7 CIRCULAR MOTION

H2 Physics 9478

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Learning Outcomes

Candidates should be able to:

(a) express angular displacement in radians.

(b) show an understanding of and use the concept of angular velocity.

(c) recall and use $v = r\omega$ to solve problems.

(d) show an understanding of centripetal acceleration in the case of uniform motion in a circle, and qualitatively describe motion in a curved path (arc) as due to a resultant force that is both perpendicular to the motion and centripetal in direction.

(e) recall and use centripetal acceleration $a = r\omega^2$ and $a = v^2/r$ to solve problems.

(f) recall and use $F = mr\omega^2$ and $F = mv^2/r$ to solve problems.

7.1 Introduction

Circular motion is a common occurrence in our daily lives. The second hand of a clock goes round in a circle in 60 seconds while the Earth orbits the Sun once in 365 days. You probably have also experienced circular motion when you sat on a carousel or in the Singapore Flyer. As physics students, you want to go beyond the physical experience. You want to understand what causes an object to move in a circular motion and the physical quantities associated with it.

For a body moving in a circular path, it must already be travelling with a certain linear speed and at the same time, there must be a force applied on it that is directed towards the centre of the circular path. As the body tries to move off in a straight line, the applied force towards the centre pulls the body inwards, causing it to move in a curve. The continuous application of this centre-pointing force allows the body to move in a circular path. This force is aptly given the name centripetal force where "centri" means centre and "petal" means pointing in Latin.

The H2 Physics syllabus classifies circular motion into two categories: *uniform* and *non-uniform* circular motion.

Uniform circular motion can be described as the motion of a body in a circle at a constant speed. As the body moves in a circle, it is continuously changing its direction. At all instances, the body's velocity is tangent to the circle. For the magnitude of the velocity (speed) to remain constant, the resultant force on the body must continuously act perpendicular to its velocity. Hence the resultant force only has a centripetal component which points radially towards the centre of the circular path, with no component in the tangential direction.

In a non-uniform circular motion, the speed of the body is varying. This occurs when the resultant force on the body has both centripetal (radial) and tangential components. The tangential component of the resultant force causes the speed to vary. The centripetal component continuously changes the direction of the body.

In this topic, you will solve problems on uniform circular motion in a horizontal plane and uniform as well as non-uniform circular motion in a vertical plane.

7.2 Kinematics of Uniform Circular Motion

It is not possible to apply the kinematics equations for constant acceleration to uniform circular motion. This is because for a body moving in circular motion, its path is not straight and the direction of its acceleration is constantly changing implying that acceleration varies. Hence, we need a new set of equations with an understanding of certain important quantities.

Consider a body P moving in a uniform circle of radius *r* about point O as shown in Fig. 7.1. Body P has an angular velocity ω and a linear velocity *v* at any point in its path.

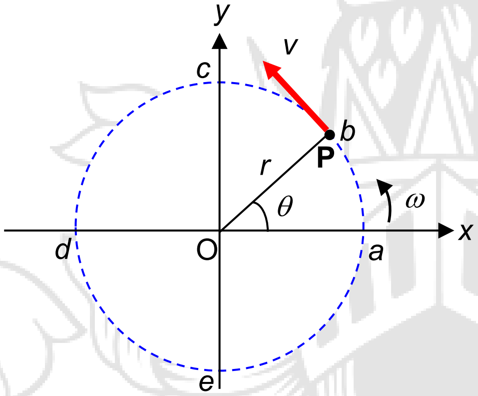


Fig. 7.1

Points *a*, *b*, *c*, *d* and *e* are points on its circular path. At time $t = 0$, body P is at point *a*. After a time interval of Δt, body P has rotated through an angle of θ to point *b*. Body P makes one complete revolution when it rotates from points *a* to *c* to *d* to *e* and back to *a*.

❖ Period and Frequency

Period

The **period** of a body in circular motion is the time taken for it to make one complete revolution.

S.I. unit: **s**

The period of body P is the total time it takes to movefrom positions *a* to *c* to *d* to *e* and back to *a* once.

Frequency

The **frequency** of a body in circular motion is the number of revolutions made per unit time.

S.I. unit: **Hz**

Period T and frequency f are related by the equation

$$\boxed{f = \frac{1}{T}}$$

❖ Angular Displacement

Angular Displacement

Angular displacement is the angle a body rotates through with respect to its initial position.

[Angular displacement Geogebra animation]

S.I. unit: **radian**.

The angular displacement of body P in the time interval of Δt is θ.

Radian

The **radian** is the angle subtended by an arc length equal to the radius of the arc.

In general, any angle θ measured **in radians** is defined by the relation:

$$\boxed{\theta = \frac{s}{r}}$$

where s is the arc length and r is the radius of the circle.

When the body moves one complete circle, the arc length s would be the circumference of the circle, which is equal to $2\pi r$. Therefore,

$$\theta = \frac{2\pi r}{r} = 2\pi$$

Since the body moves through 360° in one complete circle,

$$2\pi \text{ rad} = 360°$$
$$\pi \text{ rad} = 180°$$

In general, $$x \text{ rad} = x \times \frac{180°}{\pi}$$

Conversely, $$y° = y \times \frac{\pi}{180} \text{ rad}$$

Example 1

Fill in the blanks.

