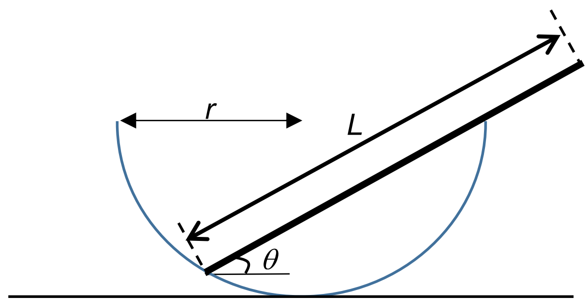
Forces

1 **A Level S Paper N00 Q2**

The figure shows a smooth, hemispherical bowl of radius r, fixed to a horizontal table.



A friend gives you a uniform, smooth rod of length L and weight W, and challenges you to find an angle θ (the angle the rod makes with horizontal) such that the rod will rest in equilibrium on the rounded rim of the bowl.

(a) Draw a labelled diagram showing the three forces acting on the rod when it is in equilibrium. Make clear the directions and lines of action of these forces. [3]

(b)

By resolving these forces vertically and horizontally and by taking moments about the lower end of the rod, show that the condition for equilibrium is

$$4r\cos 2\theta = L\cos\theta$$ [5]

[Hint: $\cos(A-B) = \cos A\cos B + \sin A\sin B$]

2 A block of mass M sits on a plane that is inclined at an angle θ to the horizontal as shown in Fig 1.1.

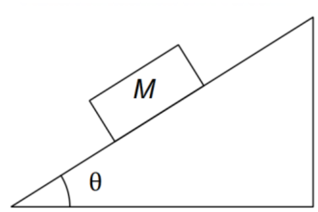


Fig 2.1

(a) Assume that the frictional force between the block and plane is large enough to keep the block at rest and the coefficient of static friction is μ.

(i) Determine the horizontal components of the normal and friction forces acting on the block.

(ii) Determine the angle θ at which these horizontal components are a maximum.

(b) A horizontal force $F = Mg$ is applied on the block, as shown in **Fig 1.2**. Assume that the frictional force between the block and plane is large enough to keep the block at rest and the coefficient of static friction is μ.

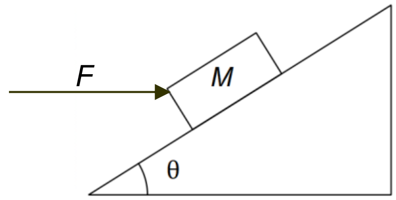


Fig 1.2

(i) Determine the normal and friction forces acting on the block.

(ii) Show that the range of θ for which the block will remain at rest is given by:

$$\frac{1-\mu}{1+\mu} \le \tan\theta \le \frac{1+\mu}{1-\mu}.$$

3 **A Level S Paper N02 Q7**

(a) A smooth wire XYZ is bent to form a right angle at Y and is fixed in a vertical plane, as shown in Fig. 3.1. XY makes an angle of 30° to the horizontal.

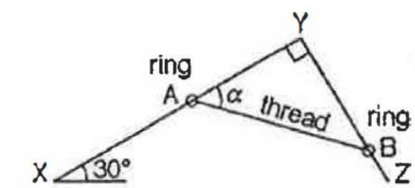


Fig. 3.1

Rings A and B, of mass m_1 and m_2 respectively, are placed on the wire and are connected by a light thread. The rings move without friction on the wire. When the system is in equilibrium, the thread makes an angle α with XY, as shown.

(i) Ring A is in equilibrium under the action of three forces. List these forces, stating the direction of each. Sketch a vector triangle of the three forces. Label the sides and angle of the triangle. Explain why it is not possible to obtain the value of α from this triangle; [5]

(ii) Ring B is also in equilibrium under the action of three forces. Sketch a labeled vector triangle of these forces. [2]

(iii) Use your answers in (i) and (ii) to find an expression for α in terms of m_1 and m_2. [3]

[$\alpha = \tan^{-1}\dfrac{\sqrt{3}m_2}{m_1}$]

(b) A rope is passed several times round a fixed cylinder, as shown in Fig. 3.2.

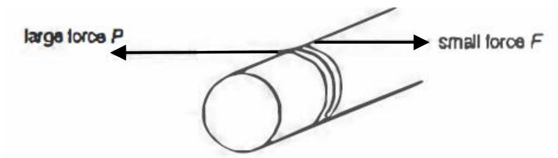


Fig. 3.2.

