H3 Mathematics (9820)
Combinatorics 2
Principle of Inclusion and Exclusion (PIE)

1 The Principle of Inclusion and Exclusion (PIE)

Recall that when the sets A and B are disjoint, $|A \cup B| = |A| + |B|$ (by the Addition Principle).

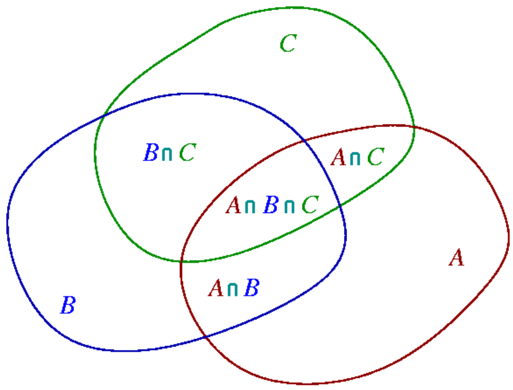
However, when the sets A and B are not disjoint, how do we find the value of $|A \cup B|$? What if we have more than 2 sets?

Principle of Inclusion and Exclusion (PIE)

For 2 sets: $|A \cup B| = |A| + |B| - |A \cap B|$

If A and B are disjoint sets, then $|A \cup B| = |A| + |B|$ (AP)

For 3 sets: $|A \cup B \cup C| = |A| + |B| + |C| - \left(|A \cap B| + |A \cap C| + |B \cap C|\right) + |A \cap B \cap C|$



Useful result

For a real number r, the **floor function** $\lfloor r \rfloor$ denotes the greatest integer that is smaller than or equal to r. In general, for any two natural numbers n, k with $k \le n$, the number of integers in the set $\{1, 2, \ldots, n\}$ which are divisible by k is given by $\left\lfloor \dfrac{n}{k} \right\rfloor$.

Example 1

Find the number of integers from the set $\{1,\ 2,\ \dots,\ 1000\}$ which are divisible by 3 or 5.

Let $S=\{1,\ 2,\ \dots,\ 1000\}$,

$A=\{x\in S\,|\,x \text{ is divisible by } 3\}$, and

$B=\{x\in S\,|\,x \text{ is divisible by } 5\}$.

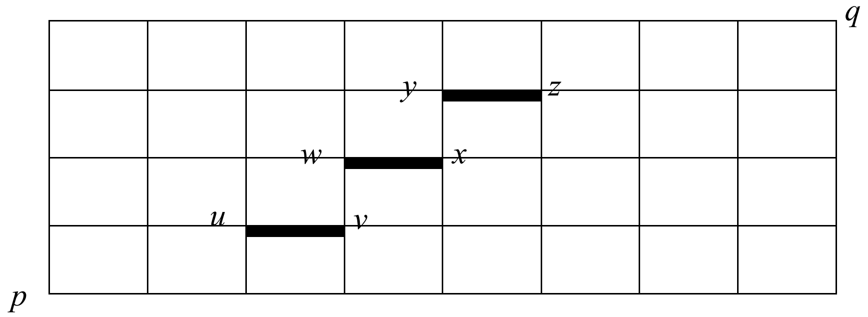
Example 2

Find the number of positive divisors of at least one of the numbers 5400 and 18000.

Let $A = \{x \in \mathbb{N}: x \text{ is a divisor of } 5400\}$ and $B = \{x \in \mathbb{N}: x \text{ is a divisor of } 18000\}$.

Example 3 (Self-Reading)

The diagram below shows a 4 by 8 rectangular grid with two specified corners p and q and three specified segments uv, wx and yz.