Angle in degrees	Angle in radians (in terms of π)
360	
90	
180	
45	
30	
1	

Example 2

The laser in a CD player is 5.0 cm from the central of the disc. What length of the disc is scanned by the laser when the disc turns through an angle of 0.45 radians?

❖ Angular Velocity

Angular Velocity

Angular velocity of a body is defined as the rate of change of its angular displacement with respect to time.

Angular velocity Geogebra animation

Angular velocity, $\boxed{\omega = \frac{d\theta}{dt}}$

where θ is the angular displacement and t is the elapsed time.

S.I. unit: **rad s^{-1}**

Angular velocity is a **vector** – it has both magnitude and direction. The direction is described as "clockwise" or "anti-clockwise".

For a body in uniform circular motion, its angular velocity is constant.

❖ Relationship between Angular Velocity and Linear Velocity

Differentiating $s = r\,\theta$ with respect to t and since r is a constant,

$$\frac{ds}{dt} = r\frac{d\theta}{dt}$$

$$\boxed{\therefore\ v = r\omega}$$

where $v = \frac{ds}{dt}$ and $\omega = \frac{d\theta}{dt}$.

Linear velocity v is also known as the **tangential velocity**.

Linear or tangential velocity is a vector quantity. Linear speed is the magnitude of linear velocity.

For a body in uniform circular motion, its linear speed is constant.

❖ Relationship between Angular Velocity, Period and Frequency

Time taken for one complete revolution, period = T

Angular displacement for one complete revolution = 2π

From the definition, $\omega = \frac{d\theta}{dt}$

Therefore, $\boxed{\omega = \frac{2\pi}{T}}$

Since $T = \frac{1}{f}$, $\boxed{\omega = 2\pi f}$

❖ Relationship between Period and Linear Speed

Linear speed v of a body moving in a uniform circular motion,

$$v = \frac{\text{circumference of the circle}}{T}$$

Therefore, $$\boxed{T = \frac{2\pi r}{v}}$$

Example 3

Alice and Bob are riding on a merry-go-round, which is rotating at a constant angular velocity. Alice stands on a point twice as far from the centre of the platform as Bob.

Which of the following statements is correct?

A. Alice's linear velocity is twice of Bob's.
B. Alice's linear velocity is the same as Bob's.
C. Alice's linear velocity is half of Bob's.
D. Alice's linear velocity is a quarter of Bob's.

7.3 Dynamics of Uniform Circular Motion

❖ Presence of a Resultant Force

For a body moving in a uniform circular motion, both its linear speed v and angular velocity ω are constant. However, a resultant force must act on this body. Why is this so? The following derivation provides the proof.

Uniform circular motion Geogebra animation

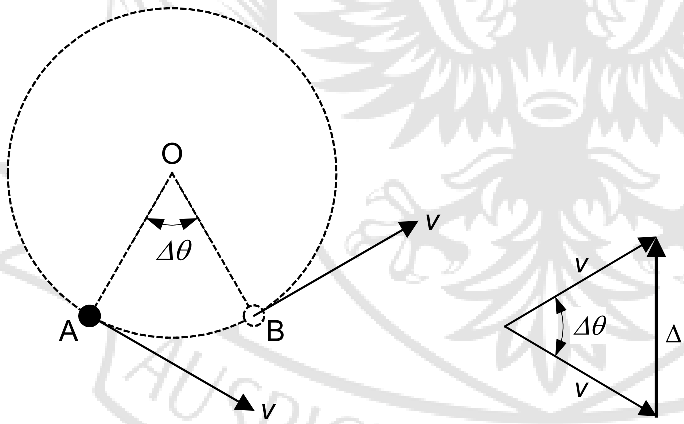
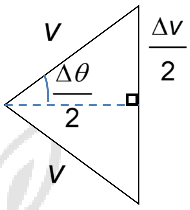


Fig. 7.2

Consider a body moving from A to B with a constant speed v. In a short time Δt , the body experiences a small angular displacement of $\Delta\theta$. The change in velocity Δv is shown in the Fig. 7.2 on the right.

$$\sin\frac{\Delta\theta}{2} = \frac{\frac{\Delta v}{2}}{v}$$

$$\Delta v = 2 \times v \sin\frac{\Delta\theta}{2}$$



For small values of $\Delta\theta$, measured in radians,

$$\sin\frac{\Delta\theta}{2} \approx \frac{\Delta\theta}{2} \quad \text{(small angle approximation)}$$

$$\therefore \Delta v \approx v\Delta\theta$$

Dividing both sides by Δt,

$$\frac{\Delta v}{\Delta t} \approx v\frac{\Delta\theta}{\Delta t}$$

For infinitesimal small values of $\Delta\theta$ and Δt, the above approximation becomes equality,

$$\frac{dv}{dt} = v\frac{d\theta}{dt}$$

$$a = v\omega$$

Since $v = r\omega$, the magnitude of the **centripetal acceleration** can be given by

$$\boxed{a = v\omega = r\omega^2 = \frac{v^2}{r}}$$

When a body describes a circular path, the direction of motion of the body is constantly changing. This means the velocity of the body is also constantly changing. Since there is a rate of change of velocity, the body is accelerating. According to Newton's Second Law, a resultant force must act on the body to cause this acceleration. This resultant force, which always points towards the centre of the circular path, is the *centripetal force*.