Rope and cylinder are both rough. A large pulling force Pis applied at the left-hand end of the rope. The rope can be prevented from slipping by applying a much smaller force Fat the right-hand end. Fig. 3.3 shows a small arc of a rope, subtending a small angle $\delta\theta$ at the centre of a cylinder.

When this section of the rope is in equilibrium under tension *T*, the normal force between the rope and the surface of the cylinder has magnitude *N*.

(i) Find an expression for *N*, in terms of *T* and $\delta\theta$.

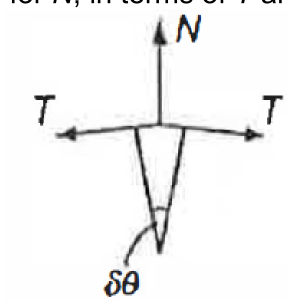


Fig. 3.3

(ii) The force at the left-hand end of the arc in Fig. 3.3 is increased to $T + \delta T$. This gives rise to a frictional force, tangential to the cylinder, of magnitude *μN*, where *μ* is a constant. User your answer to **(i)** to show that the condition for this section of the rope to stay in equilibrium is

$$\mu\ T\ \delta\theta = \delta T$$

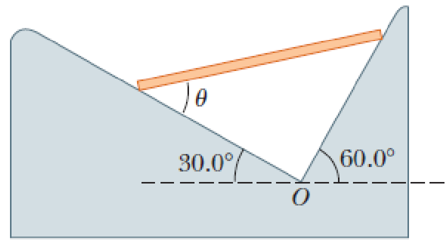
(iii) In a particular application, the rope is wound three times round the cylinder. The constant μ is 0.40. The pulling force *P* at the left-hand end of the rope is 3.0×10^4 N.

Calculate the force *F* at the right-hand end of the rope required just to prevent movement of the rope from right to left. [5]

[16 N]

4 **Serway 7th Ed P12.44**

A uniform rod of weight F_g and length L is supported at its ends by a frictionless trough as shown.



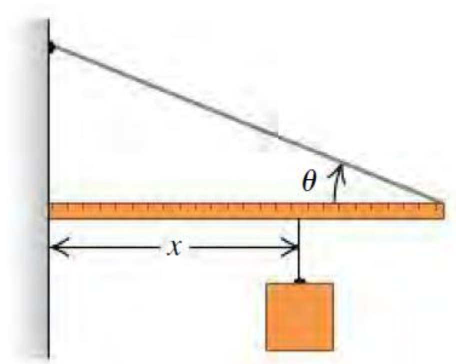
(a) Show that the centre of gravity of the rod must be vertically over point O when the rod is in equilibrium.

(b) Determine the equilibrium value of angle θ.

[60°]

5 **Freedman P11.66**

One end of a uniform meter stick is placed against a vertical wall. The other end is held by a lightweight cord that makes an angle θ with the stick. The coefficient of static friction between the end of the meter stick and the wall is 0.40.



(a) What is the maximum value the angle θ can have if the stick is to remain in equilibrium?

[22°]

(b) Let the angle θ be 15°. A block of the same weight as the meter stick is suspended from the stick, as shown at a distance x from the wall. What is the minimum value of x for which the stick will remain in equilibrium?

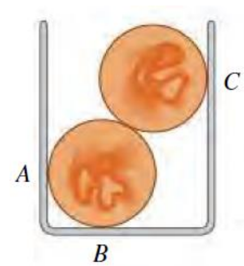
[30.2 cm]

(c) When θ = 15°, how large must the coefficient of static friction be so that the block can be attached 10 cm from the left end of the stick without causing it to slip?

[0.625]

6 **Freedman P11.75**

Two uniform, 75.0 g marbles 2.00 cm in diameter are stacked as shown in a container that is 3.00 cm wide.



(a) Find the force that the container exerts on the marbles at the points of contact A, B, and C.

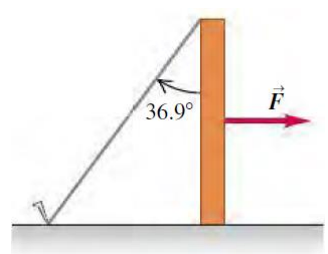
[0.424 N, 1.47 N, 0.424 N]

(b) What force does each marble exert on each other?

[0.848 N]

7 **<u>Freedman P11.90</u>**

One end of a post weighing 400 N and with height h rests on a rough horizontal surface with μ_s = 0.30. The upper end is held by a rope fastened to the surface and making an angle of 36.9° with the post. A horizontal force F is exerted on the post as shown.