Find in the grid

(i) the number of shortest p-q routes;

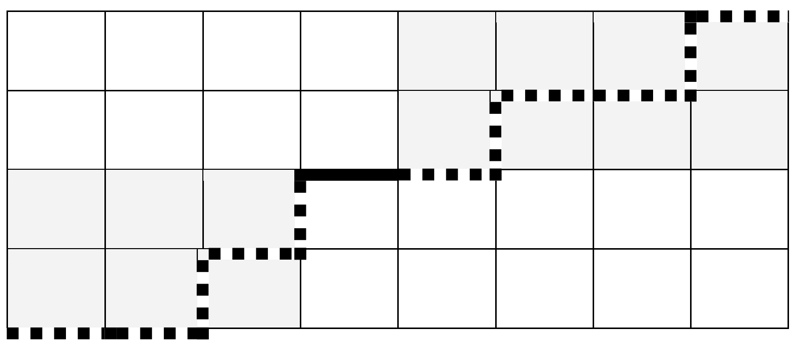
(ii) the number of shortest p-q routes which pass through wx;

(iii) the number of shortest p-q routes which pass through at least one of the segments uv, wx and yz;

(iv) the number of shortest p-q routes which do not pass through any of the segments uv, wx and yz.

(i) Number of shortest p-q routes is given by $\binom{4+8}{4} = \binom{12}{4} = 495$.

(ii) As shown in the diagram below, a shortest p-q route passing through wx consists of a shortest p-w route (in a 2 by 3 grid), the segment wx and a shortest x-q route (in a 2 by 4 grid). Thus, the number of shortest p-q routes passing through wx is given by $\binom{2+3}{2} \cdot 1 \cdot \binom{2+4}{2} = \binom{5}{2}\binom{6}{2} = 150$.



(iii) Let A, B and C be the set of shortest p-q routes which pass through uv, wx and yz respectively.

$|A| = \binom{1+2}{1}\binom{3+5}{3} = \binom{3}{1}\binom{8}{3} = 168$,

$$|B| = \binom{5}{2}\binom{6}{2} = 150 \quad \text{(from (ii)), and}$$

$$|C| = \binom{3+4}{3}\binom{1+3}{1} = \binom{7}{3}\binom{4}{1} = 140.$$

Next, we compute $|A \cap B|$, $|A \cap C|$ and $|B \cap C|$.

Observe that $A \cap B$ is the set of shortest *p-q* routes passing through both *uv* and *wx*. Any such shortest *p-q* route consists of a shortest *p-u* route, the route *uvwx* and a shortest *x-q* route. Thus,

$$|A \cap B| = \binom{3}{1} \cdot 1 \cdot \binom{6}{2} = \binom{3}{1}\binom{6}{2} = 45.$$

Likewise, we obtain $\quad |B \cap C| = \binom{5}{2}\binom{4}{1} = 40.$

And each route in $A \cap C$ consists of a shortest *p-u* route, the segment *uv*, a shortest *v-y* route, the segment *yz* and a shortest *z-q* route, which gives

$$|A \cap C| = \binom{3}{1}\binom{3}{2}\binom{4}{1} = 36.$$

Finally, we evaluate $|A \cap B \cap C|$. Each route in $A \cap B \cap C$ is a *p-q* route consisting of a shortest *p-u* route, the route *uvwxyz* and a shortest *z-q* route. Thus,

$$|A \cap B \cap C| = \binom{3}{1}\binom{4}{1} = 12.$$

By (PIE),

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - \left(|A \cap B| + |A \cap C| + |B \cap C|\right) + |A \cap B \cap C| \\ &= 168 + 150 + 140 - \left(45 + 36 + 40\right) + 12 = 349 \end{aligned}$$

(iv) By (CP), **(i)** – **(iii)**. i.e. 495 – 349 = 146 (see below for reason) □

Let S be a set and A, B and C be subsets of S.
Then

$$\begin{aligned} S \setminus (A \cup B \cup C) &= \overline{A \cup B \cup C} \\ &= \overline{(A \cup B) \cup C} \\ &= \overline{A \cup B} \cap \overline{C} \qquad \text{(by De Morgan's Laws: } \overline{A \cup B} = \overline{A} \cap \overline{B}, \overline{A \cap B} = \overline{A} \cup \overline{B}\text{)} \\ &= \overline{A} \cap \overline{B} \cap \overline{C} \end{aligned}$$