Another way of understanding this is, according to Newton's First Law of motion, the body will continue to move at its constant speed in the same direction unless there is a resultant force acting on it. Hence, there must be a resultant force acting on the body to constantly change its direction of motion.

❖ Direction of the Resultant Force

For the body to move at constant speed, the resultant force cannot have any component in the direction of the motion (tangential direction) of the body. Otherwise, it will increase or decrease the speed of the body.

Therefore, the direction of this resultant force must point towards the centre of the circle such that it is always perpendicular to its instantaneous velocity. This resultant force only changes the direction of the velocity of the body but does not change its speed.

Since the resultant force points towards the centre, this means the acceleration of the body also points towards the centre. From a vector diagram, the direction of the change in velocity (acceleration) is radially towards the centre of the circular path.

❖ Magnitude of the Resultant Force

For a body moving in a uniform circular motion, the magnitude of the resultant force acting on it is also constant. The linear speed and the radius of the circular path are both constant. Hence, the centripetal force F can be given by

$$\boxed{F = mv\omega = mr\omega^2 = \frac{mv^2}{r}}$$

❖ Work Done by the Resultant Force

For a body moving in a uniform circular motion, since the resultant force towards the centre is always perpendicular to its instantaneous velocity, it is also always perpendicular to the instantaneous displacement of the body.

Since the displacement in the direction of the resultant force is zero, the work done by the resultant force is zero. The kinetic energy and hence the speed of the body remains constant.

❖ Centripetal Acceleration and Centripetal Force

For a body in circular motion, it experiences acceleration and a resultant force must act on it. The resultant acceleration and the resultant force can be resolved to its perpendicular components. One component along the radius and the other along the tangent to the circular path.

The radial component of the resultant acceleration directed towards the centre is called the **centripetal acceleration** and the corresponding resultant force is called the **centripetal force**.

In a uniform circular motion, the tangential components (parallel to the velocity) of the resultant acceleration and the resultant force are zero. Hence the resultant acceleration and the resultant force only have the centripetal components. This means the resultant acceleration and the resultant force point towards the centre of the circle.

In a **non-uniform** circular motion, both the tangential and centripetal components of the resultant acceleration and resultant force are not zero. This means the resultant acceleration and the resultant force do not point towards the centre of the circle.

The magnitude of the **centripetal acceleration** is given by:

$$\boxed{a_c = v\omega = r\omega^2 = \frac{v^2}{r}}$$

where $v = r\omega$.

By Newton's Second Law, the **centripetal force** acting on the object is

$$F_c = ma_c = mv\omega = mr\omega^2 = m\frac{v^2}{r}$$

where F_c is the centripetal force, m is the mass of the body and a_c is the centripetal acceleration.

The centripetal force is not another type of force. It is the centripetal or radial component of the resultant force on the body in circular motion. It can be provided by the tension in a string, gravitational force of attraction between two masses, frictional force between two surfaces or a combination of forces, depending on the context.

The centripetal force should not be drawn in a free body diagram in addition to the other forces acting on the body.

If the centripetal force is removed, the object no longer moves in its circular path. It will leave the circular path tangentially, in the direction of its instantaneous velocity, according to Newton's First Law of Motion.

Example 4

Assuming the Earth orbits round the Sun at a constant speed of 29.9×10^3 m s^{-1}, find the centripetal acceleration of the Earth.

7.4 Problem Solving Strategy for Uniform Circular Motion

1. Draw a free-body diagram of the body undergoing circular motion.
2. Identify the centre of the circular path.
3. Resolve each force along two perpendicular directions. One of the directions must be along the radius of the circular path. The other direction will be perpendicular to the radius.
4. Determine the net force F_{net} towards the centre of the circular path.

   Equate it to the centripetal force $F_{net} = ma_c = mr\omega^2$ (or $\frac{mv^2}{r}$).
5. For a *uniform horizontal* circular motion, the body does not experience acceleration in the direction perpendicular to the plane of the circular motion (vertical direction). Since there is no resultant force in the vertical direction, another equation can be formed by equating the sum of the upward forces to the sum of the downward forces.

