SUMMARY: Graphs and Transformations

1 Systematic Curve Sketching $y = \mathrm{f}(x)$

1. Intercepts	Let $x=0$, find y. Let $y=0$, find x.
2. Linear Asymptotes	If there exists a constant c such that $\mathrm{f}(x) \to +\infty$ (or $-\infty$) as $x \to c$, then the line $\boldsymbol{x = c}$ is a ***vertical asymptote*** of the graph $y = \mathrm{f}(x)$.
	If there exists a constant k such that $\mathrm{f}(x) \to k$ as $x \to +\infty$ (and/or $x \to -\infty$), then the line $\boldsymbol{y = k}$ is a ***horizontal asymptote*** of the graph $y = \mathrm{f}(x)$.
	If $y \to \mathrm{g}(x)$ as $x \to +\infty$ (and/or $x \to -\infty$), where $\mathrm{g}(x)$ is a linear function, then $\boldsymbol{y = \mathrm{g}(x)}$ is an ***oblique asymptote*** of the graph $y = \mathrm{f}(x)$.
3. Stationary Points $(a, \mathrm{f}(a))$	***First Derivative Test*** (see table below); ***Second Derivative Test*** (see table below)

First Derivative Test

x	a^-	a	a^+	Shape	Conclusion
$\frac{\mathrm{d}y}{\mathrm{d}x}$	$+$	0	$-$		$(a, \mathrm{f}(a))$ is a ***maximum*** point.
	$-$	0	$+$		$(a, \mathrm{f}(a))$ is a ***minimum*** point.
	$+$	0	$+$		$(a, \mathrm{f}(a))$ is a ***stationary point of inflexion***.
	$-$	0	$-$		$(a, \mathrm{f}(a))$ is a ***stationary point of inflexion***.

Second Derivative Test

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$	$+$	$(a, \mathrm{f}(a))$ is a ***minimum*** point.
	$-$	$(a, \mathrm{f}(a))$ is a ***maximum*** point.
	0	No conclusion. In this case the first derivative test must be done. This does **not** imply a point of inflexion.

2 Graphs

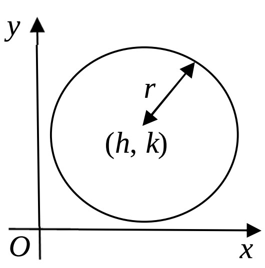
1. Conic Sections

Skill(s) used: Completing the square

Note: To label asymptotes!

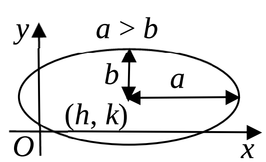
Circle

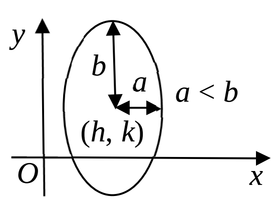
$(x-h)^2 + (y-k)^2 = r^2$



Ellipse

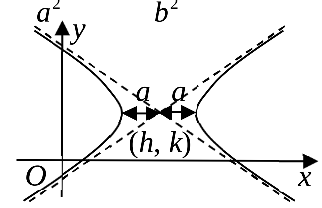
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$



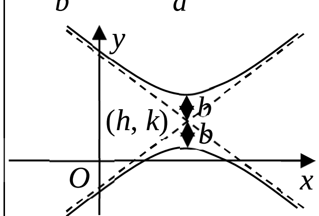


Hyperbola

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

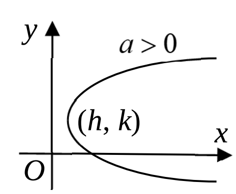


$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

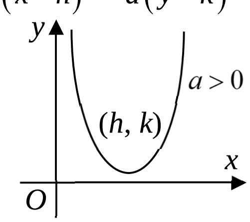


Parabola

$(y-k)^2 = a(x-h)$



$(x-h)^2 = a(y-k)$



2 Graphs (cont'd)

To obtain asymptotes of hyperbola: $\dfrac{(x-h)^2}{a^2}-\dfrac{(y-k)^2}{b^2}=\pm 1 \Leftrightarrow \dfrac{(y-k)^2}{b^2}=\dfrac{(x-h)^2}{a^2}\pm 1$

As $x\to\pm\infty$, $\dfrac{(y-k)^2}{b^2}\to\dfrac{(x-h)^2}{a^2}$ Thus $y-k\to\pm\dfrac{b}{a}(x-h)$ as $x\to\pm\infty$

The asymptotes of these hyperbolas are $y-k=\pm\dfrac{b}{a}(x-h)$.