It follows that

$$\boxed{\begin{gathered} |\overline{A} \cap \overline{B} \cap \overline{C}| = |S \setminus (A \cup B \cup C)| = |S| - |A \cup B \cup C|. \\ \text{or} \\ |\overline{A} \cap \overline{B} \cap \overline{C}| = |S| - (|A| + |B| + |C|) + (|A \cap B| + |A \cap C| + |B \cap C|) - |A \cap B \cap C| \end{gathered}}$$

Tip on when to use PIE

A useful consideration on whether to apply PIE is to ask if the set A to be counted can be split into sets A_i, $i = 1, 2, \ldots, n$, such that either $A = A_1 \cup A_2 \cup \ldots \cup A_n$ or $A = \overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n}$. Whether we choose the first or the second depends on whether it is easier to find the cardinality of the unions of A_i's or the cardinality of the intersections of $\overline{A_i}$'s.

Example 4 (onto mappings)

Let $X = \{1, 2, 3, 4, 5\}$ and $Y = \{1, 2, 3\}$.

(i) How many mappings are there from X to Y?

(ii) How many onto mappings are there from X to Y?

(i) Number of mappings from X to Y is given by 3^5.

(ii) Let S be the set of mappings from X to Y.

Let A be the set of mappings from X to $Y \setminus \{1\}$, B be the set of mappings from X to $Y \setminus \{2\}$, and C be the set of mappings from X to $Y \setminus \{3\}$. Then $\overline{A}$ is the set of mappings from X to Y which contain '1' in Y as an image, $\overline{B}$ is the set of mappings from X to Y which contain '2' in Y as an image, and $\overline{C}$ is the set of mappings from X to Y which contain '3' in Y as an image.

It follows that $\overline{A} \cap \overline{B} \cap \overline{C}$ is the set of mappings from X to Y which contain '1', '2' and '3' in Y as images; that is, $\overline{A} \cap \overline{B} \cap \overline{C}$ is the set of *onto* mappings from X to Y.

$|S| = 3^5$ by **(i)**.

Since A is the set of mappings from $\{1, 2, 3, 4, 5\}$ to $\{2, 3\}$, $|A| = 2^5$.
Likewise, $|B| = 2^5$ and $|C| = 2^5$.

As $A \cap B$ is the set of mappings from $\{1, 2, 3, 4, 5\}$ to $\{3\}$, $|A \cap B| = 1^5 = 1$.
Likewise, $|A \cap C| = 1$ and $|B \cap C| = 1$.

Finally, observe that $A \cap B \cap C$ is the set of mappings from X to $Y \setminus \{1, 2, 3\}$ $(= \phi)$
Thus, $A \cap B \cap C = \phi$ and so $|A \cap B \cap C| = 0$.

Thus,

$$\begin{aligned} |\overline{A} \cap \overline{B} \cap \overline{C}| &= |S| - (|A| + |B| + |C|) + (|A \cap B| + |A \cap C| + |B \cap C|) - |A \cap B \cap C| \\ &= 3^5 - \left(2^5 + 2^5 + 2^5\right) + (1 + 1 + 1) - 0 = 150 \end{aligned}$$ □

General Statement of the Principle of Inclusion and Exclusion

From the PIE for 2 and 3 sets, we observe the following patterns on the right-hand sides of the identities.

For the sum of the terms within the first grouping, we have:

n	Sum	Number of terms in the sum
2	$\lvert A_1 \rvert + \lvert A_2 \rvert$	$2 = \binom{2}{1}$
3	$\lvert A_1 \rvert + \lvert A_2 \rvert + \lvert A_3 \rvert$	$3 = \binom{3}{1}$

For the sum of terms within the second grouping, we have:

n	Sum	Number of terms in the sum
2	$\lvert A_1 \cap A_2 \rvert$	$1 = \binom{2}{2}$
3	$\lvert A_1 \cap A_2 \rvert + \lvert A_1 \cap A_3 \rvert + \lvert A_2 \cap A_3 \rvert$	$3 = \binom{3}{2}$

For the sum of terms within the third grouping, we have:

n	Sum	Number of terms in the sum
2	None	0
3	$\lvert A_1 \cap A_2 \cap A_3 \rvert$	$1 = \binom{3}{3}$

We also notice that the groupings alternate in sign, beginning with a (+) sign.

