中正中学 义顺

CHUNG CHENG HIGH SCHOOL (YISHUN)

2024 Preliminary Examination

Secondary Four Express / Five Normal Academic

CANDIDATE NAME	MARKING SCHEME		
CLASS		INDEX NUMBER	

ADDITIONAL MATHEMATICS 4049/01

Paper 1

22 August 2024

2 hours 15 minutes

Candidates answer on the Question Paper.

No Additional Materials are required.

READ THESE INSTRUCTIONS FIRST

Write your name, class and index number on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need of clear presentation in your answers. Up to 2 marks may be deducted for improper presentation.

The number of marks is given in brackets [ ] at the end of each question or part question.

For Examiner's Use		
Presentation Deduction		– 1 / – 2
TOTAL	90	

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial Expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGNOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} bc \sin A$$

1 **(a)** Given that the line $y = 2x - k$ meets the curve $y = \frac{k}{4}x^2 - 2kx + 1$, find the range of values of k. [4]

$$\frac{k}{4}x^2 - 2kx + 1 = 2x - k$$

$$\frac{k}{4}x^2 - 2kx + 1 - 2x + k = 0$$

$$\frac{k}{4}x^2 - 2kx - 2x + k + 1 = 0$$

$$a = \frac{k}{4},\ b = -2k - 2,\ c = k + 1$$

Line meets curve,

$$b^2 - 4ac \geq 0$$

$$(-2k-2)^2 - 4\left(\frac{1}{4}k\right)(k+1) \geq 0$$

$$4k^2 + 8k + 4 - k^2 - k \geq 0$$

$$3k^2 + 7k + 4 \geq 0$$

$$(3k+4)(k+1) \geq 0$$

$$k \leq -1\frac{1}{3} \quad \text{or} \quad k \geq -1$$

(b) A student claimed that when $k = -\frac{1}{2}$, the curve does not intersect the line. By showing your workings clearly, explain whether the statement is valid. [2]

No intersection,

$$b^2 - 4ac < 0$$

$$3k^2 + 7k + 4 < 0$$

$$(3k+4)(k+1) < 0$$

$$-1\frac{1}{3} < k < -1$$

Since $k = -\frac{1}{2}$ does not lies within the range. The statement is not valid.

2 A triangle has a base of $\left(\frac{1}{\sqrt{2}}-\frac{\sqrt{32}}{2}+\frac{16}{\sqrt{48}}\right)$ m and a height of h m. Given that the area of the triangle is $\left(2\sqrt{2}-\sqrt{3}\right)$ m^2, find, without using calculator, the value of h in the form $\frac{a\sqrt{6}+b}{5}$ where a and b are integers. [5]

Simplifying base:

$$\frac{1}{\sqrt{2}}-\frac{\sqrt{32}}{2}+\frac{16}{\sqrt{48}}$$

$$=\frac{1}{\sqrt{2}}-\frac{4\sqrt{2}}{2}+\frac{16}{4\sqrt{3}}$$

$$=\frac{1}{\sqrt{2}}-\frac{2\sqrt{2}}{1}+\frac{4}{\sqrt{3}}$$

$$=\frac{\sqrt{3}}{\sqrt{6}}-\frac{4\sqrt{3}}{\sqrt{6}}+\frac{4\sqrt{2}}{\sqrt{6}}$$

$$=\frac{4\sqrt{2}-3\sqrt{3}}{\sqrt{6}}$$

$$=\frac{4\sqrt{2}-3\sqrt{3}}{\sqrt{6}}\times\frac{\sqrt{6}}{\sqrt{6}}$$

$$=\frac{8\sqrt{3}-9\sqrt{2}}{6}$$

$\text{Area}=\frac{1}{2}\times\frac{8\sqrt{3}-9\sqrt{2}}{6}\times h$ $2\sqrt{2}-\sqrt{3}=\frac{8\sqrt{3}-9\sqrt{2}}{12}\times h$ $h=\frac{12\left(2\sqrt{2}-\sqrt{3}\right)}{8\sqrt{3}-9\sqrt{2}}$ $h=\frac{24\sqrt{2}-12\sqrt{3}}{8\sqrt{3}-9\sqrt{2}}\times\frac{8\sqrt{3}+9\sqrt{2}}{8\sqrt{3}+9\sqrt{2}}$ $h=\frac{192\sqrt{6}+432-288-108\sqrt{6}}{192-162}$	$h=\frac{84\sqrt{6}+144}{30}$ $h=\frac{6\left(14\sqrt{6}+24\right)}{30}$ $h=\frac{14\sqrt{6}+24}{5}$