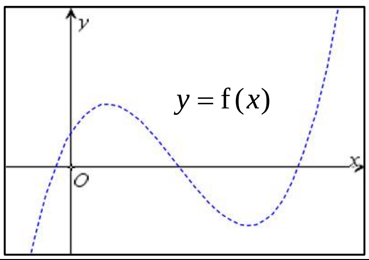
2. Modulus Functions

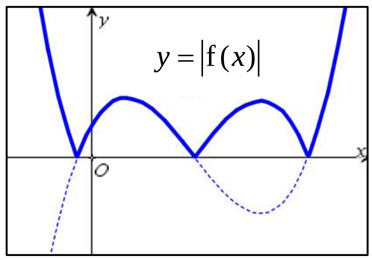
$$|\mathrm{f}(x)|=\begin{cases}\mathrm{f}(x), & \mathrm{f}(x)\geq 0\\ -\mathrm{f}(x), & \mathrm{f}(x)<0\end{cases}$$

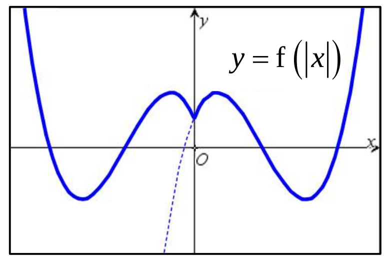
1. Keep the positive *y* portion(s) of $y=\mathrm{f}(x)$.
2. Reflect the negative *y* portion(s) of $y=\mathrm{f}(x)$ in the *x*-axis.
3. Delete the negative *y* portion(s) of $y=\mathrm{f}(x)$.

$$\mathrm{f}\left(|x|\right)=\begin{cases}\mathrm{f}(x), & x\geq 0\\ \mathrm{f}(-x), & x<0\end{cases}$$

1. Keep the positive *x* portion(s) of $y=\mathrm{f}(x)$.
2. Delete the negative *x* portion(s) of $y=\mathrm{f}(x)$.
3. Reflect the positive *x* portion(s) of $y=\mathrm{f}(x)$ in the *y*-axis.







3. Rational Functions

Skill(s) used: Long division

Let $\mathrm{f}(x)=\dfrac{\mathrm{P}(x)}{\mathrm{Q}(x)}$ where the degrees of $\mathrm{P}(x)$ and $\mathrm{Q}(x)$ are less than or equal to 2.

If degree of $\mathbf{P}(x)$ < degree of $\mathbf{Q}(x)$, then $\mathrm{f}(x)=\dfrac{\mathrm{P}(x)}{\mathrm{Q}(x)}$ is a *proper fraction.*

- Horizontal asymptote: *x*-axis (i.e. $y=0$)

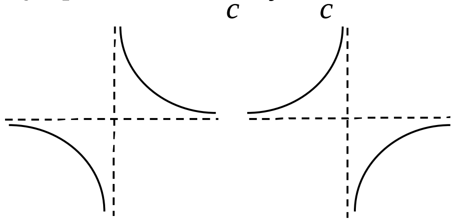
If degree of $\mathbf{P}(x)$ ≥ degree of $\mathbf{Q}(x)$, then $\mathrm{f}(x)=\dfrac{\mathrm{P}(x)}{\mathrm{Q}(x)}$ is an *improper fraction*.

- Long division $\Rightarrow$ $\mathrm{f}(x)=px+q+\dfrac{\mathrm{R}(x)}{\mathrm{Q}(x)}$, where $\dfrac{\mathrm{R}(x)}{\mathrm{Q}(x)}$ is a proper fraction
- $p=0 \Rightarrow$ horizontal asymptote $y=q$; $p\neq 0 \Rightarrow$ oblique asymptote $y=px+q$

In both cases, solve $\mathrm{Q}(x)=0$ to find vertical asymptote(s)

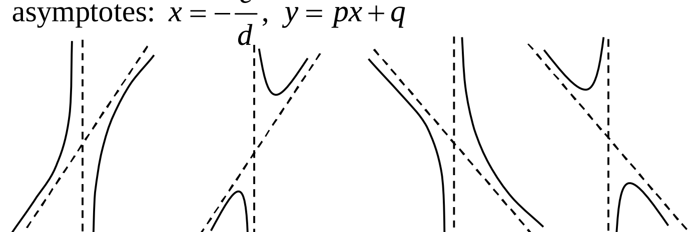
$y=\dfrac{ax+b}{cx+d}$

asymptotes: $x=-\dfrac{d}{c}$, $y=\dfrac{a}{c}$



$y=\dfrac{ax^2+bx+c}{dx+e}=\left(px+q\right)+\dfrac{r}{dx+e}$, $a\neq 0$

asymptotes: $x=-\dfrac{e}{d}$, $y=px+q$



3 Simple Transformations

The following table summarises simple transformations for the case where $a > 0$.

