NANYANG JUNIOR COLLEGE

JC2 Preliminary Examination

Higher 2

FURTHER MATHEMATICS **9649/01**

Paper 1 **15th September 2025**

3 hours

Additional Materials: Printed Answer Booklet

List of Formulae and Results (MF27)

READ THESE INSTRUCTIONS FIRST

Answer **all** the questions.
Write your answers on the Printed Answer Booklet. Follow the instructions on the front cover of the answer booklet.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are **not** allowed in a question, you must present the mathematical steps using mathematical notations and not calculator commands.
You must show all necessary working clearly.

The number of marks is given in brackets [ ] at the end of each question or part question.

1 Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{36}{35}y\left(1-\frac{y}{180}\right)-h,$$

where h is a positive constant.

(a) Given that $\frac{\mathrm{d}y}{\mathrm{d}x}\geq 0$ for some values of y, find the range of values of h. [2]

(b) For $h=25$, find the equilibrium values of y, stating whether they are stable or unstable. [3]

2 The curve C has equation $y=x\cos 2x$, $0<x\leq\frac{\pi}{2}$, and has a single turning point at P.

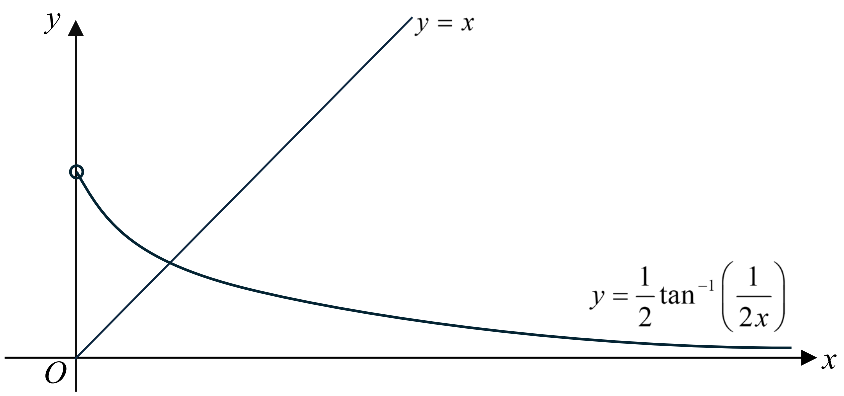
(a) Show that the x-coordinate of P is a solution of the equation $x-\frac{1}{2}\tan^{-1}\left(\frac{1}{2x}\right)=0$. [2]

(b) Show that the equation in part **(a)** has a root α between 0.3 and 0.5. [1]

The iterative formula $x_{n+1}=\frac{1}{2}\tan^{-1}\left(\frac{1}{2x_n}\right)$ with $x_1=0.3$ is used to find α.

(c) Find, correct to 5 decimal places, the values of x_2, x_3 and x_4. [2]

The diagram below shows the graphs of $y=x$ and $y=\frac{1}{2}\tan^{-1}\left(\frac{1}{2x}\right)$ for $x>0$.



(d) Using the sketch in your answer booklet, show how the convergence to the root α takes place. Indicate clearly on your diagram the positions of x_1, x_2, x_3 and x_4 in relation to α. [2]

3 It is known that the equation $\mathrm{f}(x)=0,$ where $\mathrm{f}(x)=\sqrt{x}-2x^2+2,$ has a root β.

To approximate β, linear interpolation is used on the interval $[0,k]$ to get an initial approximation x_0, followed by the Newton-Rhapson method.

(a) If $k=4$, find x_0. [1]

(b) Use the Newton-Raphson method repeatedly with the initial approximation found in part **(a)** to determine β correct to 3 decimal places, justifying the correctness of the result obtained. [3]

(c) For $k>\beta$, linear interpolation is used to obtain x_0 as in part **(a)**. Find the range of values of *k* such that the Newton-Raphson method would fail. [3]

4 The prices per unit weight, in dollars, of two commodities, coffee (C_n) and tea (T_n), in the month *n* are modelled by the following recurrence relations:

$$\begin{cases} C_n=0.8C_{n-1}+0.2T_{n-1}+5 \\ T_n=0.1C_{n-1}+0.7T_{n-1}+3 \end{cases}$$

where $C_0=20$ and $T_0=15$.

(a) Denoting the equilibrium prices of coffee and tea by C^* and T^* respectively, find the values of C^* and T^*. [2]

(b) Using the substitution $X_n=C_n-C^*$ and $Y_n=T_n-T^*$, derive the second order recurrence relation for X_n. [3]

(c) Hence find C_n in terms of *n*. [3]

5. A solid of revolution is formed when a circle of radius r, with centre at $(R,0)$, is rotated about the y-axis through 2π radians, where $R > r > 0$.

