RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 5

C6A: Differentiation Techniques (Additional Practice)

1 Differentiate each of the following with respect to x:

(a) $(x-1)^2 \mathrm{e}^{2x}$

(b) $\dfrac{\cos\left(x+\frac{\pi}{4}\right)}{\sqrt{x}}$

(c) $\dfrac{1}{1+\sin x}$

(d) $\left(\ln\left(\dfrac{2}{x}\right)\right)\tan^{-1}(\sqrt{x})$

(e) $\ln\sqrt{\dfrac{1+x^2}{1-x^2}}$

(f) $\mathrm{e}^{x\ln x}$

(g) $\mathrm{e}^{\tan^{-1}\sqrt{3}x}$

(h) $x^2 \sec^3 2x$

(i) $\cot x \operatorname{cosec} 2x$

(j) $\sin^{-1}(\tan x)$

(k) $x^2 \cos^{-1}(\mathrm{e}^x)$

(l) $x^{\sin x}$

Solution:	
(a)	$\dfrac{\mathrm{d}}{\mathrm{d}x}\left[(x-1)^2 \mathrm{e}^{2x}\right] = 2(x-1)\mathrm{e}^{2x} + (x-1)^2 \mathrm{e}^{2x} \times 2$ $= 2(x-1)\mathrm{e}^{2x}(1+x-1)$ $= 2x(x-1)\mathrm{e}^{2x}$
(b)	$\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\dfrac{\cos(x+\frac{\pi}{4})}{\sqrt{x}}\right] = \dfrac{\sqrt{x}\left[-\sin\left(x+\frac{\pi}{4}\right)\right] - \cos\left(x+\frac{\pi}{4}\right)\left[\frac{1}{2}x^{-\frac{1}{2}}\right]}{\sqrt{x}^2}$ $= -\dfrac{1}{2(\sqrt{x})^3}\left[2x\sin\left(x+\frac{\pi}{4}\right) + \cos\left(x+\frac{\pi}{4}\right)\right]$

(c)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1+\sin x}\right)=-(1+\sin x)^{-2}\cos x$ $=-\frac{\cos x}{(1+\sin x)^2}$
(d)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(\frac{2}{x}\right)\tan^{-1}\sqrt{x}\right)=-\frac{1}{x}\tan^{-1}\sqrt{x}+\left(\ln\frac{2}{x}\right)\left(\frac{\frac{1}{2}x^{-\frac{1}{2}}}{1+\sqrt{x}^2}\right)$ $=-\frac{1}{x}\tan^{-1}\sqrt{x}+\frac{1}{2\sqrt{x}\left(1+x\right)}\ln\frac{2}{x}$
(e)	$\frac{\mathrm{d}}{\mathrm{d}x}\ln\sqrt{\frac{1+x^2}{1-x^2}}=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}[\ln(1+x^2)-\ln(1-x^2)]$ $=\frac{1}{2}\left[\frac{2x}{1+x^2}-\frac{(-2x)}{1-x^2}\right]$ $=\frac{2x}{1-x^4}$
(f)	$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x\ln x})=\mathrm{e}^{x\ln x}\left(1\cdot\ln x+x\cdot\frac{1}{x}\right)$ $=\mathrm{e}^{x\ln x}\left(1+\ln x\right)$
(g)	$\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{e}^{\tan^{-1}\sqrt{3}x}]=\mathrm{e}^{\tan^{-1}\sqrt{3}x}\left[\frac{1}{1+\left(\sqrt{3}x\right)^2}.\sqrt{3}\right]$ $=\frac{\sqrt{3}}{1+3x^2}\mathrm{e}^{\tan^{-1}\sqrt{3}x}$
(h)	$\frac{\mathrm{d}}{\mathrm{d}x}(x^2\sec^3 2x)=2x\sec^3 2x+x^2(3\sec^2 2x)(\sec 2x\tan 2x)(2)$ $=2x\sec^3 2x\left(1+3x\tan 2x\right)$
(i)	$\frac{\mathrm{d}}{\mathrm{d}x}(\cot x\,\mathrm{cosec}2x)=-\mathrm{cosec}^2 x\,\mathrm{cosec}2x+\cot x\left(-\mathrm{cosec}2x\cot 2x\right)(2)$ $=-\mathrm{cosec}2x\left(\mathrm{cosec}^2 x+2\cot x\cot 2x\right)$
(j)	$\frac{\mathrm{d}}{\mathrm{d}x}[\sin^{-1}(\tan x)]=\frac{\sec^2 x}{\sqrt{1-\tan^2 x}}$
(k)	$\frac{\mathrm{d}}{\mathrm{d}x}[x^2\cos^{-1}(\mathrm{e}^x)]=2x\cos^{-1}(\mathrm{e}^x)+x^2\left(-\frac{1}{\sqrt{1-(\mathrm{e}^x)^2}}.\mathrm{e}^x\right)$ $=x\left[2\cos^{-1}(\mathrm{e}^x)-\frac{x\mathrm{e}^x}{\sqrt{1-\mathrm{e}^{2x}}}\right]$
(l)	Let $y=x^{\sin x}$.