Transformation	Description of transformation	Geometrically
TRANSLATION		
$y = \mathrm{f}(x) \to y = \mathrm{f}(x-a)$ Replace x by $x - a$.	The graph of $y = \mathrm{f}(x-a)$ is the **translation** of the graph of $y = \mathrm{f}(x)$ **by *a* units in the positive *x*-direction**.	$(x, y) \to (x+a, y)$
$y = \mathrm{f}(x) \to y = \mathrm{f}(x+a)$ Replace x by $x + a$.	The graph of $y = \mathrm{f}(x+a)$ is the **translation** of the graph of $y = \mathrm{f}(x)$ **by *a* units in the negative *x*-direction**.	$(x, y) \to (x-a, y)$
$y = \mathrm{f}(x) \to y - a = \mathrm{f}(x)$ i.e. $y = \mathrm{f}(x) + a$ Replace y by $y - a$.	The graph of $y = \mathrm{f}(x) + a$ is the **translation** of the graph of $y = \mathrm{f}(x)$ **by *a* units in the positive *y*-direction**.	$(x, y) \to (x, y+a)$
$y = \mathrm{f}(x) \to y + a = \mathrm{f}(x)$ i.e. $y = \mathrm{f}(x) - a$ Replace y by $y + a$.	The graph of $y = \mathrm{f}(x) - a$ is the **translation** of the graph of $y = \mathrm{f}(x)$ **by *a* units in the negative *y*-direction**.	$(x, y) \to (x, y-a)$
SCALING		
$y = \mathrm{f}(x) \to y = \mathrm{f}\left(\frac{x}{a}\right)$ Replace x by $\frac{x}{a}$.	The graph of $y = \mathrm{f}\left(\frac{x}{a}\right)$ is the **scaling** of the graph of $y = \mathrm{f}(x)$ **by factor *a* parallel to the *x*-axis**.	$(x, y) \to (ax, y)$
$y = \mathrm{f}(x) \to \frac{y}{a} = \mathrm{f}(x)$ i.e. $y = a\mathrm{f}(x)$ Replace y by $\frac{y}{a}$.	The graph of $y = a\mathrm{f}(x)$ is the **scaling** of the graph of $y = \mathrm{f}(x)$ **by factor *a* parallel to the *y*-axis**.	$(x, y) \to (x, ay)$
REFLECTION		
$y = \mathrm{f}(x) \to y = \mathrm{f}(-x)$ Replace x by $-x$.	The graph of $y = \mathrm{f}(-x)$ is the **reflection** of the graph of $y = \mathrm{f}(x)$ **in the *y*-axis.**	$(x, y) \to (-x, y)$
$y = \mathrm{f}(x) \to y = -\mathrm{f}(x)$ Replace y by $-y$.	The graph of $y = -\mathrm{f}(x)$ is the **reflection** of the graph of $y = \mathrm{f}(x)$ **in the *x*-axis.**	$(x, y) \to (x, -y)$