(a) If the force F is applied at the midpoint of the post, what is the largest value it can have without causing the post to slip?

[400 N]

(b) How large can the force be without causing the post to slip if its point of application is $\dfrac{6}{10}$ the way from the ground to the top of the post?

[750 N]

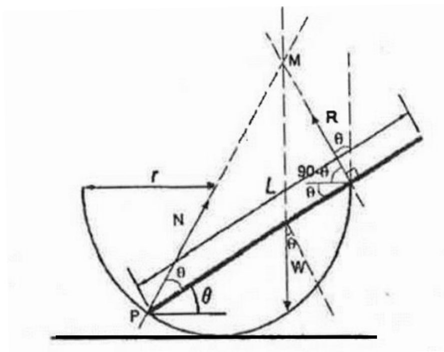
(c) Show that if the point of application of the force is too high, the post cannot be made to slip, no matter how great the force. Find the critical height for the point of application.

[0.71h]

Forces

1 A Level S Paper N00 Q2

(a) The labelled diagram is shown in Fig 1.2. below.



W refers to the weight of the rod, which acts through its CG.
N refers to the normal reaction force by the inner surface of the bowl on the lower end of the rod.
R refers to the reaction force by the rim of the bowl on the rod.

For a body to be in equilibrium, the lines of action of all the forces must meet at one point M.

(b) The vertical component of N is Nsin2θ; the horizontal component of N is Ncos2θ.
The vertical component of R is Rcosθ; the horizontal component of R is Rsinθ.

Since the body is in equilibrium, there is no net force in either the horizontal or vertical direction. Hence,

$$N\cos 2\theta = R\sin\theta \qquad (1)$$
$$N\sin 2\theta + R\cos\theta = W \qquad (2)$$

Eliminating N from (1) and (2),

$$\frac{R\sin\theta\sin 2\theta}{\cos 2\theta} + R\cos\theta = W$$

Multiplying by cos2θ throughout,

$$R\sin\theta\sin 2\theta + R\cos\theta\cos 2\theta = W\cos 2\theta$$

Since cos(A–B)=cosAcosB+sinAsinB,

$$R\cos\theta = W\cos 2\theta \qquad (3)$$

Taking moments about the lower end of the rod P,

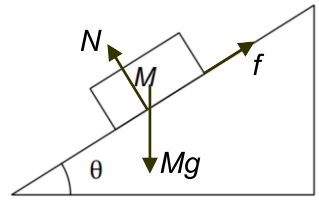
anticlockwise moment of R = clockwise moment of Wcosθ

$$\mathrm{R}\times 2\mathrm{r}\cos\theta = W\cos\theta\times\frac{L}{2} \qquad (4)$$

Substituting Rcosθ in (3) into (4),

$$2rW\cos 2\theta = W\cos\theta\times\frac{L}{2}$$
$$4r\cos 2\theta = L\cos\theta$$

2 **(a)** **(i)** Resolving forces along and perpendicularly to the plane,



$f - Mg\sin\theta = 0 \;\Rightarrow\; f = Mg\sin\theta$

$N - Mg\cos\theta = 0 \;\Rightarrow\; N = Mg\cos\theta$

Horizontal component of friction:

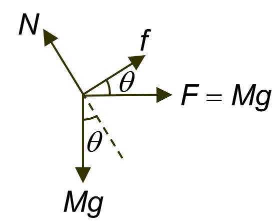
$f\cos\theta = Mg\sin\theta\cos\theta$

Horizontal component of the normal contact force:

$N\sin\theta = Mg\cos\theta\sin\theta$

(ii) Since $f\cos\theta = N\sin\theta = Mg\sin\theta\cos\theta = \dfrac{Mg}{2}\sin 2\theta$, the maximum occurs when $\sin 2\theta = 1 \;\Rightarrow\; \theta = 45°$

(b) **(i)** Resolving forces along and perpendicularly to the plane,



$F\cos\theta + f - Mg\sin\theta = 0$

$\Rightarrow f = Mg\sin\theta - Mg\cos\theta$

$N - F\sin\theta - Mg\cos\theta = 0$

$\Rightarrow N = Mg\cos\theta + Mg\sin\theta$

(ii) For the block to remain at rest, we must have $|f| \le \mu N$, or

$$Mg\left|\sin\theta - \cos\theta\right| \le \mu Mg(\cos\theta + \sin\theta)$$

If $\sin\theta \ge \cos\theta$,

$$\sin\theta - \cos\theta \le \mu(\cos\theta + \sin\theta)$$

Divide throughout by $\cos\theta$:

$$\tan\theta - 1 \le \mu(1 + \tan\theta) \;\Rightarrow\; \tan\theta \le \frac{1+\mu}{1-\mu}.$$