	Taking ln on both sides, $\ln y = (\sin x)\ln x$. Differentiate with respect to x, $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = (\cos x)(\ln x) + \frac{\sin x}{x}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = x^{\sin x}\left[\frac{\sin x}{x} + (\cos x)(\ln x)\right]$ Alternatively, $y = \mathrm{e}^{\ln x^{\sin x}} = \mathrm{e}^{(\sin x)\ln x}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{(\sin x)\ln x}\left[(\cos x)\ln x + (\sin x)(\frac{1}{x})\right] = x^{\sin x}\left[(\cos x)\ln x + \frac{\sin x}{x}\right]$

2 Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(x\sqrt{9-x^2} + 9\sin^{-1}\left(\frac{x}{3}\right)\right)$, leaving your answer in the form $a\sqrt{b-x^2}$, where the values of a and b are to be found.

Solution:
$\frac{\mathrm{d}}{\mathrm{d}x}(x\sqrt{9-x^2} + 9\sin^{-1}(\frac{x}{3})) = \sqrt{9-x^2} + x(\frac{1}{2})(9-x^2)^{-\frac{1}{2}}(-2x) + \frac{9}{\sqrt{9-x^2}}$ $= \sqrt{9-x^2} + \frac{9-x^2}{\sqrt{9-x^2}} = 2\sqrt{9-x^2}$ $\therefore a = 2,\ b = 9.$

3 For each of the following, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y.

(a) $x^3 + 2y^3 + 3xy = 0$

(b) $x + \tan\left(xy\right) = 0$

(c) $\cos(xy) - \ln(x+y) = 0$

Solution:	
(a)	$x^3 + 2y^3 + 3xy = 0$ Differentiating w.r.t. x, $3x^2 + 6y^2\frac{\mathrm{d}y}{\mathrm{d}x} + 3x\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = 0$ $(6y^2 + 3x)\frac{\mathrm{d}y}{\mathrm{d}x} = -3x^2 - 3y$ $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x^2 + y}{x + 2y^2}$

(b)	$x+\tan(xy)=0$ Differentiating w.r.t. x, $1+\sec^2(xy)\left[x\dfrac{\mathrm{d}y}{\mathrm{d}x}+y\right]=0$ $x\sec^2(xy)\dfrac{\mathrm{d}y}{\mathrm{d}x}=-1-y\sec^2(xy)$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{1+y\sec^2(xy)}{x\sec^2(xy)}$
(c)	$\cos(xy)-\ln(x+y)=0$ Differentiating w.r.t. x, $-\sin(xy)\left[x\dfrac{\mathrm{d}y}{\mathrm{d}x}+y\right]-\dfrac{1+\dfrac{\mathrm{d}y}{\mathrm{d}x}}{x+y}=0$ $-x(x+y)\sin(xy)\dfrac{\mathrm{d}y}{\mathrm{d}x}-y(x+y)\sin(xy)-1-\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{1+y(x+y)\sin(xy)}{1+x(x+y)\sin(xy)}$

4 JPJC Promo 9758/2021/Q3

It is given that $x^2y-\tan^{-1}y=\frac{3}{4}\pi$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. [4]