Suppose now that we are given n finite sets: $A_1, A_2, \ldots, A_n$. By generalizing the above observations, what identity would you expect for $\lvert A_1 \cup A_2 \cup \ldots \cup A_n \rvert$?

The first grouping should be the sum of $\binom{n}{1} = n$ terms, each involving a single set:

$$\lvert A_1 \rvert + \lvert A_2 \rvert + \ldots + \lvert A_n \rvert;$$

or, in abbreviation, $$\sum_{i=1}^{n} \lvert A_i \rvert.$$

The second grouping should be the sum of $\binom{n}{2}$ terms, each involving the intersection of two sets:

$$| A_1 \cap A_2 | + | A_1 \cap A_3 | + \ldots + | A_{n-1} \cap A_n |;$$

in abbreviation, $\sum_{i<j} | A_i \cap A_j |$.

The third grouping should be the sum of $\binom{n}{3}$ terms, each involving the intersection of three sets:

$$| A_1 \cap A_2 \cap A_3 | + | A_1 \cap A_2 \cap A_4 | + \ldots + | A_{n-2} \cap A_{n-1} \cap A_n |;$$

in abbreviation, $\sum_{i<j<k} | A_i \cap A_j \cap A_k |$.

This pattern continues until the final, nth grouping with $\binom{n}{n} = 1$ term involving the intersection of n sets.

Bearing in mind that the groupings alternate in sign, beginning with a '+' sign, one would expect that the following holds:

General Statement of the Principle of Inclusion and Exclusion (PIE)

For n sets:

$$\left|A_1 \cup A_2 \cup \ldots \cup A_n\right| = \sum_{i=1}^{n} \left|A_i\right| - \sum_{i<j} \left|A_i \cap A_j\right| + \sum_{i<j<k} \left|A_i \cap A_j \cap A_k\right| - \ldots + (-1)^{n+1} \left|A_1 \cap A_2 \cap \ldots \cap A_n\right|$$

Alternative Formulation: Let $A_1, ..., A_n$ be some events. Then,

$$\left|P(A_1 \cup A_2 \cup \ldots \cup A_n)\right| = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \ldots$$
$$+ (-1)^{n+1} P(A_1 \cap A_2 \cap \ldots \cap A_n)$$

Proof by induction:

For $n=1$, statement is trivial.

For $n=2$, observe that if X and Y are disjoint sets, then $|X \cup Y| = |X| + |Y|$. So $|X \cup Y| = |X \cup (Y \setminus X)| = |X| + |Y \setminus X|$ and $|Y| = |(Y \cap X) \cup (Y \setminus X)| = |Y \cap X| + |Y \setminus X|$, where in both cases, the first equality stems from the equality of sets and the second equality stems from our observation.

Substituting the second equation into the first provides the proof.

Suppose that the formula is true for n. In the case of (n+1), we group the first n sets together, apply the inductive hypothesis when $n=2$ and then use the distributivity of intersections:

$$\begin{aligned}|A_1 \cup A_2 \ldots \cup A_{n+1}| &= |A_1 \cup A_2 \ldots \cup A_n| + |A_{n+1}| - |(A_1 \cup A_2 \ldots \cup A_n) \cap A_{n+1}| \\ &= |A_1 \cup A_2 \ldots \cup A_n| + |A_{n+1}| - |(A_1 \cap A_{n+1}) \cup (A_2 \cap A_{n+1}) \ldots \cup (A_n \cap A_{n+1})| \ldots (*)\end{aligned}$$

The first and last terms are unions of n sets, so applying the inductive hypothesis,

$$\begin{aligned}|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i<j\le n} |A_i \cap A_j| + \sum_{i<j<k\le n} |A_i \cap A_j \cap A_k| - \ldots \\ + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n| \ldots (1)\end{aligned}$$

$$\begin{aligned}&|(A_1 \cap A_{n+1}) \cup (A_2 \cap A_{n+1}) \ldots \cup (A_n \cap A_{n+1})| \\ &= \sum_{i=1}^{n} |(A_i \cap A_{n+1})| - \sum_{i<j\le n} |A_i \cap A_j \cap A_{n+1}| + \ldots (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n \cap A_{n+1}| \ldots (2)\end{aligned}$$

The first group in (1) contains all sets with index at most n; the remaining set with index (n+1) is in (*). The second group in (1) contains all the intersections of 2 sets with index at most n while the first group in (2) contains the intersections of 2 sets, one of which has index (n+1). Together (note their opposite signs in (*)), they account for all the intersections of 2 sets.

