H3 Mathematics Problem Solving

Lecture 3: Looking Back, Expanding, and Thinking About Thinking

Dr. Ho Weng Kin

Reconnecting to Lecture 2

Last lesson, we studied:

- mathematical resources;
- heuristics;
- structure recognition;
- the importance of checking and extending.

Reconnecting to Lecture 2

Last lesson, we studied:

- mathematical resources;
- heuristics;
- structure recognition;
- the importance of checking and extending.

Today we move from **solving problems**
to **understanding how we solve problems**.

Book link: Chapter 3; Chapter 4; Chapter 5

Lecture 3 Roadmap

1. Discuss Homework (2)
2. Extract the key counting structures
3. Look back, check and expand
4. Introduce Schoenfeld's framework
5. Apply Schoenfeld's framework to a problem

Lecture 3 Roadmap

1. Discuss Homework (2)
2. Extract the key counting structures
3. Look back, check and expand
4. Introduce Schoenfeld's framework
5. Apply Schoenfeld's framework to a problem

Big Theme

Problem solving is not just about getting an answer.
It is about developing control over mathematical thinking.

Homework (2): Problem 1

H3 2018 Question 4

A clothes shop sells a particular make of T-shirt in four different colours. The shopkeeper has a large number of T-shirts of each colour.

A customer wishes to buy seven T-shirts.

(i)(a) In how many ways can he do this?

(i)(b) In how many ways can he do this if he buys at least one of each colour?

Book link: Chapter 3, Exercises

T-shirt Problem: Part (i)(a)

Let

$$x_1 + x_2 + x_3 + x_4 = 7,$$

where x_i is the number of T-shirts bought in colour i.

T-shirt Problem: Part (i)(a)

Let

$$x_1 + x_2 + x_3 + x_4 = 7,$$

where x_i is the number of T-shirts bought in colour i.
We need the number of non-negative integer solutions.

T-shirt Problem: Part (i)(a)

Let

$$x_1 + x_2 + x_3 + x_4 = 7,$$

where x_i is the number of T-shirts bought in colour i.
We need the number of non-negative integer solutions.
By stars and bars:

$$\binom{7+4-1}{4-1} = \binom{10}{3} = 120.$$

T-shirt Problem: Part (i)(a)

Let

$$x_1 + x_2 + x_3 + x_4 = 7,$$

where x_i is the number of T-shirts bought in colour i.
We need the number of non-negative integer solutions.
By stars and bars:

$$\binom{7+4-1}{4-1} = \binom{10}{3} = 120.$$

Heuristic

Use a suitable representation: turn colours into variables.

T-shirt Problem: Part (i)(b)

Now the customer buys at least one of each colour:

$$x_1, x_2, x_3, x_4 \geq 1.$$

T-shirt Problem: Part (i)(b)

Now the customer buys at least one of each colour:

$$x_1, x_2, x_3, x_4 \geq 1.$$

Let

$$y_i = x_i - 1.$$

Then

$$y_1 + y_2 + y_3 + y_4 = 3, \qquad y_i \geq 0.$$

T-shirt Problem: Part (i)(b)

Now the customer buys at least one of each colour:

$$x_1, x_2, x_3, x_4 \geq 1.$$

Let

$$y_i = x_i - 1.$$

Then

$$y_1 + y_2 + y_3 + y_4 = 3, \qquad y_i \geq 0.$$

Hence the number of ways is

$$\binom{3+4-1}{4-1} = \binom{6}{3} = 20.$$

T-shirt Problem: Part (i)(b)

Now the customer buys at least one of each colour:

$$x_1, x_2, x_3, x_4 \geq 1.$$

Let

$$y_i = x_i - 1.$$

Then

$$y_1 + y_2 + y_3 + y_4 = 3, \qquad y_i \geq 0.$$

Hence the number of ways is

$$\binom{3+4-1}{4-1} = \binom{6}{3} = 20.$$

Heuristic

Reduce a constrained problem to an unconstrained one.

T-shirt Problem: Arrangement Version

The shopkeeper places seven T-shirts in a line.

(ii)(a) In how many ways can she do this?

(ii)(b) In how many ways can she do this if no two T-shirts of the same colour are next to each other?

(ii)(c) Use inclusion-exclusion to count arrangements using at least one T-shirt of each colour.

T-shirt Problem: Part (ii)(a)

Each of the seven positions has 4 choices.