7.5 Uniform Horizontal Circular Motion

❖ **Ball Moving on a Smooth Horizontal Surface**

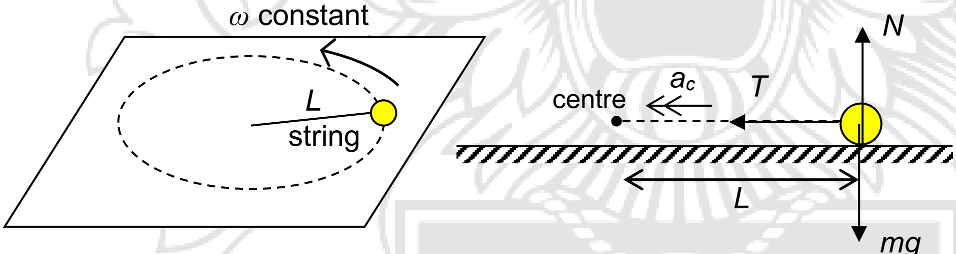


Fig. 7.3

A ball of mass *m* is tied to one end of a string of length *L*. The other end of the string is fixed to a point on a smooth table surface. The ball is made to rotate about the fixed point with speed *v*.

Fig. 7.3 shows the free-body diagram of the ball, where *N* is the normal contact force exerted by the table. The resultant force acting on the ball is the tension *T* in the string. Since T is the only force directed towards the centre of the circular path, it provides the centripetal force.

Radially towards the centre: $T = ma_c = mr\omega^2 = m\frac{v^2}{r}$, where $r = L$.

Vertically, forces are in equilibrium: $N = mg$.

❖ **Conical Pendulum**

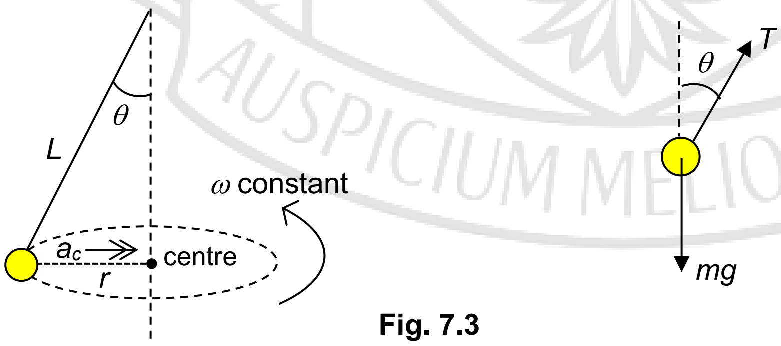


Fig. 7.3

A pendulum of length L and mass m is suspended from one end. The pendulum is swung such that it describes a horizontal circular path with speed v and makes an angle θ with the vertical.

Fig. 7.3 shows the free-body diagram of the pendulum. The resultant force acting on the pendulum is the radial component of the tension T in the string, $T\sin\theta$. Since this is the only force directed towards the centre of the circular path, it provides the centripetal force.

Radially towards the centre: $T\sin\theta = ma_c = mr\omega^2 = m\dfrac{v^2}{r}$ ..... (1)

where $r = L\sin\theta$

Vertically, forces are in equilibrium: $T\cos\theta = mg$ ..... (2)

$\dfrac{(1)}{(2)}$: $\tan\theta = \dfrac{v^2}{rg}$

❖ Aircraft Going Round a Horizontal Circular Turn

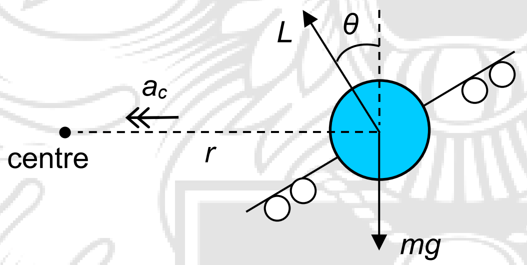


Fig. 7.4

An aircraft of mass m, moving with speed v, turns in a circular path of radius r in a horizontal plane. The forces acting on the aircraft are the lift force L which acts perpendicularly to the wingspan of the aircraft and its weight mg, as shown in Fig. 7.4.

The aircraft needs to be tilted inwards towards the centre of the circular path such that the lift force L makes an angle θ with the vertical. This allows the radial component $L\sin\theta$ to provide the centripetal force.

Radially towards the centre: $L\sin\theta = ma_c = mr\omega^2 = m\dfrac{v^2}{r}$ ..... (1)

Vertically, forces are in equilibrium: $L\cos\theta = mg$ ..... (2)

