2026 J2 H2 FM - Linear Algebra (Set 3)

1 In this question, V denotes the set of vectors of the form $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$, where a, b, c and d are real numbers. You may assume that V forms a linear space under the usual operations of vector addition and multiplication by a scalar.

(a) Show that the subset of V for which $a+b-d=c^2$ does not form a linear space. [1]

(b) Determine whether or not the subset of V for which both $a+4b=2c+d$ and $a-2b=c+d$ forms a linear space. [2]

(c) Determine whether or not the subset of V for which $a+4b=2c+d$ or $a-2b=c+d$ forms a linear space. [2]

(d) State the dimension of the subspace of V such that $3a+2b-c+d=0$ and provide a basis for this subspace. [3]

2 An *orthogonal* matrix $\mathbf{Q}$ is one such that

$$\mathbf{Q}^{\mathrm{T}}\mathbf{Q}=\mathbf{I},$$

where $\mathbf{I}$ is the identity matrix.

(a) Find the possible determinants of an orthogonal matrix. [3]

For any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, it can be shown that $\mathbf{u}\cdot\mathbf{v}=\mathbf{u}^{\mathrm{T}}\mathbf{v}$.

(b) Suppose $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are the column vectors (in that order) of a 3 × 3 orthogonal matrix $\mathbf{Q}$ with real entries. Use the above result to show that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are unit vectors that are pairwise perpendicular. [4]

Suppose $\mathbf{v}_1=\frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $\mathbf{v}_2=\frac{1}{3}\begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_3=\frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$, and the linear transformation T is defined as follows:

$$\mathrm{T}(\mathbf{u})=\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \text{ if and only if } \mathbf{u}=\lambda_1\mathbf{v}_1+\lambda_2\mathbf{v}_2+\lambda_3\mathbf{v}_3,$$

where λ_1, λ_2 and λ_3 are real constants.

(c) The column vectors of an orthogonal matrix are called *orthonormal* vectors. Given that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ as defined above are orthonormal vectors, find the matrix $\mathbf{A}$ representing the linear transformation T. [2]

(d) Hence find the subspace of $\mathbb{R}^3$ whose range space under T is given by $\left\{\begin{pmatrix} p \\ q \\ p+q \end{pmatrix} : p, q \in \mathbb{R}\right\}$. [3]

3 Let V be the vector space of all real polynomials of degree at most n, and let $D: V \to V$ be the linear transformation representing differentiation, that is, $D(p) = \dfrac{\mathrm{d}p}{\mathrm{d}x}$.

(a) Explain why D is a linear transformation. [1]

(b) State bases for the kernel and range of D. [2]

(c) Give a reason why the set of real polynomials of degree n is not a vector space. [1]

4 **Do not use a calculator in answering this question.**

The linear transformation T, from $\mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$\mathrm{T}: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{where } \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ where } a, b, c \text{ and } d \text{ are constants.}$$

It is given that the transformation under T has a line L_1 of invariant points.

(a) Explain why $\mathbf{A}$ has an eigenvalue equal to 1. [1]

(b) Hence show that $ad + 1 = a + d + bc$. [2]

It is given that $\mathbf{A} = \dfrac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$.

(c) Find an eigenvector $\mathbf{e_1}$ that corresponds to eigenvalue 1 of $\mathbf{A}$ and the Cartesian equation of L_1. [2]

It is given that the transformation under T has an invariant line L_2.

(d) By finding the other eigenvalue and its corresponding eigenvector $\mathbf{e_2}$ of $\mathbf{A}$, give the Cartesian equation of L_2. [3]

(e) Show that L_1 and L_2 are perpendicular. [1]

(f) By considering your answer to part **(c), (d), (e)** and the image of $\alpha\mathbf{e_1} + \beta\mathbf{e_2}$ when it is transformed under T, where $\alpha, \beta \in \mathbb{R}$, describe the geometrical transformation T explaining your answer. [3]