T-shirt Problem: Part (ii)(a)

Each of the seven positions has 4 choices.
Hence

$$4^7 = 16384.$$

T-shirt Problem: Part (ii)(a)

Each of the seven positions has 4 choices.
Hence

$$4^7 = 16384.$$

Resource

Product rule:

$$m_1 m_2 \cdots m_k$$

counts successive independent choices.

T-shirt Problem: Part (ii)(b)

No two adjacent T-shirts have the same colour.

T-shirt Problem: Part (ii)(b)

No two adjacent T-shirts have the same colour.

- First position: 4 choices.
- Each subsequent position: 3 choices.

T-shirt Problem: Part (ii)(b)

No two adjacent T-shirts have the same colour.

- First position: 4 choices.
- Each subsequent position: 3 choices.

Hence

$$4 \cdot 3^6 = 2916.$$

T-shirt Problem: Part (ii)(b)

No two adjacent T-shirts have the same colour.

- First position: 4 choices.
- Each subsequent position: 3 choices.

Hence

$$4 \cdot 3^6 = 2916.$$

Heuristic

Build the arrangement from left to right.

T-shirt Problem: Why Inclusion-Exclusion?

We want arrangements of 7 T-shirts in a line using **all four colours**.

T-shirt Problem: Why Inclusion-Exclusion?

We want arrangements of 7 T-shirts in a line using **all four colours**.
Counting this directly is awkward.

T-shirt Problem: Why Inclusion-Exclusion?

We want arrangements of 7 T-shirts in a line using **all four colours**.
Counting this directly is awkward.
Instead, count the complement:

arrangements missing at least one colour.

T-shirt Problem: Why Inclusion-Exclusion?

We want arrangements of 7 T-shirts in a line using **all four colours**.
Counting this directly is awkward.
Instead, count the complement:

$$\text{arrangements missing at least one colour.}$$

Key Idea

Start from all arrangements, then remove those that fail the condition.

T-shirt Problem: Inclusion-Exclusion

Let A_i be the set of arrangements in which colour i is missing.

T-shirt Problem: Inclusion-Exclusion

Let A_i be the set of arrangements in which colour i is missing.
Total arrangements:

$$4^7.$$

T-shirt Problem: Inclusion-Exclusion

Let A_i be the set of arrangements in which colour i is missing.
Total arrangements:

$$4^7.$$

Subtract arrangements missing one specified colour:

$$\binom{4}{1}3^7.$$

T-shirt Problem: Inclusion-Exclusion

Let A_i be the set of arrangements in which colour i is missing.
Total arrangements:

$$4^7.$$

Subtract arrangements missing one specified colour:

$$\binom{4}{1}3^7.$$

But arrangements using only two colours have been subtracted twice, so add them back:

$$\binom{4}{2}2^7.$$

T-shirt Problem: Inclusion-Exclusion

Let A_i be the set of arrangements in which colour i is missing.
Total arrangements:

$$4^7.$$

Subtract arrangements missing one specified colour:

$$\binom{4}{1}3^7.$$

But arrangements using only two colours have been subtracted twice, so add them back:

$$\binom{4}{2}2^7.$$

Arrangements using only one colour have now been added back too much, so subtract:

$$\binom{4}{3}1^7.$$

T-shirt Problem: Inclusion-Exclusion

Therefore the required number is

$$4^7 - \binom{4}{1}3^7 + \binom{4}{2}2^7 - \binom{4}{3}1^7 = 8400.$$

From Example to Principle

In the T-shirt problem, we:

- counted all arrangements;
- subtracted those missing one colour;
- added back those missing two colours;
- subtracted those missing three colours.

From Example to Principle

In the T-shirt problem, we:

- counted all arrangements;
- subtracted those missing one colour;
- added back those missing two colours;
- subtracted those missing three colours.

Observation

We alternated between subtraction and addition.

From Example to Principle

In the T-shirt problem, we:

- counted all arrangements;
- subtracted those missing one colour;
- added back those missing two colours;
- subtracted those missing three colours.

Observation

We alternated between subtraction and addition.

This is not a coincidence.

Inclusion-Exclusion Principle

Let $A_1, A_2, \ldots, A_n$ be subsets of a universal set U.

Inclusion-Exclusion Principle

Let $A_1, A_2, \ldots, A_n$ be subsets of a universal set U.
Then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1}|A_1 \cap \cdots \cap A_n|.$$

Inclusion-Exclusion Principle

Let $A_1, A_2, \ldots, A_n$ be subsets of a universal set U.
Then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n|.$$

Meaning

To count elements in the union:

- add individual sizes;
- subtract overlaps;
- add back over-subtracted parts;
- continue alternating.