$\dfrac{(1)}{(2)}$: $\tan\theta = \dfrac{v^2}{rg}$

❖ Cyclist Moving Round a Horizontal Circular Track

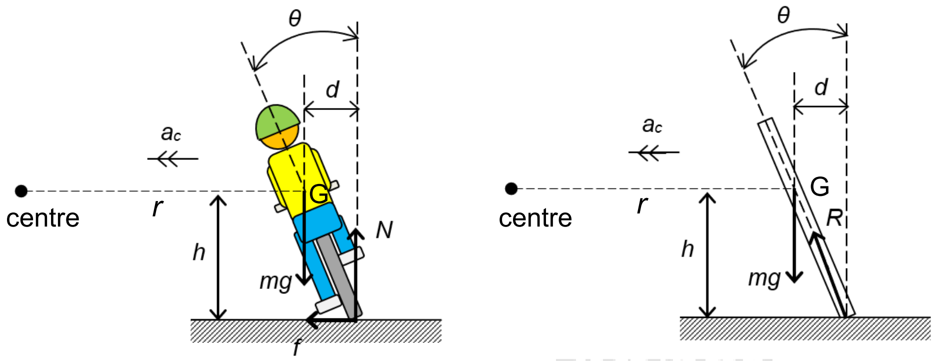


Fig. 7.5

A cyclist riding a bicycle is moving with speed v around a horizontal circular track of radius r. The forces acting on the cyclist and the bicycle are the normal contact force N and friction f experienced by the track as well as the total weight mg of the cyclist and bicycle, as shown in Fig. 7.5. The resultant of N and f is the total contact force R by the road. R acts at an angle θ to the vertical.

The cyclist and the bicycle need to tilt inwards towards the centre of the circular path to ensure that they do not rotate about their centre of gravity G. Friction f produces a clockwise moment about G. When tilted inwards, the normal contact force N produces an equal anticlockwise moment about G so that the resultant moment about G is zero. R also needs to act through G for rotational equilibrium.

The resultant force acting on the cyclist and bicycle is f or $R\sin\theta$. The direction of f is towards the centre of the circular path and it provides the centripetal force.

Radially towards the centre: $R\sin\theta = f = ma_c = mr\omega^2 = m\dfrac{v^2}{r}$ ----- (1)

Vertically, forces are in equilibrium: $R\cos\theta = N = mg$ ----- (2)

$\dfrac{(1)}{(2)}$: $\tan\theta = \dfrac{f}{N} = \dfrac{v^2}{rg}$

❖ Car Travelling Round a Circular Bend on a Flat Horizontal Road

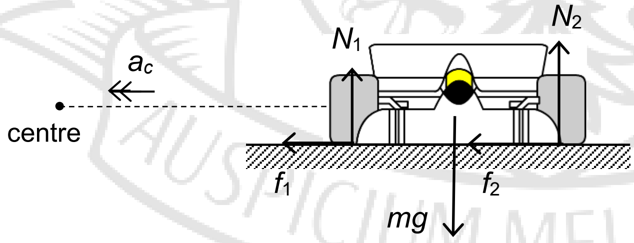


Fig. 7.6

A car of mass m is moving with speed v around a circular bend of radius r on a flat horizontal road. A normal contact force and a sideward frictional force by the road act on each of the four tyres.

The forces on the front and rear tyres can be added up vectorially such that there are only two normal contact forces, N_1 and N_2 and two sideward friction f_1 and f_2 as shown in Fig. 7.6.

The resultant force acting on the car is $f_1 + f_2$. The direction of $f_1 + f_2$ is towards the centre of the circular path and it provides the centripetal force.

Radially towards the centre: $f_1 + f_2 = ma_c = mr\omega^2 = m\dfrac{v^2}{r}$

Vertically, forces are in equilibrium: $N_1 + N_2 = mg$

As the speed of the car increases, the centripetal acceleration increases. The centripetal force needed also increases to maintain the same radius of circular motion. This means the sideward friction $f_1 + f_2$ which provides the centripetal force must increase.

If the speed of the car increases to a value where the centripetal force that is required to keep the car on the circular path is greater than the maximum sideward friction that can be provided by the road, the car will skid off the road.

When going round sharp bends where the radius of curvature is small and the centripetal force required is large or when the road is wet where friction is reduced, there is greater danger of skidding. To minimise this danger, the road is sloped or banked at an angle to the horizontal, instead of being flat.

❖ **Car Travelling Round a Circular Bend on a Banked Road**

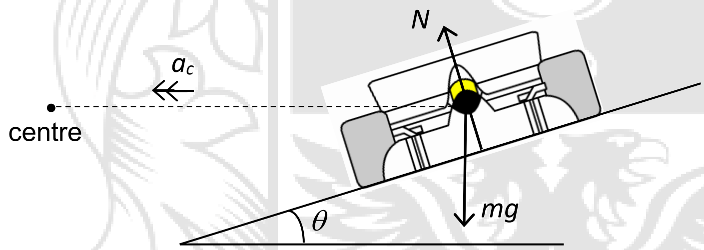


Fig. 7.7

When the road is banked, the normal contact force on the car is tilted at an angle to the vertical and it helps to provide the centripetal force. This reduces the sideward friction that is needed to provide the centripetal force.

With reduced friction, it cuts down wear on the road surface and on the tyres. It also reduces noise and enhances safety.

If banking is ideal, a car of mass m travelling round a bend of radius r on a banked road at speed v_{ideal} in a horizontal circular path, does not need any sideward friction to provide the centripetal force.

The resultant force acting on the car is the radial component of the normal contact force *N*, $N\sin\theta$. Since the direction of $N\sin\theta$ is towards the centre of the circular path, $N\sin\theta$ provides the centripetal force.

