$ By definition, $\mathrm{P}(x) = \mathrm{hQh}^{-1}(x)$ $\therefore \mathrm{hQh}^{-1}(x)$ $= \mathrm{P}(x)$ $= \mathrm{gg}(x)$ $= \mathrm{hfh}^{-1}\mathrm{g}(x)$ $= \mathrm{hfh}^{-1}\mathrm{hfh}^{-1}(x)$ $= \mathrm{hffh}^{-1}(x)$ $\therefore \mathrm{Q}(x)$ $= \mathrm{ff}(x)$ $= 1 - \lvert 2(1 - \lvert 2x - 1\rvert) - 1\rvert$ $= 1 - \lvert 2 - 2\lvert 2x - 1\rvert - 1\rvert$ $= 1 - \big\lvert 1 - \lvert 4x - 2\rvert\big\rvert$ $\mathrm{D_Q} = \mathrm{D_{ff}} = \mathrm{D_f} = [0, 1]$ $\therefore \mathrm{Q}(x) = 1 - \big\lvert 1 - \lvert 4x - 2\rvert\big\rvert,\ 0 \le x \le 1$	This question may also prove challenging. This question mainly tests candidates on cancellation laws for function composition, in which the identities $\mathrm{ff}^{-1}(x) = x$ and $\mathrm{f}^{-1}\mathrm{f}(x) = x$ are used. Before any cancellation can even begin, candidates must first and foremost obtain the result $\mathrm{P}(x) = \mathrm{gg}(x)$, which should already be expected upon completing **(b)**. Afterwards, substituting $\mathrm{g}(x) = \mathrm{hfh}^{-1}(x)$ to the composite function $\mathrm{gg}(x)$ will yield the identity $\mathrm{hQh}^{-1}(x) = \mathrm{hffh}^{-1}(x)$, at which point it is acceptable to directly deduce that $\mathrm{Q}(x) = \mathrm{ff}(x)$. As the question asks for the topological conjugate "in a similar form", the final answer cannot be simply left as $\mathrm{ff}(x)$ and therefore must be further simplified. Note that the domain of Q must also be stated. As an aside, topological conjugacy demonstrates the concept of cancellation property as seen in *group theory*, which is a branch of mathematics that explores algebraic structures. Considering its abstract nature, group theory is quite the reach from any JC mathematics syllabus. Still, the use of cancellation laws is evidently prevalent in mathematics, ranging from H2 Mathematics in composition cancellation, to H2 Further Mathematics in square matrix diagonalisation, and even further beyond.

12 (a) [2]	Finding two points on the satellite orbit, $\rightarrow t = \frac{1}{4}\pi, \mathbf{q_1} = \frac{3}{2}\cos\left(\frac{1}{2}\pi\right)\mathbf{i} + 3\sin\left(\frac{1}{2}\pi\right)\mathbf{j} + \frac{3\sqrt{3}}{2}\cos\left(\frac{1}{2}\pi\right)\mathbf{k} = 3\mathbf{j}$ $\rightarrow t = \frac{7}{4}\pi, \mathbf{q_2} = \frac{3}{2}\cos(2\pi)\mathbf{i} + 3\sin(2\pi)\mathbf{j} + \frac{3\sqrt{3}}{2}\cos(2\pi)\mathbf{k} = \frac{3}{2}\mathbf{i} + \frac{3\sqrt{3}}{2}\mathbf{k}$ $\therefore$ Normal of the satellite orbit $= \begin{pmatrix}0\\3\\0\end{pmatrix} \times \begin{pmatrix}3/2\\0\\3\sqrt{3/2}\end{pmatrix} = \begin{pmatrix}9\sqrt{3/2}\\0\\-9/2\end{pmatrix} \rightarrow \begin{pmatrix}\sqrt{3}\\0\\-1\end{pmatrix}$	This question is leaning towards assessing three-dimensional vectors, modified to include parameters. This preamble question aims to highlight that the given parametric vector gives a two-dimensional orbit, which would therefore lie on a plane. Since the orbit is centred at the origin, the plane would also contain it, and therefore with two more points on the orbit the normal of the planar orbit can be determined.