If $\sin\theta \le \cos\theta$,

$$-(\sin\theta - \cos\theta) \le \mu(\cos\theta + \sin\theta)$$

We will thus have

$$-\tan\theta + 1 \le \mu(1 + \tan\theta) \;\Rightarrow\; \tan\theta \ge \frac{1-\mu}{1+\mu}.$$

Putting both ranges together,

$$\frac{1-\mu}{1+\mu} \le \tan\theta \le \frac{1+\mu}{1-\mu}.\ \text{(shown)}$$

3 **A Level S Paper N02 Q7**

(a) **(i)** Since the rings move without friction on the wire, the reaction force RA exerted by the wire on the rings acts perpendicular to the wire. The gravitational force m_1g acts vertically downwards. The tension acts along the direction of the wire, away from the ring. The forces acting on both rings are summarized in Fig. 3.1.

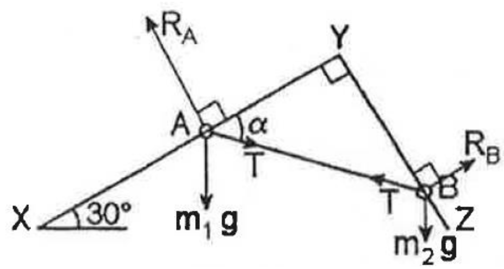


Fig. 3.1.

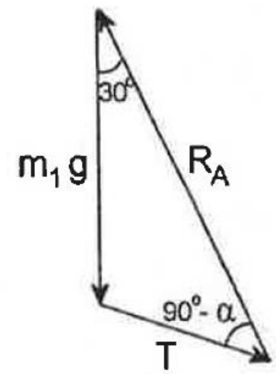


Fig. 3.2.

The forces acting on ring A can be rearranged into a vector triangle as in Fig. 3.2. Applying the sine rule,

$$\frac{m_1 g}{\sin(90° - \alpha)} = \frac{T}{\sin 30°}$$

$$\frac{m_1 g}{\cos\alpha} = 2T$$

$$\cos\alpha = \frac{m_1 g}{2T}$$

α cannot be obtained because T is not known.

(ii) Referring to Fig. 3.1 again, the forces acting on ring B can be rearranged into a vector triangle as in Fig. 3.3.

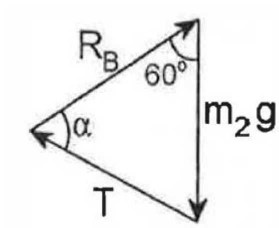


Fig. 3.3

(iii) Applying the sine rule again in Fig. 3.3,

$$\frac{m_2 g}{\sin\alpha} = \frac{T}{\sin 60°}$$
$$= \frac{2T}{\sqrt{3}}$$
$$\sin\alpha = \frac{\sqrt{3}\, m_2 g}{2T} \qquad (1)$$

From (i) $$\cos\alpha = \frac{m_1 g}{2T} \qquad (2)$$

Dividing (1) by (2):

$$\tan\alpha = \frac{\sqrt{3} m_2}{m_1}$$
$$\alpha = \tan^{-1}\left(\frac{\sqrt{3} m_2}{m_1}\right)$$

(b) **(i)** Since the three forces are in equilibrium, they can be rearranged into a closed triangle (Fig. 3.4).

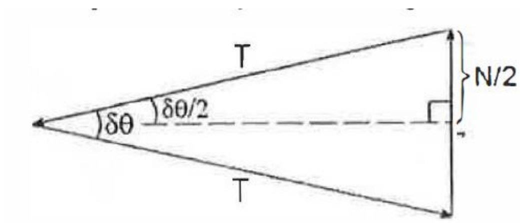


Fig. 3.4

$\dfrac{\left(N/2\right)}{T} = \sin\dfrac{\delta\theta}{2} \approx \dfrac{\delta\theta}{2}$ since $\delta\theta$ is small. Therefore, $\dfrac{N}{2} = T\dfrac{\delta\theta}{2}$ and $N = T\delta\theta$.