3 Show that $3x^2 - 5x + 21$ is always positive for all real values of x. [3]

Method 1:	**Method 2:**
$b^2 - 4ac$ $= (-5)^2 - 4(3)(21)$ $= -227 < 0$ Since discriminant < 0, there is no real roots Since the coefficient of $x^2 > 0$, the graph lies entirely above the x- axis Hence, $3x^2 - 5x + 21$ is always positive for all real values of x.	$3x^2 - 5x + 21$ $= 3\left(x^2 - \frac{5}{3}x + 7\right)$ $= 3\left(\left(x - \frac{5}{6}\right)^2 - \left(\frac{5}{6}\right)^2 + 7\right)$ $= 3\left(\left(x - \frac{5}{6}\right)^2 - \frac{25}{36} + 7\right)$ $= 3\left(x - \frac{5}{6}\right)^2 + \frac{227}{12}$ Since $\left(x - \frac{5}{6}\right)^2 \geq 0$ for all real values of x $3\left(x - \frac{5}{6}\right)^2 + \frac{227}{12} > 0$ Hence, $3x^2 - 5x + 21$ is always positive for all real values of x.

(ii) The curve $y = ax^n$, where a and n are constants, passes through $(2,\ 48)$, $(3,\ 108)$ and $(k,\ 192)$. Find the values of a, n and k where $k > 0$. [3]

$48 = 2^n a$ $a = \frac{48}{2^n}$ ---- (1) $108 = 3^n a$ $a = \frac{108}{3^n}$ ---- (2) $\frac{48}{2^n} = \frac{108}{3^n}$ $\frac{3^n}{2^n} = \frac{108}{48}$ $\left(\frac{3}{2}\right)^n = \frac{9}{4}$ $\left(\frac{3}{2}\right)^n = \left(\frac{3}{2}\right)^2$ Therefore $n = 2$	When $n = 2$, $a = \frac{48}{2^2}$ $a = 12$ $y = 12x^2$ $192 = 12k^2$ $16 = k^2$ $k = 4$ or $k = -4$ (rej)

4 **(a)** Given that $\mathrm{f}(x)=\dfrac{(2x+3)}{\sqrt{4x-3}}$, find $\mathrm{f}'(x)$. [3]

Method 1:	**Method 2:**
$\mathrm{f}(x)=\dfrac{(2x+3)}{\sqrt{4x-3}}$ $\mathrm{f}'(x)=\dfrac{\sqrt{4x-3}(2)-(2x+3)\frac{1}{2}(4x-3)^{-\frac{1}{2}}(4)}{\left(\sqrt{4x-3}\right)^2}$ $\mathrm{f}'(x)=\dfrac{\left(\sqrt{4x-3}\right)(2)-(2x+3)(4x-3)^{-\frac{1}{2}}(2)}{4x-3}$ $\mathrm{f}'(x)=\dfrac{(4x-3)^{-\frac{1}{2}}\left[(4x-3)(2)-(2x+3)(2)\right]}{4x-3}$ $\mathrm{f}'(x)=\dfrac{[8x-6-4x-6]}{(4x-3)^{\frac{3}{2}}}$ $\mathrm{f}'(x)=\dfrac{4x-12}{(4x-3)^{\frac{3}{2}}}$	$\mathrm{f}(x)=(2x+3)(4x-3)^{-\frac{1}{2}}$ $\mathrm{f}'(x)=(2)(4x-3)^{-\frac{1}{2}}+(2x+3)\left(-\frac{1}{2}\right)(4x-3)^{-\frac{3}{2}}(4)$ $\mathrm{f}'(x)=(4x-3)^{-\frac{3}{2}}\left[(2)(4x-3)+(2x+3)(-2)\right]$ $\mathrm{f}'(x)=(4x-3)^{-\frac{3}{2}}[8x-6-4x-6]$ $\mathrm{f}'(x)=\dfrac{4x-12}{(4x-3)^{\frac{3}{2}}}$