Radially towards the centre: $N\sin\theta = ma_c = mr\omega^2 = m\dfrac{{v_{ideal}}^2}{r}$ ..... (1)

Vertically, forces are in equilibrium: $N\cos\theta = mg$ ..... (2)

$\dfrac{(1)}{(2)}$: $\tan\theta = \dfrac{{v_{ideal}}^2}{rg}$

For a particular bend if the road is banked at this angle θ and the car travels at v_{ideal}, the car can go round the bend without skidding and with no need of friction.

If the actual speed of the car is greater than v_{ideal}, it will need a larger centripetal force. The sideway friction will act down the slope to increase the centripetal force needed. This also means that the car tends to slide up the banked road at a greater speed.

If the actual speed of the car is smaller than v_{ideal}, it will need a smaller centripetal force. The sideway friction will act up the slope to decrease the centripetal force needed. This also means that the car tends to slide down the banked road at a smaller speed.

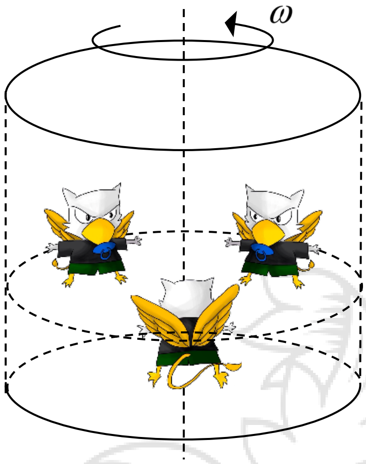
Example 5

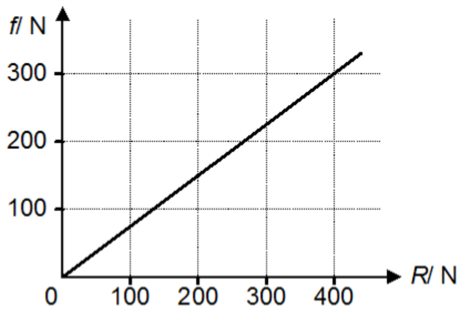
A pendulum bob of mass 0.15 kg suspended from a fixed point moves in a horizontal circular motion of radius 1.82 m as shown. The bob moves at a steady speed, taking 18.0 s to make 10 complete revolutions.

Calculate
(a) the centripetal acceleration of the bob,
(b) the tension in the thread.

Example 6

Three baby Griffles, each of mass 30 kg, are in a large vertical cage of radius 5.0 m, spinning about a vertical axis through its centre. When the cage is spinning sufficiently fast, the floor is dropped and the Griffles are stuck to the wall. The graph shows the variation in frictional force *f* with normal contact force *R* between the baby Griffles and the wall.





Determine the minimum angular speed at which the cage must be rotated for the Griffles to remain in position when the floor is dropped.

(Take g = 10 m s^{-2})

7.6 Vertical Circular Motion

When a body moves in a circular path in the vertical plane, its height changes as the body rises and falls. This means the gravitational potential energy of the body changes, which in turn causes the kinetic energy of the body to change. When kinetic energy changes, it means that the speed of the body changes. In such a case, the body is performing non-uniform circular motion.

However, in certain scenarios, the speed of the body can be made to remain constant even when the gravitational potential energy is changing. This is possible if there is work done by an external force on the system.

In analysing vertical circular motion, since there must still be a centripetal force on the body, $F_c = ma_c$ can be applied. In addition, the conservation of energy is also often applied.

❖ Uniform: Speed of Body Constant

Consider a ball of mass m attached to a light rigid rod of length L that is moving at constant speed v in a circle in the vertical plane. The radius r of the circular path is the length of the rod L. The forces acting on the ball are tension T due to the rod and its weight mg.

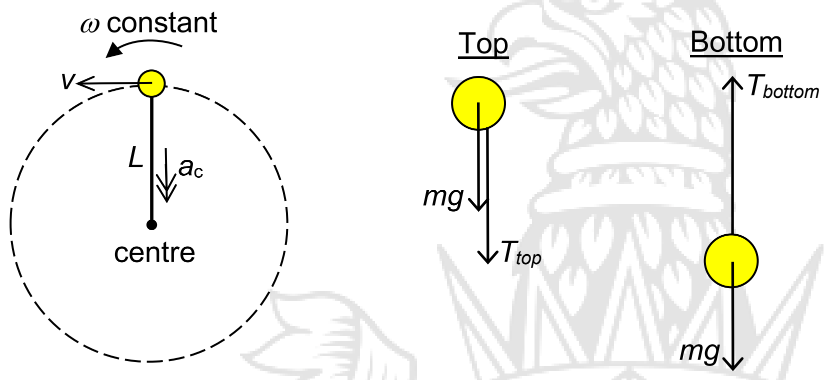


Fig. 7.8

Referring to Fig. 7.8, when the ball is at the top of the circular path, radially towards the centre of the circle,

$$T_{top} + mg = \frac{mv^2}{r}$$

$$T_{top} = \frac{mv^2}{r} - mg \quad \text{..... (1)}$$

When the ball is at the bottom of the circular path, radially towards the centre of the circle,

$$T_{bottom} - mg = \frac{mv^2}{r}$$

$$T_{bottom} = \frac{mv^2}{r} + mg \quad \text{..... (2)}$$

Fig. 7.9 shows the ball when the rod is at any other angle θ with the vertical during its circular motion.