(ii) At equilibrium,

$$T + \delta T - \mu N = T\ .$$

Hence, $$\delta T = \mu N$$

Since $\delta\theta\ N = T\delta\theta$, $\qquad \delta T = \mu N = \mu T\delta\theta$

(iii) 1. As $\delta\theta$ approaches zero, the equation in **(ii)** can be rewritten as

$$dT = \mu T\delta\theta$$
$$\frac{1}{T}dT = \mu d\theta \qquad (1)$$

The LHS of the equation (1) can be integrated with limits from T_1 = F to T_2 = P = 3.0 × 10^4 N; while the RHS can be integrated with limits from θ = 0 to θ = 6π (wound three times).

$$\int \frac{1}{T}\, dT = \mu \int d\theta$$

$$\ln T = \mu\theta = 0.40\ \theta$$

$$\ln(3.0 \times 10^4) - \ln F = 0.40\ (6\pi - 0)$$

$$F = 15.94 = 16\ \text{N}$$

4 Serway 7th Ed. P12.44

P12.44 (a) Just three forces act on the rod: forces perpendicular to the sides of the trough at A and B, and its weight. The lines of action of A and B will intersect at a point above the rod. They will have no torque about this point. The rod's weight will cause a torque about the point of intersection as in Figure 12.52(a), and the rod will not be in equilibrium unless the center of the rod lies vertically below the intersection point, as in Figure 12.52(b). All three forces must be concurrent. Then the line of action of the weight is a diagonal of the rectangle formed by the trough and the normal forces, and the rod's center of gravity is vertically above the bottom of the trough.

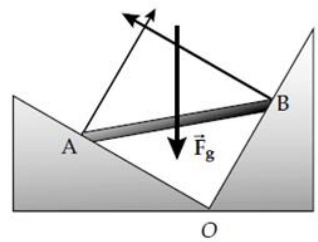


FIG. P12.44(a)

(b) In Figure (b), $\overline{AO}\cos 30.0° = \overline{BO}\cos 60.0°$ and

$$L^2 = \overline{AO}^2 + \overline{BO}^2 = \overline{AO}^2 + \overline{AO}^2\left(\frac{\cos^2 30.0°}{\cos^2 60.0°}\right)$$

$$\overline{AO} = \frac{L}{\sqrt{1+\left(\frac{\cos 30°}{\cos 60°}\right)^2}} = \frac{L}{2}$$

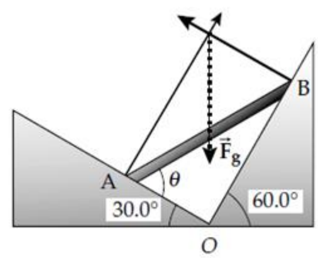


FIG. P12.44(b)

So

$$\cos\theta = \frac{\overline{AO}}{L} = \frac{1}{2} \text{ and } \theta = \boxed{60.0°}$$

5 Freedman P11.66

11.66. IDENTIFY: Apply $\sum \tau_z = 0$ to the meterstick.

SET UP: The wall exerts an upward static friction force f and a horizontal normal force n on the stick. Denote the length of the stick by l. $f = \mu_s n$.

EXECUTE: **(a)** Taking torques about the right end of the stick, the friction force is half the weight of the stick, $f = w/2$. Taking torques about the point where the cord is attached to the wall (the tension in the cord and the friction force exert no torque about this point), and noting that the moment arm of the normal force is $l\tan\theta$, $n\tan\theta = w/2$. Then, $(f/n) = \tan\theta < 0.40$, so $\theta < \arctan(0.40) = 22°$.

(b) Taking torques as in part (a), $fl = w\dfrac{l}{2} + w(l-x)$ and $nl\tan\theta = w\dfrac{l}{2} + wx$. In terms of the coefficient of friction μ_s, $\mu_s > \dfrac{f}{n} = \dfrac{l/2 + (l-x)}{l/2 + x}\tan\theta = \dfrac{3l - 2x}{l + 2x}\tan\theta$. Solving for x, $x > \dfrac{l}{2}\dfrac{3\tan\theta - \mu_s}{\mu_s + \tan\theta} = 30.2$ cm.

(c) In the above expression, setting $x = 10$ cm and $l = 100$ cm and solving for μ_s gives

$$\mu_s > \frac{(3 - 20/l)\tan\theta}{1 + 20/l} = 0.625.$$

EVALUATE: For $\theta = 15°$ and without the block suspended from the stick, a value of $\mu_s \geq 0.268$ is required to prevent slipping. Hanging the block from the stick increases the value of μ_s that is required.