(b) Find the set of values of x for $y=\ln\left(\dfrac{3+8x}{8x-5}\right)$ to be a decreasing function. [3]

$y=\ln\left(\dfrac{3-8x}{8x-5}\right)$ $y=\ln(3+8x)-\ln(8x-5)$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{8}{3+8x}-\dfrac{8}{8x-5}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{64x-40-64x-24}{(3+8x)(8x-5)}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{-64}{(3+8x)(8x-5)}$	For decreasing function, $\dfrac{\mathrm{d}y}{\mathrm{d}x}<0$ $\dfrac{-64}{(3+8x)(8x-5)}<0$ $(3+8x)(8x-5)>0$ $x<-\dfrac{3}{8}$ or $x>\dfrac{5}{8}$

5 **(a)** Explain why $\dfrac{2x^3-4x^2+13x-10}{(x^2+4)(x-2)}$ is considered an improper fraction. [1]

The degree of x in the numerator is the same as the degree of x in denominator thus, it is an improper fraction.

(b) Express $\dfrac{2x^3-4x^2+13x-10}{(x^2+4)(x-2)}$ in the form of $A+\dfrac{Bx+C}{(x^2+4)(x-2)}$ and hence, express it in partial fractions. [6]

Using long division:

$$\frac{2x^3-4x^2+13x-10}{(x^2+4)(x-2)}=2+\frac{5x+6}{(x^2+4)(x-2)}$$

$$\frac{5x+6}{(x^2+4)(x-2)}=\frac{Ax+B}{(x^2+4)}+\frac{C}{(x-2)}$$

$$5x+6=(Ax+B)(x-2)+C(x^2+4)$$

When $x=2$,

$16=8C$

$C=2$

Comparing x^2:

$0=A+C$

$A=-2$

Comparing constant:

$6=-2B+4C$

$6=-2B+8$

$B=1$

$$\frac{2x^3-4x^2+13x-10}{(x^2+4)(x-2)}=2+\frac{1-2x}{(x^2+4)}+\frac{2}{(x-2)}$$

6 **(a)** Given that $6x^2 - 9x + 18 = A(2x+1)(x-1) + B(x-2) + C$ for all values of x, find the values of A, B and C. [3]

$6x^2 - 9x + 18 = A(2x+1)(x-1) + B(x-2) + C$ Comparing x^2: $6 = 2A$ $A = 3$ Comparing x: $-9 = -A + B$ $B = -6$	Let $x = 1$ $6 - 9 + 18 = -B + C$ $C = 15 + B$ $C = 9$

(b) Given that $\mathrm{f}(x) = 2x^3 - 11x^2 + 3x + 36$, show that $2x+3$ is a factor of $\mathrm{f}(x)$ and hence solve the equation $\mathrm{f}(x) = 0$. [5]

$\mathrm{f}(x) = 2x^3 - 11x^2 + 3x + 36$ Let $(2x+3)$ be a factor of $f(x)$ $\mathrm{f}(-1.5) = 2(-1.5)^3 - 11(-1.5)^2 + 3(-1.5) + 36$ $= -6.75 - 24.75 - 4.5 + 36$ $= 0$ Since $\mathrm{f}(-1.5) = 0$, by factor theorem, $2x+3$ is a factor of $\mathrm{f}(x)$. $2x^3 - 11x^2 + 3x + 36 = (2x+3)(ax^2 + bx + c)$ Comparing x^3: $A = 1$ Comparing constant: $36 = 3c$ $c = 12$ Comparing x^2: $-11 = 3a + 2b$ $b = -7$	$2x^3 - 11x^2 + 3x + 36 = (2x+3)(x^2 - 7x + 12)$ $\mathrm{f}(x) = (2x+3)(x^2 - 7x + 12)$ $\mathrm{f}(x) = 0$ $(2x+3)(x^2 - 7x + 12) = 0$ $(x-4)(x-3)(2x+3) = 0$ $x = 4$ or $x = 3$ or $x = -1\frac{1}{2} / -1.5$ or $2x^3 - 11x^2 + 3x + 36 = (x-4)(2x^2 + bx - 9)$