Inclusion-Exclusion: 4 Sets

For four sets A_1, A_2, A_3, A_4:

$$|\bigcup A_i| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - |A_1 \cap A_2 \cap A_3 \cap A_4|.$$

Inclusion-Exclusion: 4 Sets

For four sets A_1, A_2, A_3, A_4:

$$|\bigcup A_i| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - |A_1 \cap A_2 \cap A_3 \cap A_4|.$$

Connection

In the T-shirt problem:

- A_i = arrangements missing colour i

Inclusion-Exclusion: 4 Sets

For four sets A_1, A_2, A_3, A_4:

$$|\bigcup A_i| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - |A_1 \cap A_2 \cap A_3 \cap A_4|.$$

Connection

In the T-shirt problem:

- $A_i =$ arrangements missing colour i

Insight

We were already using this principle.

Why Does It Work?

When we add $|A_i|$:

- elements in multiple sets are counted multiple times.

Why Does It Work?

When we add $|A_i|$:

- elements in multiple sets are counted multiple times.

When we subtract $|A_i \cap A_j|$:

- we correct the double counting.

Why Does It Work?

When we add $|A_i|$:

- elements in multiple sets are counted multiple times.

When we subtract $|A_i \cap A_j|$:

- we correct the double counting.

But then:

- elements in three sets are subtracted too many times.

Why Does It Work?

When we add $|A_i|$:

- elements in multiple sets are counted multiple times.

When we subtract $|A_i \cap A_j|$:

- we correct the double counting.

But then:

- elements in three sets are subtracted too many times.

Key Pattern

Alternate $+$ and $-$ to correct overcounting.

What Did We Use?

In the T-shirt problem, we used:

- suitable representation;
- stars and bars;
- product rule;
- complement counting;
- inclusion-exclusion.

What Did We Use?

In the T-shirt problem, we used:

- suitable representation;
- stars and bars;
- product rule;
- complement counting;
- inclusion-exclusion.

Counting is organised thinking.

Homework (2): Problem 2

H3 2017 Question 5

(i) Explain why the number of ways to distribute r distinct objects into 2 distinct boxes such that neither box is empty is

$$2^r - 2.$$

(ii) Let $S(r, n)$ denote the number of ways to distribute r distinct objects into n identical boxes such that no box is empty.

Book link: Chapter 3, Exercises

Distinct Objects into Two Distinct Boxes

Each object has 2 choices.

Distinct Objects into Two Distinct Boxes

Each object has 2 choices.
So the total number of distributions is

$$2^r.$$

Distinct Objects into Two Distinct Boxes

Each object has 2 choices.
So the total number of distributions is

$$2^r.$$

But two distributions are not allowed:

- all objects in Box 1;
- all objects in Box 2.

Distinct Objects into Two Distinct Boxes

Each object has 2 choices.
So the total number of distributions is

$$2^r.$$

But two distributions are not allowed:

- all objects in Box 1;
- all objects in Box 2.

Hence the number is

$$2^r - 2.$$

Distinct Objects into Two Distinct Boxes

Each object has 2 choices.
So the total number of distributions is

$$2^r.$$

But two distributions are not allowed:

- all objects in Box 1;
- all objects in Box 2.

Hence the number is

$$2^r - 2.$$

Heuristic

Count all possibilities, then remove bad cases.

Identical Boxes: Why More Subtle?

For distinct boxes:

$$\{a, b\} \mid \{c\} \quad \text{and} \quad \{c\} \mid \{a, b\}$$

are different.

Identical Boxes: Why More Subtle?

For distinct boxes:

$$\{a, b\} \mid \{c\} \quad \text{and} \quad \{c\} \mid \{a, b\}$$

are different.

For identical boxes, they are the same distribution.

Identical Boxes: Why More Subtle?

For distinct boxes:

$$\{a, b\} \mid \{c\} \quad \text{and} \quad \{c\} \mid \{a, b\}$$

are different.
For identical boxes, they are the same distribution.

Key Distinction

Identical boxes remove labels.
So we must count structures, not placements.

Deriving the Recurrence

We want to show:

$$S(r,3) = 2^{r-2} - 1 + 3S(r-1,3).$$

Deriving the Recurrence

We want to show:

$$S(r,3) = 2^{r-2} - 1 + 3S(r-1,3).$$

Choose one special object, say object r.