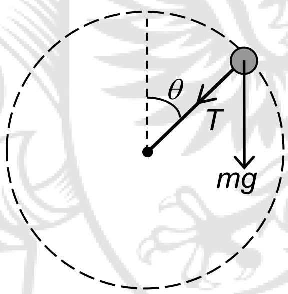


Fig. 7.9

Radially towards the centre,

$$T + mg\cos\theta = \frac{mv^2}{r}$$

$$T = \frac{mv^2}{r} - mg\cos\theta \quad \text{..... (3)}$$

Tension in the rod is dependent on θ and it varies continuously throughout the circular motion while speed of the ball remains constant. Tension is smallest when the ball is at the top and largest when the ball is at the bottom. Note that equation (3) reduces to (1) and (2) when $\theta =$ 0° and 180° respectively.

As the ball goes round in the vertical circular path, its gravitational potential energy changes continuously. Its kinetic energy and hence linear speed are expected to change based on the conservation of energy. To keep the kinetic energy and speed constant, work must be done on the system.

When the ball moves from the bottom to the top of the circle, to ensure that kinetic energy does not decrease due to an increase in gravitational potential energy, there must be positive work done by an external force on the system. One possible way is to apply a torque to the rod in the same direction as the rotation.

Similarly, when the ball moves from the top of the circle to the bottom, to ensure that kinetic energy does not increase due to a decrease in gravitational potential energy, there must be negative work done by an external force on the system. One possible way is to apply a torque to the rod in the opposite direction to the rotation.

The conservation of energy for the whole system can be written as

initial G.P.E. + work done by external force = final G.P.E.

❖ Non-uniform: Speed of Body not Constant

Consider a roller coaster carriage of mass m moving round in a vertical circular track of radius r in a loop-a-loop. Assume the carriage does not have an engine and moves freely. Its motion is purely due to the interchanges between its kinetic energy and gravitational potential energy. Hence its speed changes continuously. The forces acting on the carriage are the normal contact force N by the track and its weight mg.

Non-uniform circular motion Geogebra animation

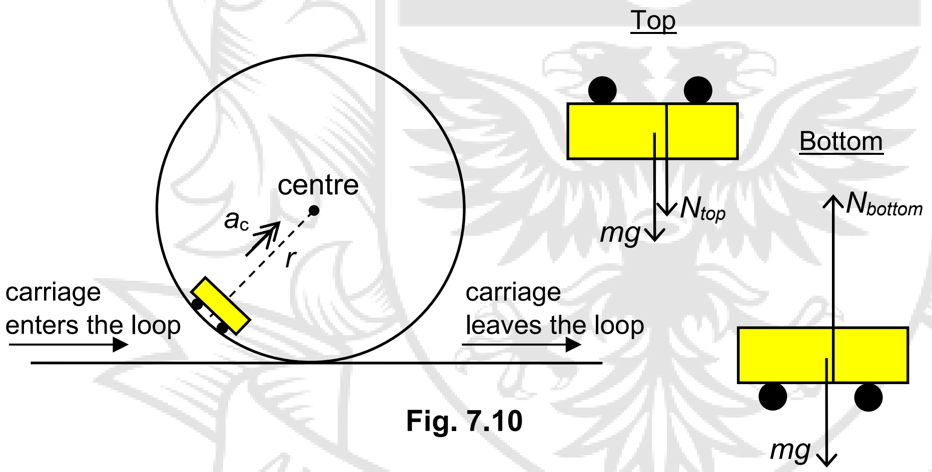


Fig. 7.10

Referring to Fig. 7.10, when the carriage is at the top of the circular path, radially towards the centre of the circle,

$$N_{top} + mg = \frac{mv_{top}^{2}}{r}$$

$$N_{top} = \frac{mv_{top}^{2}}{r} - mg \qquad \text{----- (1)}$$

When the carriage is at the bottom of the circular path, radially towards the centre of the circle,

$$N_{bottom} - mg = \frac{m{v_{bottom}}^2}{r}$$

$$N_{bottom} = \frac{m{v_{bottom}}^2}{r} + mg \quad \text{----- (2)}$$

When the carriage moves from the bottom to the top of the loop, there is an increase in gravitational potential energy. By the conservation of energy, kinetic energy of the carriage will decrease by the same amount. This means $v_{bottom} > v_{top}$.

Comparing equations (1) and (2), we can conclude that $N_{bottom} > N_{top}$. Furthermore, from the equations it can be seen that if the speed of the carriage decreases, the normal contact force decreases.