Similarly, the third group in (1) contains all the intersections of 3 sets with index at most n while the second group in (2) contains the intersections of 3 sets, one of which has index (n+1). Together, they account for all the intersections of 3 sets. This continues until the last term: the nth group in (2) containing the intersection of all (n+1) sets. So,

$$\begin{aligned}|A_1 \cup A_2 \cup \ldots \cup A_{n+1}| = \sum_{i=1}^{n+1} |A_i| - \sum_{i<j\le n+1} |A_i \cap A_j| + \sum_{i<j<k\le n+1} |A_i \cap A_j \cap A_k| - \ldots \\ + (-1)^{n+2} |A_1 \cap A_2 \cap \ldots \cap A_n + 1| \text{ (shown)}\end{aligned}$$

$\square$

Example 5

How many ways are there to arrange n (≥ 2) married couples in a row so that at least one couple are next to each other?

Denote the n couples by $H_1, W_1, H_2, W_2, \ldots, H_n, W_n$.

Thus, when $n = 4$ for example, the following is a possible arrangement:
$W_2H_1H_4\underline{W_3H_3}H_2W_1W_4$, $H_4\underline{H_2W_2}W_4H_3\underline{W_1H_1}H_4$.

For each $i = 1, 2, \ldots, n$, let A_i be the set of arrangements of the n couples such that H_i and W_i are adjacent (next to each other). The problem is thus to find $\left|A_1 \cup A_2 \cup \ldots \cup A_n\right|$.

To compute $\sum_{i=1}^{n} | A_i |$, we first consider $| A_1 |$. A_1 is the set of arrangements of n couples such that H_1 and W_1 are adjacent. This is the same as arranging the $2n - 1$ objects:
$H_1W_1,\ H_2,\ W_2,\ \ldots,\ H_n,\ W_n$

in a row where H_1W_1 can be permuted in two ways: H_1W_1 and W_1H_1. Thus, $\left|A_1\right| = 2 \cdot (2n-1)!$

Similarly, $\left|A_i\right| = 2 \cdot (2n-1)!$ for each $i = 2, 3, \ldots, n$. Thus,

$$\sum_{i=1}^{n} | A_i | = \binom{n}{1} \cdot 2 \cdot (2n-1)!.$$

To compute $\sum_{i<j} | A_i \cap A_j |$, we first consider $\left|A_1 \cap A_2\right|$. $A_1 \cap A_2$ is the set of arrangements of the n couples such that H_1 and W_1 are adjacent and H_2 and W_2 are adjacent. This is the same as arranging the $2n - 2$ objects:
$H_1W_1, H_2W_2, \ldots, H_n, W_n$

in a row where both H_1W_1 and H_2W_2 can be permuted by themselves. Thus,
$\left|A_1 \cap A_2\right| = 2^2 \cdot (2n-2)!$

Similarly, for $1 \leq i < j \leq n$, $\left|A_i \cap A_j\right| = 2^2 \cdot (2n-2)!$. Thus,

$$\sum_{i<j} | A_i \cap A_j | = \binom{n}{2} \cdot 2^2 \cdot (2n-2)!.$$

Likewise,

$$\sum_{i<j<k} | A_i \cap A_j \cap A_k | = \binom{n}{3} \cdot 2^3 \cdot (2n-3)!$$

and so on.

