2025 Raffles Institution H3 Mathematics Preliminary Examinations

1	The function f satisfies $\mathrm{f}(x)-k\mathrm{f}(1-x)=x$ for all real values of x, where $k \neq \pm 1$.		
	(a)	Find $\mathrm{f}(x)$.	[2]
	Consider the equation $\mathrm{f}(x)-\mathrm{f}(1-x)=\mathrm{g}(x)$. $\quad (*)$		
	(b)	Explain why $\mathrm{g}(x)+\mathrm{g}(1-x)=0$ is a necessary condition for there to be a solution f to $(*)$.	[2]
	It is now given that there is a solution f to $(*)$.		
	(c)	If $\mathrm{g}(x)$ is a polynomial with degree $k \in \mathbb{Z}^+$, explain why k cannot be even.	[2]
	(d)	If g is a linear function, give a possible $\mathrm{g}(x)$.	[2]
	(e)	Find a solution f to $(*)$ when $\mathrm{g}(x)=(2x-1)^3$.	[2]

2	Let z be a complex number such that $z^3=1,\ z \neq 1$.		
	(a)	Show that $1+z+z^2=0$ and hence find all the solution to $z^3=1,\ z\neq 1$.	[2]
	Let w deote the solution to $z^3=1$ with $\mathrm{Im}(w)>0$. The complex numbers a, b and c are represented by the points A, B and C respectively in the Argand diagram such that ABC is an equilateral triangle described in the anticlockwise sense. Let m be the complex number representing the point M, the midpoint of AB.		
	(b)	Show that $c-m=k(b-m)$, where k is a complex number to be determined.	[2]
	(c)	Hence show that $c=b+w(b-a)$.	[2]
	(d)	Deduce that $a+bw+cw^2=0$.	[2]
	Let PQR be a triangle, and construct equilateral triangles XQP, YRQ, ZPR on the exterior of triangle PQR. Let G_1, G_2, G_3 be the centroids of triangles XQP, YRQ, ZPR respectively.		
	(e)	Using the fact that the complex number representing the centroid of a triangle is given by the arithmetic mean of the complex numbers representing the vertices of the triangle, show that triangle $G_1G_2G_3$ is equilateral.	[4]

3 Let m and n be nonnegative integers and

$$I_{m,n}(x)=\int_0^x \frac{t^m}{(1+t^2)^{n+1}}\,\mathrm{d}t.$$

(a) Prove that for $m\geq 2$ and $n\geq 1$,

$$I_{m,n}(x)=-\frac{x^{m-1}}{2n(1+x^2)^n}+\frac{m-1}{2n}I_{m-2,n-1}(x).$$

[3]

(b) State an inequality in terms of m and n such that $\lim\limits_{x\to\infty}\dfrac{x^{m-1}}{(1+x^2)^n}=0$. [1]

(c) **(i)** Show that $\lim\limits_{x\to\infty} I_{0,1}=\dfrac{\pi}{4}$. [3]

(ii) Hence find the exact volume of revolution when the region bounded by the curve $y=\left(\dfrac{x}{1+x^2}\right)^2$ and the x-axis is completely rotated about the x-axis. [4]

4 In this question, we will prove the following identity using various methods.

$$\binom{n}{0}\binom{m}{k}-\binom{n}{1}\binom{m-1}{k}+\cdots+(-1)^n\binom{n}{n}\binom{m-n}{k}=\binom{m-n}{k-n} \text{ ----- } (*)$$

for positive integers $m\geq k\geq n$.

(a) **(i)** Use a combinatorial argument to explain why for positive integers $n>i\geq 1$,

$$\binom{n}{i}=\binom{n-1}{i}+\binom{n-1}{i-1}.$$

[2]

(ii) Prove (*) using mathematical induction on n. [4]

(b) By considering the number of ways to select a team of k players from a group of m players such that all n star players are included in the team of k players and the principle of inclusion and exclusion, show (*). [4]

(c) **(i)** Write down the binomial expansion of $(y-1)^n$ in the form $\sum\limits_{r=0}^{n}\mathrm{g}(r,n)y^{n-r}$, where g is a function of r and n to be determined. [1]

(ii) By considering the coefficient of the term x^k in the expansion of $\left((1+x)-1\right)^n(1+x)^{m-n}$, prove (*). [3]