12 (b) [2]	<u>**Method 1: Using the vector position of the starspot**</u> Since the starspot orbit has no **k**-component, $\rightarrow$ plane of starspot orbit : $z = 0$ $\rightarrow$ normal of the starspot orbit $= \begin{pmatrix}0\\0\\1\end{pmatrix}$ <u>**Method 2: Using two distinct points on the orbit (similar to (a))**</u> Finding two points on the starspot orbit, $\rightarrow t = 0, \mathbf{p_1} = \cos(0)\mathbf{i} + \sin(0)\mathbf{j} = \mathbf{i}$ $\rightarrow t = \frac{\pi}{2}, \mathbf{p_2} = \cos\left(\frac{\pi}{2}\right)\mathbf{i} + \sin\left(\frac{\pi}{2}\right)\mathbf{j} = \mathbf{j}$ $\therefore$ Normal of the plane of the starspot orbit $= \begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\-1\end{pmatrix} \rightarrow \begin{pmatrix}0\\0\\1\end{pmatrix}$ There is no real k for which $\begin{pmatrix}0\\0\\1\end{pmatrix} = k\begin{pmatrix}\sqrt{3}\\0\\-1\end{pmatrix}$ $\rightarrow$ The normal of the starspot orbit is not parallel to that of the satellite orbit $\therefore$ The two orbits are contrasting because they are not coplanar.	Candidates may use similar deduction as done in **(a)** to find the normal of the starspot orbit, which is most natural. As there is no “hence” keyword, a much faster method is available upon considering that the **k**-component of the starspot orbit is 0, which directly gives the cartesian equation of the orbital plane. The subsequent part tests on relationships in three-dimensional space, in particular the concept of coplanar lines and/or the relationship between two planes. For a positive marking, evidence of such awareness should manifest through specific keywords such as “coplanar”, “normal” and “parallel” in the explanation the contrast between the two orbits.

12 (c) [3]	$\cos\theta = \dfrac{\mathbf{p}\cdot\mathbf{q}}{\lvert\mathbf{p}\rvert\lvert\mathbf{q}\rvert}$ $= \dfrac{\frac{3}{2}\cos\left(t+\frac{1}{4}\pi\right)\cos(t) + 3\sin\left(t+\frac{1}{4}\pi\right)\sin(t)}{\left\lvert\sqrt{\cos^2 t+\sin^2 t}\right\rvert\left\lvert\sqrt{\frac{9}{4}\cos^2\left(t+\frac{1}{4}\pi\right)+9\sin^2\left(t+\frac{1}{4}\pi\right)+\frac{27}{4}\cos^2\left(t+\frac{1}{4}\pi\right)}\right\rvert}$ $= \dfrac{\frac{3}{2}\left[\cos\left(t+\frac{1}{4}\pi\right)\cos(t)+\sin\left(t+\frac{1}{4}\pi\right)\sin(t)\right]+\frac{3}{2}\sin\left(t+\frac{1}{4}\pi\right)\sin(t)}{\left\lvert\sqrt{\cos^2 t+\sin^2 t}\right\rvert\left\lvert\sqrt{9\cos^2\left(t+\frac{1}{4}\pi\right)+9\sin^2\left(t+\frac{1}{4}\pi\right)}\right\rvert}$ $= \dfrac{\frac{3}{2}\cos\left(t+\frac{1}{4}\pi-t\right)+\frac{3}{2}\left(-\frac{1}{2}\right)\left[\cos\left(2t+\frac{1}{4}\pi\right)-\cos\left(\frac{1}{4}\pi\right)\right]}{\left\lvert\sqrt{1}\right\rvert\left\lvert\sqrt{9}\right\rvert}$ $= \frac{1}{3}\left[\frac{3}{2}\left(\frac{\sqrt{2}}{2}\right)-\frac{3}{4}\cos\left(2t+\frac{1}{4}\pi\right)+\frac{3}{4}\left(\frac{\sqrt{2}}{2}\right)\right]$ $= \frac{3\sqrt{2}}{8}-\frac{1}{4}\cos\left(2t+\frac{1}{4}\pi\right)$	This question may prove challenging. The sheer complexity of the trigonometric functions may appear insurmountable for the easily deterred – regardless, the vector portion being assessed is rather straightforward. Candidates should very well be familiar with associating vector angles with dot products. Despite the different coefficients, candidates may find the form $\cos A\cos \boldsymbol{B}+\sin A\sin \boldsymbol{B}$ in the nominator leading towards cosine of subtracted angles. Splitting the sine product $\sin A\sin \boldsymbol{B}$ to obtain a more workable bracketing leads to the use of the extra formula to yield the angle $2t+\frac{1}{4}\pi$, which appears in the result to be shown. Meanwhile, vector moduli in the denominator can be found through simpler trigonometric identities. As an aside, note that the List of Formulae and Result (MF27) removes the sum-to-product formulae that used to appear in the previous List of Formulae (MF26).