4 Graph of $y = \dfrac{1}{f(x)}$

Graph of $y = f(x)$	Graph of $y = \dfrac{1}{f(x)}$
Let $b \neq 0$:	
$y = f(x)$ has an x-intercept at $x = a$. $y = f(x)$ has a vertical asymptote $x = a$.	$y = \dfrac{1}{f(x)}$ has a vertical asymptote $x = a$. $y = \dfrac{1}{f(x)}$ has an x-intercept $x = a$.
$y = f(x)$ has a y-intercept at $y = b$.	$y = \dfrac{1}{f(x)}$ has a y-intercept at $y = \dfrac{1}{b}$.
$y = f(x)$ has a horizontal asymptote $y = b$. (a) As $f(x) \to b^{+}$, (b) As $f(x) \to b^{-}$,	$y = \dfrac{1}{f(x)}$ has a horizontal asymptote $y = \dfrac{1}{b}$. (a) $\dfrac{1}{f(x)} \to \left(\dfrac{1}{b}\right)^{-}$. (b) $\dfrac{1}{f(x)} \to \left(\dfrac{1}{b}\right)^{+}$.
$f(x) \to +\infty$. $f(x) \to -\infty$.	$\dfrac{1}{f(x)} \to 0^{+}$. $\dfrac{1}{f(x)} \to 0^{-}$.
(a, b) is a maximum point. (a, b) is a minimum point.	$\left(a, \dfrac{1}{b}\right)$ is a minimum point. $\left(a, \dfrac{1}{b}\right)$ is a maximum point.
For checking:	
$f(x)$ increases. $f(x)$ decreases.	$\dfrac{1}{f(x)}$ decreases. $\dfrac{1}{f(x)}$ increases.
$f(x) > 0$. $f(x) < 0$.	$\dfrac{1}{f(x)} > 0$. $\dfrac{1}{f(x)} < 0$.
$y = f(x)$ has an oblique asymptote.	$y = \dfrac{1}{f(x)}$ has a horizontal asymptote $y = 0$.

4 Graph of $y = \dfrac{1}{\mathrm{f}(x)}$

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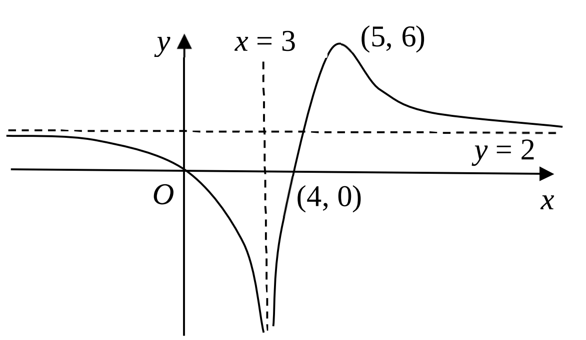

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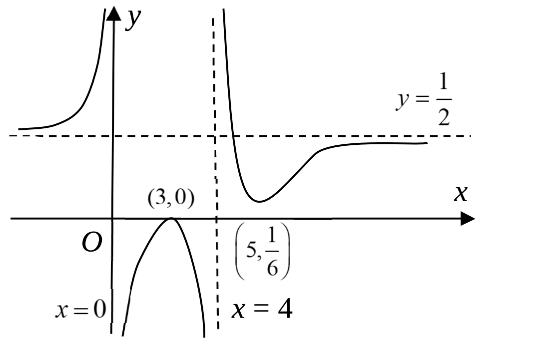

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5 Graphs of Parametric Equations $x = \mathrm{f}(t)$, $y = \mathrm{f}(t)$

1. Work with parametric equations

Key in the pair of parametric equations into GC and sketch directly via **PAR** mode.

1. Change calculator to PAR mode.
2. Ensure RADIANS mode.
3. Key in expressions for x and y.
4. Change window settings for range of t. Default window settings are: $t_{\min} = 0$, $t_{\max} = 2\pi$.

2. Convert to Cartesian equation

Convert parametric equations into Cartesian equation by eliminating the parameter t and use GC to sketch via **FUNC** mode.

Type I: Make t the subject using one parametric equation, then do substitution into the other parametric equation

Type II: Algebraic manipulations, e.g. squaring each parametric equation then adding/ subtracting the two equations; adding and subtracting both parametric equations then eliminate t; etc.

Type III: Use trigonometric identities

(a) $\sin^2\theta + \cos^2\theta = 1$

(b) $\tan^2\theta + 1 = \sec^2\theta$

(c) $\cot^2\theta + 1 = \mathrm{cosec}^2\theta$ and others in MF27

3. Find the point(s) of intersection of a curve (parametric) and a line (Cartesian)

Substitute the pair of parametric equations into the Cartesian equation to solve for the parameter t. Then obtain the values of x and y from the parametric equations using the value(s) of t found.

Curve: $x = 3t + 2$ and $y = t + \dfrac{3}{t}$ ----- (1)

Line: $9y = 2x + 14$ ----- (2)

Sub (1) into (2), $9\left(t + \dfrac{3}{t}\right) = 2(3t+2) + 14 \quad \Rightarrow 3t^2 - 18t + 27 = 0 \quad \Rightarrow 3(t-3)^2 = 0$

$\Rightarrow t = 3$

When $t = 3$, $x = 11$, $y = 4$

Hence the coordinates of the point of intersection are (11, 4).