Hence

$$| A_1 \cup A_2 \cup \ldots \cup A_n | = \sum_{r=1}^{n} (-1)^{r+1} \binom{n}{r} \cdot 2^r \cdot (2n-r)!.$$

For the case when $n = 4$, we have

$$\begin{aligned}|A_1 \cup A_2 \cup A_3 \cup A_4| &= \sum_{r=1}^{4} (-1)^{r+1} \binom{4}{r} \cdot 2^r \cdot (8-r)! \\ &= \binom{4}{1} \cdot 2 \cdot (8-1)! - \binom{4}{2} \cdot 2^2 \cdot (8-2)! + \binom{4}{3} \cdot 2^3 \cdot (8-3)! - \binom{4}{4} \cdot 2^4 \cdot (8-4)! \\ &= 26496\end{aligned}$$

□

For n sets:

Where $A_1, A_2, \ldots, A_n$ are subsets of S,

$$\begin{aligned}\left|\overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n}\right| &= |S| - |A_1 \cup A_2 \cup \ldots \cup A_n| \\ &= |S| - \underbrace{\sum_{i=1}^{n} |A_i|}_{\binom{n}{1} \text{ terms}} + \underbrace{\sum_{i<j} |A_i \cap A_j|}_{\binom{n}{2} \text{ terms}} - \underbrace{\sum_{i<j<k} |A_i \cap A_j \cap A_k|}_{\binom{n}{3} \text{ terms}} + \ldots + \underbrace{(-1)^n |A_1 \cap A_2 \cap \ldots \cap A_n|}_{\binom{n}{n} \text{ terms}}\end{aligned}$$

'Le Probleme Des Rencontres' (the matching problem)

Two decks X, Y of cards, with 52 cards each, are given. The 52 cards of X are first laid out. Those of Y are then placed randomly, one each on top of each card of X, so that 52 pairs of cards are formed. What is the probability that no cards in each pair are identical (i.e., having the same suit and rank)?

This problem was introduced and studied by the Frenchman Pierre Remond de Montmort (1678 – 1719) around the year 1708. The number of ways of distributing the cards of Y to form 52 pairs of cards with those in X is clearly 52!. Thus, to find the desired probability, we need to find out the number of ways of distributing the cards of Y such that each card in Y is placed at the top of a different card in X.

Derangements

A permutation $a_1 a_2 \cdots a_n$ of $\mathbb{N}_n = \{1, 2, \ldots, n\}$ is a **derangement** of $\mathbb{N}_n$ if $a_i \neq i$ for each $i = 1, 2, \ldots, n$. The permutation 54132 is a derangement of $\mathbb{N}_5$ but $51\underline{34}2$ and $3\underline{2}154$ are not. The following table shows all the derangements of $\mathbb{N}_n$ for $n = 1, 2, 3, 4$.

n	Derangements of $\mathbb{N}_n$
1	None
2	21
3	231, 312
4	2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321

Let S be the set of permutations $a_1a_2\cdots a_n$ of $\mathbb{N}_n$. For $i=1,\ 2,\ \ldots,\ n$, let A_i = the set of permutations in S with $a_i = i$. Then

number of derangements of $\mathbb{N}_n$, $D_n = \left|\overline{A_1}\cap\overline{A_2}\cap\ldots\cap\overline{A_n}\right|$

$$= n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\ldots+(-1)^n\frac{1}{n!}\right)$$

Let the universal set S be the set of all permutations of $\mathbb{N}_n$. $|S| = n!$.

For each $i = 1, 2, \ldots, n$, let A_i be the set of permutations $a_1a_2\cdots a_n$ in S such that $a_i = i$. Thus, $\overline{A_i}$ is the set of permutations in S such that $a_i \neq i$ and so $\overline{A_1}\cap\overline{A_2}\cap\ldots\cap\overline{A_n}$ is the set of permutations in S such that $a_i \neq i$ for all i = 1,2 , …, n, which is exactly the set of derangements of $\mathbb{N}_n$.