7 Without using a calculator, evaluate $\dfrac{\log_5 9 + 2\log_5 6 - 4\log_5 3}{\log_{25} 4}$. [4]

$$\frac{\log_5 9 + 2\log_5 6 - 4\log_5 3}{\log_{25} 4}$$

$$= \frac{\log_5 9 + \log_5 6^2 - \log_5 3^4}{\log_{25} 4}$$

$$= \frac{\log_5 (9 \times 36 \div 81)}{\log_{25} 4}$$

$$= \frac{\log_5 4}{\frac{\log_5 4}{\log_5 25}}$$

$$= \log_5 5^2 = 2\log_5 5$$

$$= 2$$

8 Simplify $2^{3x-4} \times 9^{x-3} = 6^{2(x-2)}$ in the form $\left(\dfrac{m \times n}{36}\right)^x = 9$, where m and n are integers.

Hence solve for x, giving your answer in the simplest form $\dfrac{\ln a}{\ln b}$, where a and b are integers.

[5]

$$2^{3x-4} \times 9^{x-3} = 6^{2(x-2)}$$

$$\frac{2^{3x}}{2^4} \times \frac{3^{2x}}{3^6} = \frac{6^{2x}}{6^4}$$

$$\frac{8^x}{16} \times \frac{9^x}{729} = \frac{36^x}{1296}$$

$$\frac{8^x \times 9^x}{36^x} = \frac{729 \times 16}{1296}$$

$$\left(\frac{8 \times 9}{36}\right)^x = 9$$

$$2^x = 3^2$$

$$x \ln 2 = \ln 3^2$$

$$x = \frac{\ln 9}{\ln 2}$$

9 The function f is defined, for $0° \le x \le 180°$, by $f(x) = c + a\cos bx$, where a, b and c are positive integers. Given that the amplitude of f is 4 and the period of f is $120°$,

(i) state the value of a and of b, [2]

$a = 4$

$120° = \dfrac{360°}{b}$

$b = 3$

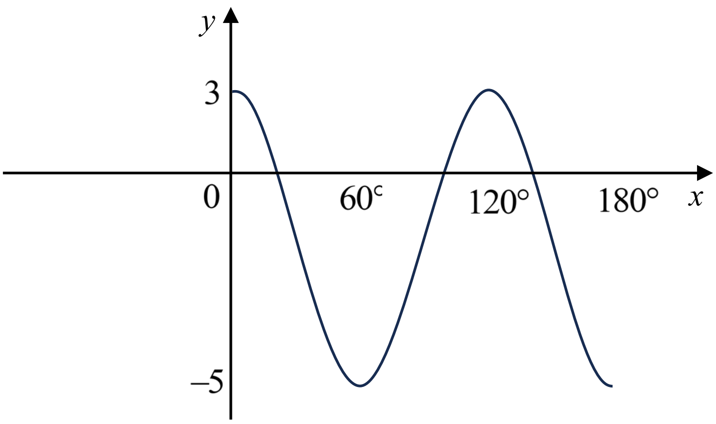
Given that the minimum value of f is -5,

(ii) find the value of c, [1]

$c - 4 = -5$

$c = -1$

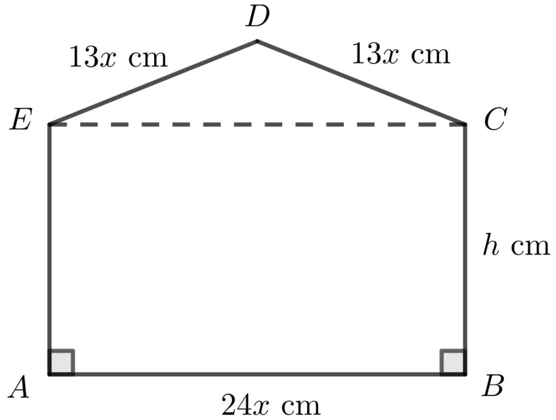
(iii) sketch the graph of f for $0° \le x \le 180°$. [3]



(iv) A line $y = k$ is drawn on the graph of f. State the range of values of k that will give 3 real roots. [1]

$-5 < k < 3$

10 The diagram shows an outline of a board, *ABCDE*, consisting of a rectangle *ABCE* of height h cm and width $24x$ cm and an isosceles triangle *CDE* in which $CD = DE = 13x$ cm. The perimeter of the logo is 480 cm.