12 (d) [3]	For $0\le\theta\le\pi$, given that $\theta\le\frac{1}{4}\pi$, $\cos\theta\ge\frac{\sqrt{2}}{2}$ $\rightarrow \frac{3\sqrt{2}}{8}-\frac{1}{4}\cos\left(2t+\frac{1}{4}\pi\right)\ge\frac{\sqrt{2}}{2}$ $\rightarrow -\frac{1}{4}\cos\left(2t+\frac{1}{4}\pi\right)\ge\frac{\sqrt{2}}{8}$ $\rightarrow \cos\left(2t+\frac{1}{4}\pi\right)\le-\frac{\sqrt{2}}{2}$ Considering principal values where equality holds, $\rightarrow 2t+\frac{1}{4}\pi=\frac{3}{4}\pi,\frac{5}{4}\pi,\frac{11}{4}\pi,\frac{13}{4}\pi$ $\rightarrow t=\frac{1}{4}\pi,\frac{1}{2}\pi,\frac{5}{4}\pi,\frac{3}{2}\pi$ Using GC, required range: $\frac{1}{4}\pi\le t\le\frac{1}{2}\pi$ or $\frac{5}{4}\pi\le t\le\frac{3}{2}\pi$	Having shown and having been given the relationship between θ and t in **(c)**, candidates need not do much in this question, apart from solving inequalities. The sketch of the cosine graph is not necessary for the mark but is nonetheless welcomed, and in fact encouraged should it be helpful for candidates.

12 (e) [4]	Implicitly differentiating with respect to t, $\rightarrow -\sin\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}t}=\dfrac{1}{2}\sin\left(2t+\dfrac{1}{4}\pi\right)$ $\rightarrow -\sin\theta\dfrac{\mathrm{d}^2\theta}{\mathrm{d}t^2}-\cos\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}t}=\cos\left(2t+\dfrac{1}{4}\pi\right)$ Setting $\dfrac{\mathrm{d}\theta}{\mathrm{d}t}=0$ for minimum θ, $\rightarrow \dfrac{1}{2}\sin\left(2t+\dfrac{1}{4}\pi\right)=0$ $\rightarrow 2t+\dfrac{1}{4}\pi=\ \pi,2\pi,3\pi,4\pi$ Considering second derivative at minimum θ, $\rightarrow \dfrac{\mathrm{d}^2\theta}{\mathrm{d}t^2}=\dfrac{-\cos\left(2t+\frac{1}{4}\pi\right)}{\sin\theta}$ $\rightarrow$ Given that $0\le\theta\le\pi,\sin\theta\ge 0\rightarrow$ minimum θ occurs when $\cos\left(2t+\dfrac{1}{4}\pi\right)<0$ $\rightarrow 2t+\dfrac{1}{4}\pi=\pi,3\pi$ $\therefore t=\dfrac{3}{8}\pi,\dfrac{11}{8}\pi$ Required distance $=\left\|\begin{pmatrix}\frac{3}{2}\cos\frac{5}{8}\pi-\cos\frac{3}{8}\pi\\ 3\sin\frac{5}{8}\pi-\sin\frac{3}{8}\pi\\ \frac{3\sqrt{3}}{2}\cos\frac{5}{8}\pi\end{pmatrix}\right\|\approx 2.31$ units	This question mainly assesses on standard maxima and minima problem as application of differentiation in vectors. As the question insists on "using calculus", candidates are expected to begin by differentiation to obtain turning points – otherwise, no positive marking can be granted. Given that candidates end up with several possible candidates for the exact orbital time t at which detection is most ideal, it should come as intuitive to rely on second differentiation to find which t values give minimum θ. With proper discussion, candidates should be able to obtain and prove the nature of these t values. The subsequent part on finding the distance at these values of t is routine and can be done very quickly using a calculator, though unaware candidates may virtuously end up calculating this distance manually, not realising that exact answers are not required.