Observe that A_1 is the set of permutations of the form $1a_2\cdots a_n$. Thus, $|A_1| = (n-1)!$.
Similarly, $|A_i| = (n-1)!$ for each $i = 2,3,\ldots,n$, and so

$$\sum_{i=1}^{n}|A_i| = n\cdot(n-1)! = \binom{n}{1}\cdot(n-1)!.$$

As $A_1\cap A_2$ is the set of permutations of the form $12a_3\cdots a_n$, we have $|A_1\cap A_2| = (n-2)!$.
Similarly, $|A_i\cap A_j| = (n-2)!$ for all $i, j\in\{1,\ 2,\ \ldots,\ n\}$ with $i<j$. There are $\binom{n}{2}$ number of ways of choosing i and j from $\{1,\ 2,\ \ldots,\ n\}$ with $i<j$, and so

$$\sum_{i<j}|A_i\cap A_j| = \binom{n}{2}\cdot(n-2)!.$$

Likewise,

$$\sum_{i<j<k}|A_i\cap A_j\cap A_k| = \binom{n}{3}\cdot(n-3)!$$

and so on.

Note that for $r = 1,2,\ldots,n$,

$$\binom{n}{r}\cdot(n-r)! = \frac{n!}{r!(n-r)!}(n-r)! = \frac{n!}{r!}.$$

The number of derangements of $\mathbb{N}_n$,

$$D_n = \left|\overline{A_1}\cap\overline{A_2}\cap\ldots\cap\overline{A_n}\right|$$

$$= |S| - \sum_{i=1}^{n} |A_i| + \sum_{i<j} |A_i \cap A_j| - \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \ldots + (-1)^n |A_1 \cap A_2 \cap \ldots \cap A_n|$$

$$= n! - \binom{n}{1} \cdot (n-1)! + \binom{n}{2} \cdot (n-2)! - \binom{n}{3} \cdot (n-3)! + \ldots + (-1)^n \binom{n}{n} \cdot (n-n)!.$$

$$= n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \ldots + (-1)^n \frac{n!}{n!}$$

$$= n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right) \qquad \square$$

Suppose we generate a permutation of $\mathbb{N}_n$ at random. The probability that this permutation is a derangement is given by $\frac{D_n}{n!}$ which by the above result is

$$\frac{D_n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}.$$

When n gets larger and larger, it is known that the quotient $\frac{D_n}{n!}$ gets closer and closer to $\frac{1}{\mathrm{e}} (\approx 0.367)$, where the constant e, called the natural exponential base, is defined by $\mathrm{e} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$. It is known that $\mathrm{e} \approx 2.718281828459045$. (The letter 'e' was chosen in honour of the great Swiss mathematician L. Euler (1707 – 1783) who made some significant contributions to the study of problems related to the limit above.)

Example 6 (2020/RI/Prelim/Q3iv)

A group of 10 people went to watch a concert, and were seated in a row. During the intermission, all 10 of them left their seats before returning. Show that the number of arrangements if each person has a different person seated to his or her right after returning is given by $\sum_{k=0}^{9}(-1)^k\binom{9}{k}(10-k)!$.

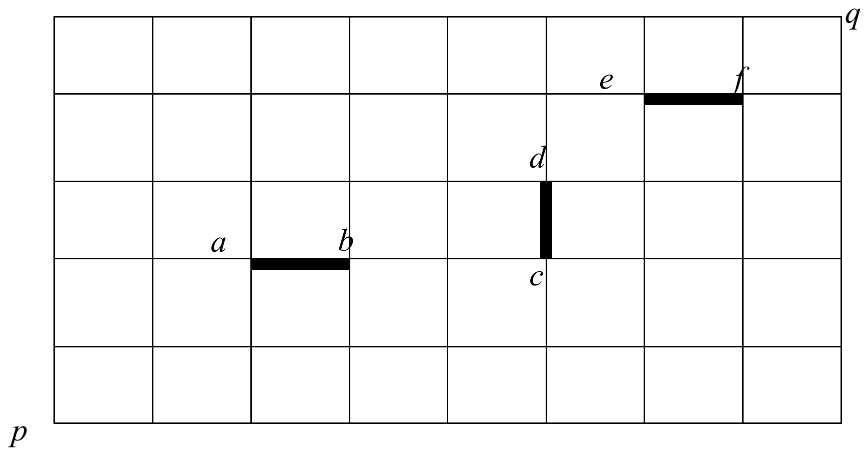
Exercise 2

1 Find the number of integers from the set {1, 2, …, 1000} which are

(i) divisible by at least one of 2, 3 and 5;

(ii) divisible by none of 2, 3 and 5.