(a) Show that the area of the board, A cm^2, is given by $A = 5760x - 540x^2$. [3]

Height of triangle $= \sqrt{(13x)^2 - (12x)^2}$

$= 5x$ cm

Perimeter of board = Perimeter of rectangle + Perimeter of Triangle

$480 = 24x + 2h + 2(13x)$

$480 = 50x + 2h$

$h = \dfrac{480 - 50x}{2}$

Area of board = Area of rectangle + Area of Triangle

$= 24xh + \frac{1}{2} \times 24x \times 5x$

$= 24x\left(\dfrac{480 - 50x}{2}\right) + 60x^2$

$= 12x(480 - 50x) + 60x^2$

$= 5760x - 600x^2 + 60x^2$

$= 5760x - 540x^2$

(b) Calculate the stationary value of A and determine whether it is a maximum or a minimum. [5]

For stationary value, $\dfrac{dA}{dx} = 0$

$\dfrac{dA}{dx} = 5760 - 1080x$

$5760 - 1080x = 0$

$x = 5\frac{1}{3}$

When $x = 5\frac{1}{3}$,

$$A = 5760\left(\frac{16}{3}\right) - 540\left(\frac{16}{3}\right)^2$$

$$A = 15360\,\text{cm}^2$$

Using 2nd derivative test,

$$\frac{d^2A}{dx^2} = -1080 < 0$$

Therefore $A = 15360$ is a maximum.

11 **(a)** If $a\cos^2\theta + b\sin^2\theta = c$, show that $\tan^2\theta = \dfrac{a-c}{c-b}$. [4]

From RHS

$$\frac{a-c}{c-b}$$

$$= \frac{a - \left(a\cos^2\theta + b\sin^2\theta\right)}{\left(a\cos^2\theta + b\sin^2\theta\right) - b}$$

$$= \frac{a\left(1-\cos^2\theta\right) - b\sin^2\theta}{a\cos^2\theta + b\left(\sin^2\theta - 1\right)}$$

$$= \frac{a\sin^2\theta - b\sin^2\theta}{a\cos^2\theta - b\left(\cos^2\theta\right)}$$

$$= \frac{\sin^2\theta\left(a-b\right)}{\cos^2\theta\left(a-b\right)}$$

$= \tan^2\theta$ (shown)

(b) **(i)** Given that $\sqrt{\left(2\cos^2 x-\sin x\right)}=2\sin x$, show that $6\sin^2 x+\sin x-2=0$. [2]

$$\sqrt{\left(2\cos^2 x-\sin x\right)}=2\sin x$$

$$2\cos^2 x-\sin x=\left(2\sin x\right)^2$$

$$2\cos^2 x-\sin x=4\sin^2 x$$

$$2\left(1-\sin^2 x\right)-\sin x=4\sin^2 x$$

$$2-2\sin^2 x-\sin x=4\sin^2 x$$

$$6\sin^2 x+\sin x-2=0$$

(ii) Hence, solve the equation $\sqrt{\left(2\cos^2 x-\sin x\right)}=2\sin x$ for $0\le\theta\le 2\pi$. [4]

$$6\sin^2 x+\sin x-2=0$$

$$\left(3\sin x+2\right)\left(2\sin x-1\right)=0$$

$$\sin x=-\frac{2}{3}\text{ (rej)}$$

Or $\sin x=\frac{1}{2}$

$$\text{basic angle}=\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$$

$$x=\frac{\pi}{6}\quad\text{or}\quad\frac{5\pi}{6}$$

12 The positive x- and y-axes are tangents to a circle C with equation of $(x-a)^2+(y-b)^2=r^2$.

(a) What conditions must apply to the constants a, b and r? [1]

a, b and r have the same value

(b) It is given that $N(1, 2)$ is a point on the circle and it is further from the centre of the circle C as compared to the origin. Show that the coordinates of the centre M are $(5,\ 5)$. [4]

Let the centre of the circle be $(x,\ x)$ and the radius be x units.