(a) Using the shell method and the substitution $u = x - R$, show that the volume V of the solid of revolution can be expressed as $V = 4\pi\int_{-r}^{r}(R+u)\sqrt{r^2-u^2}\,\mathrm{d}u$. [3]

(b) **(i)** Use Simpson's rule with five equally spaced ordinates to estimate V, expressing your answer in the form $\frac{4\pi Rr^2}{3}\left(a\sqrt{b}+c\right)$, where a, b, and c are constants to be determined. [4]

(ii) Without further calculation, explain how your answer in part **(b)(i)** would change if r is halved while R is doubled. [1]

6 **(a)** By considering Euler's formula, find the exact value of

$$\sum_{k=0}^{n-1}\cos\left(\theta+\frac{2k\pi}{n}\right)$$

where $n > 1$. [4]

(b) A regular n-sided ($n \geq 3$) polygon with vertices $A_0, A_1, \ldots, A_{n-1}$ is inscribed in a circle with centre O and radius r. The point P is on the plane of the polygon such that $OP = p$. By defining $\angle POA_k = \theta + \frac{2k\pi}{n}, k = 0,1,2,\ldots,n-1$, prove that

$$PA_0^{\ 2} + PA_1^{\ 2} + \ldots + PA_{n-1}^{\ \ 2} = n\left(r^2+p^2\right).$$ [3]

(c) The value of $PA_0^{\ 2} + PA_1^{\ 2} + \ldots + PA_{n-1}^{\ \ 2}$ is denoted by

- X when P is the centre O of the circle,
- Y when P lies on the circumference of the circle.

Find the value of $\frac{Y}{X}$. [3]

7 The curve C is defined by the equation $3ay^2 = x(a-x)^2$, where a is a positive constant. The finite region R is bounded by the curve C.

(a) Show that $\dfrac{dy}{dx} = \dfrac{(a-x)(a-3x)}{6ay}$. [2]

(b) Sketch C, indicating clearly the behaviour of the curve near the origin. Show the coordinates of the x-intercepts and turning points. [2]

(c) Show, with full working, that the length of the perimeter of region R can be expressed in the form $\dfrac{p\sqrt{q}}{r}a$, where p, q and r are positive integers. [6]

(d) Find, in terms of a, the surface area generated when region R is rotated through 2π radians about the y-axis. [3]

8 **(a)** Use the substitution $y = ux$ to find the particular solution of the differential equation

$$\frac{dy}{dx} = \frac{2y}{x} + \left(\frac{x}{y}\right)^2, \quad x > 0, \quad y > 0,$$

given that the solution curve passes through the point $(1,1)$. [5]

(b) Use two steps of Euler method, find an approximation to the value of y when $x = 2$. [3]

(c) Find the percentage error in the approximation of y obtained above, giving your answer correct to 3 decimal places. [2]

(d) Suggest two ways in which the percentage error in the approximation of y can be reduced. [2]

9 A linear transformation $\mathrm{T}:\mathbb{R}^3 \to \mathbb{R}^3$ is given by matrix $\mathbf{A}$

$$\mathbf{A} = \begin{pmatrix} 5 & -a & a \\ 12 & -11 & 12 \\ a & -a & 5 \end{pmatrix}$$

such that $\mathrm{T}(\mathbf{x}) = \mathbf{A}\mathbf{x}$, $\mathbf{x} \in \mathbb{R}^3$.

(a) **(i)** It is given that a is an integer. By using Gaussian elimination, show that $a = 5$ when dim (T) = 2. Hence, find a basis for the range space R of T. [3]

(ii) Given that $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : 2x + 3y - 5z = 1 \right\}$, find $S \cap R$. Give the geometrical interpretation of $S \cap R$ and state whether $S \cap R$ is a subspace of $\mathbb{R}^3$. [4]

(b) Given that $\mathbf{A}$ has an eigenvalue -3, show that $a^2 - 12a + 32 = 0$. [2]

Using $a = 4$ for the rest of this question.

(i) Find the remaining eigenvalue(s) of $\mathbf{A}$ and for each eigenvalue, give the full set of eigenvectors. [4]

(ii) Give a geometrical interpretation of each eigenvalue's set of eigenvectors in relation to T. [2]

(iii) State, with justification, whether the set of all eigenvectors of $\mathbf{A}$ forms a basis for $\mathbb{R}^3$. [1]

(iv) Hence find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}^3 = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$. (You do not need to evaluate $\mathbf{P}^{-1}$). [2]

10 A marine biologist monitors a large saltwater aquarium where the dissolved organic waste concentration (measured in milligram/litre) changes daily. The waste is reduced by two processes:

- Natural breakdown by microbes in the aquarium
- Probiotic treatment added every morning

Such processes convert harmful substances (e.g. ammonia) into harmless compounds that do not threaten aquatic life.

The waste concentration W_n measured at the end of day n is modelled by the following recurrence relation:

$$W_n = \left(k_d\sqrt{W_{n-1}} + k_pT\right)^2,\ n \geq 1$$

where

- T is the daily probiotic treatment dose
- k_d is the natural breakdown rate due to microbial activity
- k_p is the treatment efficacy due to the probiotic treatment
- W_0 is the initial waste concentration measurement

(a) If $k_p = 0$, describe the behaviour of the sequence W_n in the context of the question for

(i) $0 < k_d < 1$ and [2]

(ii) $k_d = 1$. [1]

Justify your answers clearly.

For the rest of the question, take $k_d = 0.5$, $k_p = 0.2$ and $W_0 = 16$.

(b) Find the value of T required to stabilise the waste concentration at 4 milligram/litre. [2]

(c) Using the substitution $I_n = \sqrt{W_n}$, solve W_n in terms of n and T. [4]

(d) Hence find the range of T for which the long-term waste concentration exceeds W_0. [2]

(e) Comment on the ecological implication of part **(d)** in the saltwater aquarium. [1]

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