

SUMMARY: Sequences and Series

Notations

a : first term of a sequence, particularly in an arithmetic (AP) or geometric progression (GP).

u_n or T_n : n th term of a sequence

S_n : sum of the first n terms of a sequence

Special sequences: Arithmetic and Geometric Sequences

	AP	GP
Type	$a, a + d, a + 2d, \dots, a + (n-1)d, \dots$	$a, ar, ar^2, \dots, ar^{n-1}, \dots$
Characteristic	Common difference, d , i.e., $d = u_n - u_{n-1} = \text{constant}$	Common ratio, r , i.e. $r = \frac{u_n}{u_{n-1}} = \text{constant}$
n th term	$u_n = a + (n-1)d$	$u_n = ar^{n-1}$
Sum of the first n terms	$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a + L)$ where last term, $L = a + (n-1)d$.	$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$, where $r \neq 1$.
Sum to infinity	Does not exist	$S_\infty = \frac{a}{1-r}$, S_∞ exists if $-1 < r < 1$

Sequences

Examples	Defined by a formula for the n th term	Defined by a relation $u_{n+1} = f(u_n)$
$\{1, 3, 5, 7, \dots\}$	$f(n) = 2n-1, n \geq 1$	$u_1 = 1, u_{n+1} = u_n + 2$ for $n \geq 1$
$\{4, 8, 16, 32, \dots\}$	$f(n) = 2^{n+1}, n \geq 1$	$u_1 = 4, u_{n+1} = 2u_n$ for $n \geq 1$
$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\right\}$	$f(n) = \frac{n}{n+1}, n \geq 1$	
$\{-2, 5, -2, 5, \dots\}$		$u_1 = -2, u_{n+1} = -u_n + 3$ for $n \geq 1$

Series and the Sigma Notation

	Using the notations S_n and u_n	Using the sigma notation
Sum of the first n terms	$S_n = u_1 + u_2 + u_3 + \dots + u_n$	$\sum_{r=1}^n u_r = u_1 + u_2 + u_3 + \dots + u_n$
n th term	$u_n = S_n - S_{n-1}$	$u_n = \sum_{r=1}^n u_r - \sum_{r=1}^{n-1} u_r$
First term	$u_1 = S_1$	$u_1 = \sum_{r=1}^1 u_r$
Sum from the m th term to the n th term where $n > m$	$u_m + u_{m+1} + u_{m+2} \dots + u_n$ $= S_n - S_{m-1}$	$\sum_{r=m}^n u_r = \sum_{r=1}^n u_r - \sum_{r=1}^{m-1} u_r$ $= u_m + u_{m+1} + u_{m+2} \dots + u_n$

Convergence

A sequence $\{u_n\}$ is convergent $\Leftrightarrow u_n \rightarrow l$ as $n \rightarrow \infty$, where l is a finite number.

i.e. $\lim_{n \rightarrow \infty} u_n = l$, a finite number. Otherwise, it is divergent.

A series S_n is convergent $\Leftrightarrow S_n \rightarrow l$ as $n \rightarrow \infty$, where l is a finite number.

i.e. $\lim_{n \rightarrow \infty} S_n = l$, a finite number. Otherwise, it is divergent.

Common series that can be evaluated immediately

	Series	Examples
1	$\sum_{r=m}^n k = \underbrace{k + k + k + \dots + k}_{n-m+1 \text{ times}} = (n-m+1)k,$ where k is a constant.	$\sum_{r=5}^{26} 3 = \underbrace{3+3+3+\dots+3}_{26-5+1=22 \text{ times}}$ $= 22 \times 3 = 66$
2	$\sum_{r=m}^n r = m + (m+1) + (m+2) + \dots + n \quad \text{(AP)}$ $= \frac{n-m+1}{2} [m+n]$	$\sum_{r=3}^{20} r = 3 + 4 + 5 + \dots + 20$ $= \frac{18}{2} (3+20)$
3	$\sum_{r=m}^n k^r = k^m + k^{m+1} + k^{m+2} + \dots + k^n \quad \text{(GP)}$ $= \frac{k^m [1 - k^{(n-m+1)}]}{1 - k}$	$\sum_{r=5}^{14} 2^r = 2^5 + 2^6 + 2^7 + \dots + 2^{14}$ $= \frac{2^5 (1 - 2^{10})}{1 - 2}$

Some series, e.g. $\sum_{r=1}^n r^2$, $\sum_{r=1}^n r^3$ have standard results that will be given should a question require their use.

Series that require formula or algebraic manipulation to evaluate

Suppose it is given (or found earlier in a question) that

$$\sum_{r=1}^n \frac{1}{r(r+2)} = \frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)} \dots\dots\dots (*)$$

1	Change of upper index Find (a) $\sum_{r=1}^{n+2} \frac{1}{r(r+2)}$ and (b) $\sum_{r=1}^{\infty} \frac{1}{r(r+2)}$. (a) $\sum_{r=1}^{n+2} \frac{1}{r(r+2)} = \frac{3}{4} - \frac{1}{2[(n+2)+1]} - \frac{1}{2[(n+2)+2]}$ $= \frac{3}{4} - \frac{1}{2(n+3)} - \frac{1}{2(n+4)}.$ (b) As $n \rightarrow \infty$, $\frac{1}{2(n+1)} \rightarrow 0$ and $\frac{1}{2(n+2)} \rightarrow 0$. $\therefore \frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)} \rightarrow \frac{3}{4}$ Hence $\sum_{r=1}^n \frac{1}{r(r+2)} \rightarrow \frac{3}{4}$ as $n \rightarrow \infty$, i.e. $\sum_{r=1}^{\infty} \frac{1}{r(r+2)} = \frac{3}{4}$.
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Substitute n in (*) by $n+2$

2	Change of lower index Find $\sum_{r=5}^n \frac{1}{r(r+2)}$. $\begin{aligned} \sum_{r=5}^n \frac{1}{r(r+2)} &= \sum_{r=1}^n \frac{1}{r(r+2)} - \sum_{r=1}^4 \frac{1}{r(r+2)} \\ &= \frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)} - \left[\frac{3}{4} - \frac{1}{2(4+1)} - \frac{1}{2(4+2)} \right] \\ &= \frac{11}{60} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}. \end{aligned}$
3	Change of $f(r)$ and index by substitution In general, $\begin{aligned} \sum_{r=1}^n f(r) &= \sum_{r=0}^{n-1} f(r+1) && \text{replace } r \text{ by } r+1 \\ &= \sum_{r=2}^{n+1} f(r-1) && \text{replace } r \text{ by } r-1 \\ &= \sum_{r=3}^{n+2} f(r-2) && \text{replace } r \text{ by } r-2 \\ &\text{etc.} \end{aligned}$ The above summations are all equivalent. To find $\sum_{r=4}^{n+3} \frac{1}{r(r-2)}$, first express what you want to find (i.e. $\sum_{r=4}^{n+3} \frac{1}{r(r-2)}$), in terms of the result for the function you already have: $\begin{aligned} \sum_{r=4}^{n+3} \frac{1}{r(r-2)} &= \sum_{r+2=4}^{r+2=n+3} \frac{1}{(r+2)(r+2-2)} && \text{replace } r \text{ by } r+2 \\ &= \sum_{r=2}^{n+1} \frac{1}{(r+2)r} \\ &= \sum_{r=2}^{n+1} \frac{1}{r(r+2)}. \end{aligned}$ Then evaluate the series using 1 and/or 2 above. [Ans: $\frac{5}{12} - \frac{1}{2(n+2)} - \frac{1}{2(n+3)}$]