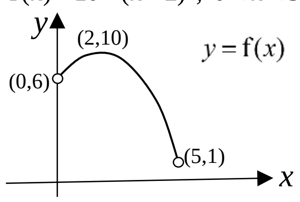
# SUMMARY: Functions

| | Function f | Inverse Function $f^{-1}$ | Composite Function gf |
|---|---|---|---|
| **Existence** | Each and every element $x$ in the domain is assigned to a unique image<br>$\Rightarrow$ "Vertical line test", whereby the line $x=a$, for **all** $a \in D_f$, cuts the graph of $y=f(x)$ **exactly once** | f must be one-one $\Rightarrow$ "Horizontal line test":<br>If the horizontal line $y=k$, for **all** $k \in \mathbb{R}$, cuts the graph of $y=f(x)$ **at most once**, then f is a one-one function.<br>To show one-one:<br>- Draw graph and write "general statement" (above)<br><br>To show NOT one-one:<br>- Give a counter-example<br>(e.g. from the graph, the line $y=9$ cuts the curve $y=f(x)$ more than once…or $f(5)=f(1)=9$…) | $R_f \subseteq D_g$ |
| **Domain** | Given | $D_{f^{-1}} = R_f$ | $D_{gf} = D_f$ |
| **Rule** | - Quadratic<br>- Modulus<br>- Rational Functions<br>- Logarithm, Exponential<br>- Trigonometric<br>- Piece-wise<br>$f(x)=\begin{cases} x & \text{for } 0<x\le 1 \\ -x^2+2 & \text{for } 1<x\le 2 \end{cases}$<br>- Periodic<br>$f(x+k)=f(x)$ for all real values of $x \Rightarrow$ f is periodic with period $k$<br>e.g. $\sin x$ is periodic with a period of $2\pi$ | **THREE STEPS:**<br>1.Let $y=f(x)$<br>2. $x=f^{-1}(y)$, i.e. Make $x$ the subject<br>- For quadratic, complete the square/use quadratic formula<br>- Choose the correct sign (if needed) based on $D_f$<br>3.Replace $y$ by $x$<br>E.g. $f(x)=x^2-6x+14,\ \ x\le 3$<br>Let $y=f(x)=x^2-6x+14=(x-3)^2+5$<br>$\Rightarrow x=3\pm\sqrt{y-5}$ ("$\pm$" important, then we choose based on $D_f$) $\Rightarrow$ since $x\le 3$, $x=3-\sqrt{y-5}$<br>$\therefore f^{-1}(x)=3-\sqrt{x-5}$, $x\ge 5$ $\left(\because D_{f^{-1}}=R_f=[5,\infty)\right)$ | $gf(x)=g\left[f(x)\right]$<br>(Sub f into g)<br>E.g. $f(x)=x^2+5,\ x\in\mathbb{R}$<br>$g(x)=x+\sin x,\ x\in\mathbb{R}$<br>$\Rightarrow gf(x)=g\left[f(x)\right]$<br>$=g(x^2+5)$<br>$=x^2+5+\sin(x^2+5)$<br>$\Rightarrow fg(x)=f\left[g(x)\right]$<br>$=f(x+\sin x)$<br>$=(x+\sin x)^2+5$ |
| **Range** | **Graphical Method:**<br>1.Draw $y=f(x)$ graph and restrict graph to the given domain of f<br>2.Read off $y$-values<br>E.g.<br>$f(x)=10-(x-2)^2,\ 0<x<5$<br><br>$\therefore R_f=(1,10]$<br>**Algebraic method:**<br>- Discriminant method<br>(*Refer to Tut 5, Q7b*) | $R_{f^{-1}} = D_f$ | To find $R_{gf}$,<br>**Method 1**:<br>1.Sketch the graphs of f and g separately<br>2.Obtain $R_f$ from the graph of f<br>3.Obtain $R_{gf}$ by reading off range of graph of g using $R_f$ as the domain<br><br>**Method 2 (usual method to find range):**<br>1.Sketch the graph of gf<br>2.Obtain the range of gf from the graph, using $D_{gf}(=D_f)$ |
| **Graphs / Others** | • Use GC but pay attention to limitations of GC<br>(for example, when sketching ln graphs) | • Graphs of $y=f(x)$ and $y=f^{-1}(x)$ are reflections of each other about the line $y=x$.<br>• Point(s) of intersection (if any) of the graphs of $y=f(x)$ and $y=f^{-1}(x)$ can be found by solving:<br>$f(x)=f^{-1}(x)$ or $f(x)=x$ or $f^{-1}(x)=x$.<br>• $ff^{-1}(x)=f^{-1}f(x)=x$, however, they are generally **different functions** as they have different domains:<br>$f^{-1}f: x\mapsto x,\ x\in D_f$ while $ff^{-1}: x\mapsto x,\ x\in D_{f^{-1}}$ | • $f^2(x)=ff(x)\ne\left[f(x)\right]^2$<br><br>• If $ff(x)=x$,<br>$\Rightarrow f^{-1}f f(x)=f^{-1}(x)$<br>$\Rightarrow f(x)=f^{-1}(x)$ |