For the carriage to complete the circular motion, it needs to be always in contact with the track. The critical case is when the carriage just loses contact with the track i.e. the normal contact force just goes to zero. Since $N_{bottom} > N_{top}$, the critical case will first happen at the top of the loop. This also means that the speed of the carriage must be sufficiently large at the top for it to remain in contact with the track.

When the carriage is at the top, radially towards the centre:

$$N_{top} + mg = \frac{m{v_{top}}^2}{r}$$

$$N_{top} = \frac{m{v_{top}}^2}{r} - mg$$

For carriage to remain in contact with the track throughout its motion around the loop,

$$N_{top} \geq 0$$

$$\frac{m{v_{top}}^2}{r} - mg \geq 0$$

$$v_{top} \geq \sqrt{rg}$$

Minimum speed of the carriage at the top to remain in contact,

$$v_{top,\min} = \sqrt{rg}$$

Assuming energy lost due to air resistance and friction are negligible, by the conservation of energy,

$$\text{total initial energy at the bottom} = \text{total final energy at the top}$$

$$\frac{1}{2}m{v_{bottom,\min}}^2 = \frac{1}{2}m{v_{top,\min}}^2 + mg(2r)$$

$$\frac{1}{2}m{v_{bottom,\min}}^2 = \frac{1}{2}m(gr) + mg(2r)$$

$$v_{bottom,\min} = \sqrt{5gr}$$

This is the minimum speed at which the carriage must enter the loop at the bottom so that it just remains in contact with the track when it is at the top.

If the carriage enters the loop with a speed larger than $\sqrt{5gr}$, it will remain well in contact with the loop throughout its circular motion.

Fig. 7.11 shows the carriage when it is at any other angle θ with the vertical during its circular motion.

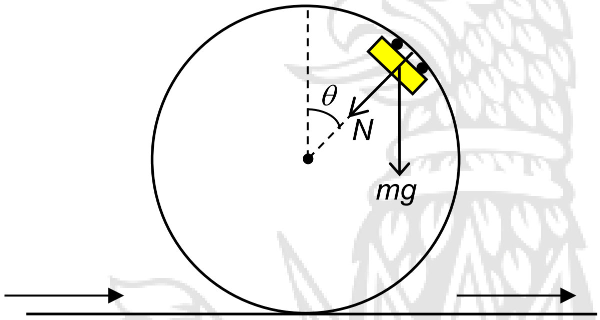


Fig. 7.11

For a carriage that is moving freely, only the normal contact force and weight acts on it. From Fig. 7.11, it can be seen that the resultant of these two forces is usually not towards the centre of the circular path but is at an angle to the radius of the circular path. This resultant force has a centripetal (radial) component that is responsible for the circular motion and a tangential component that causes the speed to vary.

Centripetal component of the resultant force,

$$F_c = ma_c$$

$$N + mg\cos\theta = \frac{mv_\theta^{\ 2}}{r}$$

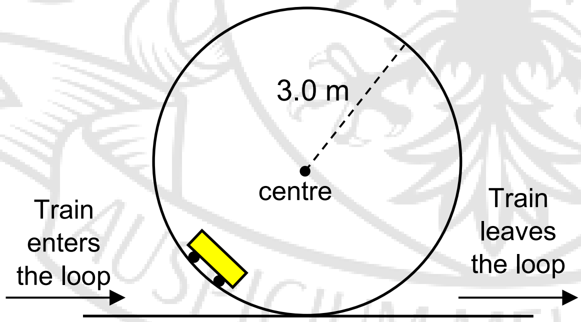
Tangential component of the resultant force,

$$F_t = ma_t$$

$$mg\sin\theta = ma_t$$

Example 7

The figure below shows a roller coaster train of mass 50 kg moving in a circular loop of radius 3.0 m. Assume that frictional forces may be neglected.



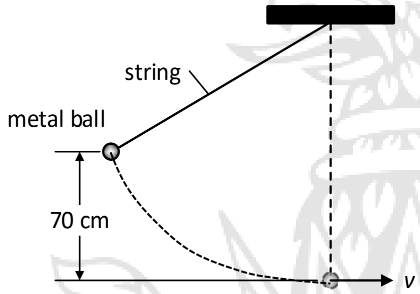
(a) What minimum speed must the train have at the top for it to be *just in contact* with the loop?

(b) What speed must the train enter at the bottom of the track so that the train just makes it over the top of the loop?

Examples of circular motion

Example 8

One end of a string is secured to the ceiling and a metal ball of mass 50 g is tied to its other end. The length of the string is 1.50 m. The ball is initially at rest in the vertical position. The ball is raised through a vertical height of 70 cm as shown in the figure below. When the ball is released, it describes a circular arc as it passes through the vertical position.



(a) Ignoring the effects of air resistance, show that the speed v of the ball is 3.7 m s^{-1} as it passes through the vertical position.
(b) Draw a labelled force diagram to show all the forces acting on the ball when the string is in the vertical position.
(c) Calculate the centripetal force on the ball when it is in the vertical position.
(d) Explain why the tension in the string when it is vertical is not equal to the weight of the ball.
(e) Calculate the tension in the string when it is vertical.