Hence, find the exact value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = 1$, given that $x > 0$. [3]

Solution:
$x^2y-\tan^{-1}y=\dfrac{3}{4}\pi$ Differentiate with respect to x, $x^2\dfrac{\mathrm{d}y}{\mathrm{d}x}+y(2x)-\dfrac{1}{1+y^2}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)=0$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}\left(\dfrac{1}{1+y^2}-x^2\right)=2xy\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2xy}{\left(\dfrac{1}{1+y^2}-x^2\right)}=\dfrac{2xy\left(1+y^2\right)}{1-x^2-x^2y^2}$.
when $y = 1$, $x^2(1)-\tan^{-1}1=\dfrac{3}{4}\pi\Rightarrow x^2-\dfrac{\pi}{4}=\dfrac{3}{4}\pi\Rightarrow x^2=\pi\Rightarrow x=\pm\sqrt{\pi}$. Since $x > 0$, $x=\sqrt{\pi}$. $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2\sqrt{\pi}\left(1+1^2\right)}{1-\pi-\pi(1)^2}=\dfrac{4\sqrt{\pi}}{1-2\pi}$ (exact).

5 SRJC/2007/1/Q5

(a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ if

(i) $y=\sqrt{\dfrac{3}{2x+1}}$

(ii) $y=\ln\left(\dfrac{x^2+x+1}{1+\cos x}\right)$

(b) Given that $y\mathrm{e}^y=(x-5y)^3$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y.

Solution:		
(a)	**(i)**	$y=\sqrt{\dfrac{3}{2x+1}}=\sqrt{3}(2x+1)^{-\frac{1}{2}}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{\sqrt{3}}{2}(2x+1)^{-\frac{3}{2}}(2)=-\dfrac{\sqrt{3}}{(2x+1)^{\frac{3}{2}}}$
	(ii)	$y=\ln\left(\dfrac{x^2+x+1}{1+\cos x}\right)=\ln(x^2+x+1)-\ln(1+\cos x)$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2x+1}{x^2+x+1}+\dfrac{\sin x}{1+\cos x}$
(b)	$y\mathrm{e}^y=(x-5y)^3$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}\mathrm{e}^y+y\mathrm{e}^y\dfrac{\mathrm{d}y}{\mathrm{d}x}=3(x-5y)^2\left(1-5\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)$ $\left[(1+y)\mathrm{e}^y+15(x-5y)^2\right]\dfrac{\mathrm{d}y}{\mathrm{d}x}=3(x-5y)^2$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{3(x-5y)^2}{(1+y)\mathrm{e}^y+15(x-5y)^2}$	

6 ACJC Promo 9758/2021/Q3 (modified)

A curve C has equation $\dfrac{x^2-4y^2}{x^2+xy^2+100}=\dfrac{1}{2}$, $x\in\mathbb{R}$, $x\neq-8$.

Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2x-y^2}{2xy+16y}$. [2]

Solution:
$2x^2-8y^2=x^2+xy^2+100$. Differentiate with respect to x, $4x-16y\dfrac{\mathrm{d}y}{\mathrm{d}x}=2x+2xy\dfrac{\mathrm{d}y}{\mathrm{d}x}+y^2\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2x-y^2}{2xy+16y}$ (shown).

7 If $\sin y = 2\sin x$, show that $\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1 + 3\sec^2 y$.

Solution:
If $\sin y = 2\sin x$, differentiating both sides with respect to x, $$\cos y\frac{\mathrm{d}y}{\mathrm{d}x} = 2\cos x$$ $$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\cos x \sec y$$ Squaring both sides, $$(\frac{\mathrm{d}y}{\mathrm{d}x})^2 = 4\cos^2 x\sec^2 y = 4(1-\sin^2 x)\sec^2 y = 4\sec^2 y - \frac{4\sin^2 x}{\cos^2 y}$$ $$= 4\sec^2 y - \frac{\sin^2 y}{\cos^2 y}, \quad \text{since } 2\sin x = \sin y$$ $$= 4\sec^2 y - \tan^2 y = 4\sec^2 y - (\sec^2 y - 1)$$ $$= 1 + 3\sec^2 y \quad \text{(Shown)}$$