2 The following diagram shows a 5 by 8 rectangular grid with two specified corners *p* and *q* and three specified segments *ab*, *cd* and *ef*.



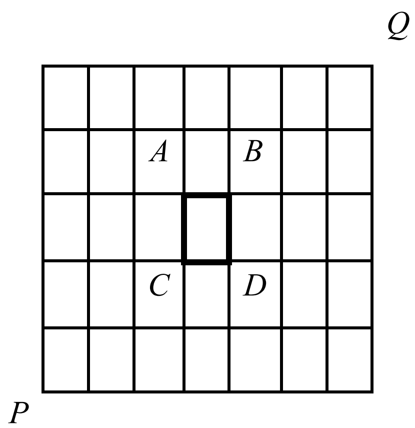
Find in the grid

(i) the number of shortest *p*-*q* routes;

(ii) the number of shortest *p*-*q* routes that pass through at least one of the segments *ab*, *cd* and *ef*;

(iii) the number of shortest *p*-*q* routes that do not pass through any of the segments *ab*, *cd* and *ef*.

3 (2019/NJC/Prelim/Q2) The following diagram shows a 5 by 7 rectangular grid with the loop *ABDC*.



Find the number of shortest routes from *P* to *Q* which do not pass through any of the segments in the loop *ABDC*. [6]

4 (2019/ASRJC/Prelim/Q5iii) A café owner has 6 chocolate, 7 strawberry and 8 vanilla muffins. Muffins of the same flavour are identical to one another. She wants to throw away these muffins into the 4 different rubbish bins in the vicinity of the café. She wants to ensure that at least 1 muffin is thrown into each bin so as to avoid detection by the members of public that she is wasting food. Find the number of ways she can do so. [5]

5 (i) Find the number of integer solutions to the equation $x + y + z = 12$ where $0 \le x \le 4$, $0 \le y \le 5$ and $0 \le z \le 6$.

(ii) Find the number of integer solutions to the equation

$$x_1 + x_2 + \ldots + x_{11} = 50$$

such that $0 \le x_r \le 9$ for each $r = 1, 2, \ldots, 11$.

6 Each of ten ladies checks her hat and umbrella in a cloakroom and the attendant gives each lady back a hat and an umbrella at random. Show that the number of ways this can be done so that no lady gets back both of her possessions is

$$\sum_{r=0}^{10} (-1)^r \binom{10}{r} \{(10-r)!\}^2 .$$

7 Show that the number of onto mappings from $\mathbb{N}_m$ to $\mathbb{N}_n$, where $m \ge n \ge 1$, is given by

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} (n-r)^m .$$

8 (2020/EJC/Prelim/Q2a) A subphrase is a sequence of letters that appears together in the exact order within the word given. For example, “JUST” and “IF” is a subphrase in the word REJUSTIFY, but not “REST”. Find the number of rearrangements possible for the 9 letters of the word REJUSTIFY in which the following phrases are not found as a subphrase of the rearrangement: “JUST’, “RYE”, “IF”. [4]

9 (2020/ASRJC/Prelim/Q8) A group of n friends attends a party where each of the friends brings along a distinct gift for gift exchange. Let D_n be the number of distinct ways for them to carry out the gift exchange such that none of them gets back their own gift.

(i) Explain why $D_n = (n-1)(D_{n-1} + D_{n-2})$, $n \ge 3$, stating the values of D_1 and D_2. [4]

(ii) Explain why there are nD_{n-1} distinct ways to carry out the gift exchange such that exactly one person gets back his own gift. [2]

(iii) A group of 5 friends organises a party, where each of them brings along a distinct gift. Using your answer to **(i)** and **(ii)**, find the number of distinct ways that they can exchange their gifts among themselves such that exactly one person gets back his own gift. [3]

(iv) Use the principle of inclusion and exclusion to show that $D_n = n!\sum_{r=0}^{n} \frac{(-1)^r}{r!}$. [4]

(v) A large group of n friends is attending another party with gift exchange taking place. For the case where $n \to \infty$, show that the probability that at most one person getting back his own gift after the gift exchange is $2\mathrm{e}^{-1}$. [3]