5	**(a)**	Let $x_1, x_2, ..., x_n, ...$ be a sequence of positive real numbers and let $$y_n = (x_1 + x_2 + ... + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}\right)$$ and $z_n = \left\lfloor \sqrt{y_n} \right\rfloor$. Note: For any real number x, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. For example, $\lfloor 2 \rfloor = 2$, $\lfloor 3.1 \rfloor = 3$.	
		(i)	Show that for all positive integers n, $y_{n+1} \geq y_n + 2\sqrt{y_n} + 1$. [3]
		(ii)	Deduce that no two terms in the sequence $z_1, z_2, ..., z_n, ...$ are the same. [2]
	(b)	Let $a_1, a_2, ..., a_n$ be real numbers such that $0 < a_1 < a_2 < ... < a_n$.	
		(i)	Show that for $i = 2, 3, ..., n$ $$\frac{a_i - a_{i-1}}{(1+a_i)^2} < \frac{1}{1+a_i} - \frac{1}{1+a_{i-1}}.$$ [2]
		(ii)	Hence, using the Cauchy-Schwarz inequality or otherwise, show that $$\left(\frac{1}{1+a_1} + ... + \frac{1}{1+a_n}\right)^2 < \frac{1}{a_1} + \frac{1}{a_2 - a_1} + ... + \frac{1}{a_n - a_{n-1}}.$$ [4]
		(iii)	Using $a_i = 2^{-m+i-1}$, show that for positive integers m, $$\left(m + \frac{1}{2}\right)^2 < 3\left(2^m\right) - 2^{1-m}.$$ [3]

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Use the information in the mathematical text to answer this Question. You should read the whole mathematical text before you start answering the questions.

A Dynamical Systems Proof of Fermat's Little Theorem

We will prove an important and foundational result in number theory using techniques from dynamical systems. The result, known as Fermat's Little Theorem, states that:

Theorem. Let p be a prime number, and a any integer. Then $a^p \equiv a \text{ (modulo } p)$.

Usually, the theorem is proved using algebraic ideas. In this note, we will prove it with a simple argument from dynamical systems on the unit interval [0, 1]. The basic idea is to count points of minimum period p of a particular dynamical system, and note that this number must be divisible by p.

For every integer n, we define the function $T_n : [0,1] \to [0,1]$ as follows:

$$T_n(x) = \begin{cases} \{nx\} & \text{if } x \neq 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Here the braces denote the fractional part of a real number. For example, {3.1415} = 0.1415, and {3} = 0. We say that x is a fixed point of T_n if $T_n(x) = x$.

If we apply this operation T_n repeatedly, some points x in [0, 1] will be periodic, in the sense that $T_n\left(T_n\left(\ldots\left(T_n(x)\right)\right)\right) = x$ for some number of iterations of T_n. We call a point k-periodic if it is mapped back to itself after k iterations, that is

$$\underbrace{T_n\left(T_n\left(\ldots\left(T_n(x)\right)\right)\right)}_{k \text{ iterations}} = x.$$

We say k is the minimal period if x is not mapped back to itself for any fewer number of iterations than k. So the 1-periodic points are fixed points. We also say that the points $x_1, x_2, \ldots, x_p$ lie in an orbit of size p if $T_a(x_1) = x_2, T_a(x_2) = x_3, \ldots, T_a(x_{p-1}) = x_p, T_a(x_p) = x_1$.

We now determine two crucial properties of the family of functions T_n:

1. Let n be an integer greater than 1. The function T_n has n fixed points in [0, 1].
2. Let a and b be positive integers. Then for all $x \in [0,1], T_a\left(T_b\left(x\right)\right) = T_{ab}(x)$.

Now if p is a prime, and a is any integer, consider the p-periodic points of T_a. These points are the fixed points of T_a iterated p times. It follows that there are $a^p - a$ points that have minimal period p. Now consider the orbit of these points. Since each point with minimal period p lies in an orbit of size p, there are $\left(a^p - a\right)/p$ orbits of size p. Since this is an integer, we have the desired result.

Adapted from Kevin Iga (2003) A Dynamical Systems Proof of Fermat's Little Theorem, Mathematics Magazine, 76:1, 48-51, DOI: 10.1080/0025570X.2003.11953946.

6	**(a)**	**(i)**	Sketch the graph of $y = T_3(x)$ for $0 \le x \le 1$. [2]
		(ii)	Give a graphical explanation why the function T_n has n fixed points. [1]
		(iii)	Find the fixed points of T_n. [2]
	(b)	**(i)**	Write down an orbit of size 2 for T_2. [1]
		(ii)	Write down an orbit of size 3 for T_3. [2]
	(c)	Show that if *a* and *b* are positive integers, then for all $x \in [0,1]$, $$T_a\left(T_b\left(x\right)\right) = T_{ab}(x).$$ [2]	
	(d)	Let *m* and *k* be positive integers. Show that for any *m*-periodic point of T_a, the minimal period *k* must divide *m*. [3]	
	(e)	Let *a* be a positive integer and *p* be a prime number.	
		(i)	Explain why there are $a^p - a$ points that have minimal period *p* amongst the fixed points of T_a iterated *p* times. [2]
		(ii)	Hence show that for all positive integers *a*, $$a^{15} \equiv a^5 + a^3 - a \text{ (modulo 15)}.$$ [3]