6 Freedman P11.75

11.75. IDENTIFY: Apply the first and second conditions of equilibrium, first to both marbles considered as a composite object and then to the bottom marble.

(a) SET UP: The forces on each marble are shown in Figure 11.75.

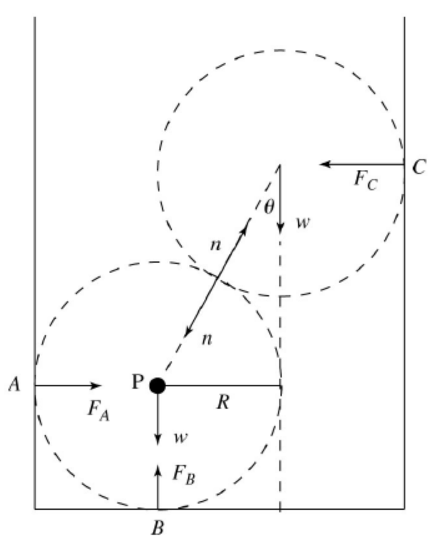


Figure 11.75

EXECUTE:

$F_B = 2w = 1.47$ N

$\sin\theta = R/2R$ so $\theta = 30°$

$\sum \tau_z = 0$, axis at P

$F_C(2R\cos\theta) - wR = 0$

$$F_C = \frac{mg}{2\cos 30°} = 0.424 \text{ N}$$

$F_A = F_C = 0.424$ N

(b) Consider the forces on the bottom marble. The horizontal forces must sum to zero, so $F_A = n\sin\theta$.

$$n = \frac{F_A}{\sin 30°} = 0.848\text{ N}$$

Could use instead that the vertical forces sum to zero

$$F_B - mg - n\cos\theta = 0$$

$$n = \frac{F_B - mg}{\cos 30°} = 0.848\text{ N},\ \text{which checks.}$$

EVALUATE: If we consider each marble separately, the line of action of every force passes through the center of the marble so there is clearly no torque about that point for each marble. We can use the results we obtained to show that $\Sigma F_x = 0$ and $\Sigma F_y = 0$ for the top marble.

7 Freedman P11.90

11.90. IDENTIFY: Apply $\Sigma\tau_z = 0$ to the post, for various choices of the location of the rotation axis.

SET UP: When the post is on the verge of slipping, f_s has its largest possible value, $f_s = \mu_s n$.

EXECUTE: **(a)** Taking torques about the point where the rope is fastened to the ground, the lever arm of the applied force is $h/2$ and the lever arm of both the weight and the normal force is $h\tan\theta$, and so

$$F\frac{h}{2} = (n - w)h\tan\theta.$$

Taking torques about the upper point (where the rope is attached to the post), $fh = F\dfrac{h}{2}$. Using $f \le \mu_s n$ and solving for F, $F \le 2w\left(\dfrac{1}{\mu_s} - \dfrac{1}{\tan\theta}\right)^{-1} = 2(400\text{ N})\left(\dfrac{1}{0.30} - \dfrac{1}{\tan 36.9°}\right)^{-1} = 400\text{ N}.$

(b) The above relations between F, n and f become $F\dfrac{3}{5}h = (n - w)h\tan\theta, f = \dfrac{2}{5}F$, and eliminating f and n and solving for F gives $F \le w\left(\dfrac{2/5}{\mu_s} - \dfrac{3/5}{\tan\theta}\right)^{-1}$, and substitution of numerical values gives 750 N to two figures.

(c) If the force is applied a distance y above the ground, the above relations become $Fy = (n - w)h\tan\theta,\ F(h - y) = fh$, which become, on eliminating n and f, $w \ge F\left[\dfrac{(1 - y/h)}{\mu_s} - \dfrac{(y/h)}{\tan\theta}\right].$

As the term in square brackets approaches zero, the necessary force becomes unboundedly large. The limiting value of y is found by setting the term in square brackets equal to zero. Solving for y gives

$$\frac{y}{h} = \frac{\tan\theta}{\mu_s + \tan\theta} = \frac{\tan 36.9°}{0.30 + \tan 36.9°} = 0.71.$$

EVALUATE: For the post to slip, for an axis at the top of the post the torque due to F must balance the torque due to the friction force. As the point of application of F approaches the top of the post, its moment arm for this axis approaches zero.