8 **CJC Promo 9740/2013/Q4**

(a) Given that $y = \tan^{-1}\sqrt{x}$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. [2]

(b) Given that $\sqrt[x]{y} = \sqrt[y]{x}$, where $x > 0$, $y > 0$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. [4]

Solution:	
(a)	$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1}\sqrt{x}\right) = \dfrac{1}{1+(\sqrt{x})^2}\cdot\dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}(1+x)}$
(b)	$\sqrt[x]{y} = \sqrt[y]{x}$ Taking logarithm on both sides, $\dfrac{1}{x}\ln y = \dfrac{1}{y}\ln x$ $y\ln y = x\ln x$ Differentiating both sides, $y\left(\dfrac{1}{y}\right)\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{\mathrm{d}y}{\mathrm{d}x}\ln y = x\left(\dfrac{1}{x}\right) + \ln x$ $(1+\ln y)\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1 + \ln x$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1+\ln x}{1+\ln y}$

9 Given the curve with equation $x^2-4xy+3y^2+1=0$, find the gradient of the curve at the point where the y coordinate is 1.

Solution:
Differentiate $x^2-4xy+3y^2+1=0$ with respect to x, $$2x-(4y+4x\frac{\mathrm{d}y}{\mathrm{d}x})+6y\frac{\mathrm{d}y}{\mathrm{d}x}=0 \text{ --- (1)}$$ When $y=1$, the equation of the curve gives $x^2-4x+4=0$, so $x=2$. Put the values into (1) to get $$4-\left(4+4(2)\frac{\mathrm{d}y}{\mathrm{d}x}\right)+6\frac{\mathrm{d}y}{\mathrm{d}x}=0.$$ So $-2\frac{\mathrm{d}y}{\mathrm{d}x}=0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=0$. Hence the gradient at (2, 1) is 0.

10 **(a)** Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t for $x=t^2-\frac{8}{t^2}$, $y=\frac{t^2}{2}+\frac{8}{t^2}$, where $t\neq 0$.

(b) **DHS Mid Year 9758/2021/Q8 (modified)**

A curve C has parametric equations

$$x=\sin 2t, \quad y=\cos^2 t+1, \quad 0\le t\le \tfrac{\pi}{2}.$$

Express $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the form $a\tan bt$, where a and b are constants to be determined. [2]

(c) **MJC Promo 9740/2013/Q2**

The parametric equations of a curve C are $x=\sin^{-1}(1-t)$, $y=\mathrm{e}^{\sqrt{2t-t^2}}$.

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t. [4]

Solution:	
(a)	$x=t^2-\frac{8}{t^2}$, $\frac{\mathrm{d}x}{\mathrm{d}t}=2t+\frac{16}{t^3}$ $y=\frac{t^2}{2}+\frac{8}{t^2}$, $\frac{\mathrm{d}y}{\mathrm{d}t}=t-\frac{16}{t^3}$ $\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\times\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{t-\frac{16}{t^3}}{2t+\frac{16}{t^3}}$ $=\frac{t^4-16}{2t^4+16}$
	$\frac{\mathrm{d}x}{\mathrm{d}t}=2\cos 2t$, $\frac{\mathrm{d}y}{\mathrm{d}t}=-2\cos t\sin t$

	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-2\cos t\sin t}{2\cos 2t}=\frac{-\sin 2t}{2\cos 2t}=-\frac{1}{2}\tan 2t$ (where $a=-\frac{1}{2},\ b=2$).
	$x=\sin^{-1}(1-t)$ $\quad y=\mathrm{e}^{\sqrt{2t-t^2}}$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{\sqrt{1-(1-t)^2}}(-1)$ $\quad \frac{\mathrm{d}y}{\mathrm{d}t}=\mathrm{e}^{\sqrt{2t-t^2}}\,\frac{1}{2}\left(2t-t^2\right)^{-\frac{1}{2}}(2-2t)$ $\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{1}{\sqrt{2t-t^2}}$ $\quad \frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{e}^{\sqrt{2t-t^2}}(1-t)}{\sqrt{2t-t^2}}$ $\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\mathrm{e}^{\sqrt{2t-t^2}}(t-1)$