$(x-1)^2+(y-2)^2=x^2$

Since $y = x$,

$(x-1)^2+(x-2)^2=x^2$

$x^2-2x+1+x^2-4x+4=x^2$

$x^2-6x+5=0$

$(x-1)(x-5)=0$

$x=1$ or $x=5$

Distance between the point $(1,\ 2)$ and the origin

$=\sqrt{(1-0)^2+(2-0)^2}$

$=\sqrt{5}$ units

Distance between the point $(1,\ 1)$ and the origin

$=\sqrt{(1-0)^2+(1-0)^2}$

$=\sqrt{2}$ units (rejected)

Distance between the point $(5,\ 5)$ and the origin

$=\sqrt{(5-0)^2+(5-0)^2}$

$=\sqrt{50}$ units

Therefore the centre of C is $(5,\ 5)$ and radius is 5 units. ---- Shown

(c) Find the equation of the tangent to the circle C at $(1,\ 2)$. [3]

Gradient = $\dfrac{5-2}{5-1}$

$= \dfrac{3}{4}$

Gradient of tangent = $-\dfrac{4}{3}$

Equation of line:

$y = -\dfrac{4}{3}x + c$ passes through $(1,\ 2)$

$2 = -\dfrac{4}{3} + c$

$c = \dfrac{10}{3}$

$y = -\dfrac{4}{3}x + \dfrac{10}{3}$ or $3y = -4x + 10$

(d) The tangent intersects the axes at the points P and Q. The point R has coordinates $\left(k^2, -k\right)$, where $k > 2$. Find the area of the quadrilateral $MPQR$ in terms of k. [4]

$0 = -4x + 10$

$x = 2.5$

$(2.5, 0)$ **and** $\left(0, \dfrac{10}{3}\right)$

Area of quadrilateral = $\dfrac{1}{2}\begin{vmatrix} 0 & 2.5 & k^2 & 5 & 0 \\ \dfrac{10}{3} & 0 & -k & 5 & \dfrac{10}{3} \end{vmatrix}$

$= \dfrac{1}{2}\left[(2.5)(-k) + 5k^2 + \dfrac{50}{3} - \left(\dfrac{25}{3} - 5k\right)\right]$

$= \dfrac{5}{2}k^2 + \dfrac{5}{4}k + \dfrac{25}{6}$ units^2

13 **(a)** Show that $\sin 2x \cot x = 2\cos^2 x$. [1]

$$\sin 2x \cot x = 2\sin x \cos x \times \frac{\cos x}{\sin x}$$
$$= 2\cos^2 x$$

(b) Hence, without using a calculator, find the value of the constant k for which

$$\int_{\frac{\pi}{18}}^{\frac{\pi}{9}} \operatorname{cosec} 6x \tan 3x \, dx = \frac{\sqrt{3}}{k}.$$ [5]

$$\int_{\frac{\pi}{18}}^{\frac{\pi}{9}} \operatorname{cosec} 6x \tan 3x \, dx = \int_{\frac{\pi}{18}}^{\frac{\pi}{9}} \frac{1}{\sin 6x \cot 3x} \, dx$$
$$= \int_{\frac{\pi}{18}}^{\frac{\pi}{9}} \frac{1}{2\cos^2 3x} \, dx$$
$$= \int_{\frac{\pi}{18}}^{\frac{\pi}{9}} \frac{1}{2} \sec^2 3x \, dx$$
$$= \left[\frac{1}{2 \times 3} \tan 3x \right]_{\frac{\pi}{18}}^{\frac{\pi}{9}}$$
$$= \frac{1}{6}\left(\tan \frac{\pi}{3} - \tan \frac{\pi}{6} \right)$$
$$= \frac{1}{6}\left(\sqrt{3} - \frac{1}{\sqrt{3}} \right)$$
$$= \frac{1}{3\sqrt{3}}$$
$$= \frac{\sqrt{3}}{9}$$

$k = 9$