12 (f) [1]	The perfect condition is not feasible. **<u>Method 1: Justify using θ to dispute collinearity</u>** Given $\cos\theta = \frac{3\sqrt{2}}{8} - \frac{1}{4}\cos\left(2t + \frac{1}{4}\pi\right)$, $\rightarrow \theta = \cos^{-1}\left(\frac{3\sqrt{2}}{8} - \frac{1}{4}\cos\left(2t + \frac{1}{4}\pi\right)\right)$ $\rightarrow$ Minimum $\theta \approx 0.676$ rad (GC) The extract states that the distance between the satellite and the starspot is the shortest possible when they are collinear with the centre of the star, which is when $\theta = 0$. However, $\theta = 0.676$ is the smallest possible θ at any orbital time, so the three points cannot be collinear. **<u>Method 2: Justify using position vectors to dispute collinearity</u>** Let $\mathbf{q} = m\mathbf{p}$, for some $m \in \mathbb{R}$. Note that $m > 1$, since the satellite is directly above the starspot. $\rightarrow \frac{3}{2}\cos\left(t + \frac{1}{4}\pi\right)\mathbf{i} + 3\sin\left(t + \frac{1}{4}\pi\right)\mathbf{j} + \frac{3\sqrt{3}}{2}\cos\left(t + \frac{1}{4}\pi\right)\mathbf{k} = m(\cos t\,\mathbf{i} + \sin t\,\mathbf{j})$ $\rightarrow$ From **k**–component**,** $m\cos\left(t + \frac{1}{4}\pi\right) = 0 \rightarrow t = \frac{1}{4}\pi, \frac{5}{4}\pi$ $(\because m > 1)$ $\rightarrow$ From **i**–component**,** substituting t gives $\cos t = \pm\frac{\sqrt{2}}{2} \neq 0 = \frac{3}{2}m\cos\left(t + \frac{1}{4}\pi\right)$ The extract states that the distance between the satellite and the starspot is the shortest possible when the satellite is directly above the starspot and that they are collinear with the centre of the star, which is when $\mathbf{q} = m\mathbf{p}$, where m is real and $m > 1$. However, such m does not exist, so the three points cannot be collinear. **<u>Method 3: Justify using orbital radius to dispute shortest possible distance</u>** As calculated in **(c)**, $\lvert\mathbf{p}\rvert = 1$ and $\lvert\mathbf{q}\rvert = 3$. The extract states that the distance between the satellite and the starspot is the shortest possible when a straight line can be drawn intersecting the two orbits simultaneously, which implies that the shortest distance would be $\lvert\mathbf{q}\rvert - \lvert\mathbf{p}\rvert = 2$ units. However, it is known from **(e)** that the shortest possible distance is 2.31 units, so the satellite cannot be directly above the starspot.	Despite the lengthy textual, the vector concept tested in this part is straightforward – collinearity. Candidates should very well be familiar with associating collinearity with parallel vectors and a common point to produce Method 1 or Method 2. Otherwise, Method 3 is acceptable (and in fact recommended as it makes use of previous results). This entire exercise on three-dimensional vectors aims to highlight that **<u>intersection does not imply coincidence</u>**. It is true that there is a straight line through O that can intersect both orbits simultaneously, but the objects themselves may not be at those intersection points on their orbit at the same time t. For better illustration, consider a point P on the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ and another point Q on the line $\mathbf{r} = \mathbf{b} + \mu\mathbf{d}$. • For the two lines to intersect, equating the two lines $\mathbf{a} + \lambda\mathbf{d} = \mathbf{b} + \mu\mathbf{d}$ should yield real (and consistent) solutions for λ and μ. This is expected knowledge under H2 Mathematics. • For points P and Q to coincide, the two lines must intersect **AND** $\lambda = \mu$. These parameters correspond to the incidence (or time, in this case) where the two points have the same position vector**.** This is not expected knowledge under H2 Mathematics. Candidates may wish to check out similar questions that appeared in past H2 Mathematics papers: • 2021/P1/Q8(c) • 2024/P1/Q12(d)