Deriving the Recurrence

We want to show:

$$S(r,3) = 2^{r-2} - 1 + 3S(r-1,3).$$

Choose one special object, say object r.
We split into two cases:

1. object r is alone in one box;
2. object r is not alone.

Deriving the Recurrence

We want to show:

$$S(r,3) = 2^{r-2} - 1 + 3S(r-1,3).$$

Choose one special object, say object r.
We split into two cases:

1. object r is alone in one box;
2. object r is not alone.

Heuristic

Divide into cases based on the role of one special object.

Case 1: The Special Object is Alone

If object r is alone in one box, then the remaining $r-1$ objects must be distributed into 2 non-empty identical boxes.

Case 1: The Special Object is Alone

If object r is alone in one box, then the remaining $r-1$ objects must be distributed into 2 non-empty identical boxes.
The number of ways is

$$S(r-1,2).$$

Case 1: The Special Object is Alone

If object r is alone in one box, then the remaining $r-1$ objects must be distributed into 2 non-empty identical boxes.
The number of ways is

$$S(r-1,2).$$

But

$$S(r-1,2)=\frac{2^{r-1}-2}{2}=2^{r-2}-1.$$

Case 2: The Special Object is Not Alone

First distribute the remaining $r-1$ objects into 3 non-empty identical boxes:

$$S(r-1,3).$$

Case 2: The Special Object is Not Alone

First distribute the remaining $r-1$ objects into 3 non-empty identical boxes:

$$S(r-1,3).$$

Then place the special object into one of the 3 boxes.

Case 2: The Special Object is Not Alone

First distribute the remaining $r-1$ objects into 3 non-empty identical boxes:

$$S(r-1,3).$$

Then place the special object into one of the 3 boxes.
This gives

$$3S(r-1,3).$$

Case 2: The Special Object is Not Alone

First distribute the remaining $r-1$ objects into 3 non-empty identical boxes:

$$S(r-1,3).$$

Then place the special object into one of the 3 boxes.
This gives

$$3S(r-1,3).$$

Therefore

$$S(r,3)=2^{r-2}-1+3S(r-1,3).$$

Proving the Modulo Result

We need to prove:

$$S(r,3) = \begin{cases} 0 \pmod 6, & r \text{ even}, \\ 1 \pmod 6, & r \text{ odd}. \end{cases}$$

Proving the Modulo Result

We need to prove:

$$S(r,3) = \begin{cases} 0 \pmod 6, & r \text{ even}, \\ 1 \pmod 6, & r \text{ odd}. \end{cases}$$

Use induction with the recurrence:

$$S(r,3) = 2^{r-2} - 1 + 3S(r-1,3).$$

Modulo Proof

If r is odd, then $r - 1$ is even, so

$$S(r-1,3) \equiv 0 \pmod{6}.$$

Also, $r - 2$ is odd, so

$$2^{r-2} \equiv 2 \pmod{6}.$$

Thus

$$S(r,3) \equiv 2 - 1 + 0 \equiv 1 \pmod{6}.$$

Modulo Proof

If r is odd, then $r-1$ is even, so

$$S(r-1,3) \equiv 0 \pmod{6}.$$

Also, $r-2$ is odd, so

$$2^{r-2} \equiv 2 \pmod{6}.$$

Thus

$$S(r,3) \equiv 2-1+0 \equiv 1 \pmod{6}.$$

If r is even, then $r-1$ is odd, so

$$S(r-1,3) \equiv 1 \pmod{6}.$$

Also, $r-2$ is even, so

$$2^{r-2} \equiv 4 \pmod{6}.$$

Thus

$$S(r,3) \equiv 4-1+3 \equiv 0 \pmod{6}.$$

What Did We Use?

In the distribution problem, we used:

- complement counting;
- distinction between labelled and unlabelled structures;
- special object argument;
- recurrence;
- induction;
- modular arithmetic.

What Did We Use?

In the distribution problem, we used:

- complement counting;
- distinction between labelled and unlabelled structures;
- special object argument;
- recurrence;
- induction;
- modular arithmetic.

Big Idea

A difficult counting problem often becomes manageable once we find the right structure.

Why Look Back?

A good problem solver does not stop after finding an answer.

Why Look Back?

A good problem solver does not stop after finding an answer.
Looking back helps us:

- check correctness;
- detect hidden assumptions;
- identify what led to success;
- find simpler or more elegant methods;
- adapt, extend, and generalise.

Why Look Back?

A good problem solver does not stop after finding an answer.
Looking back helps us:

- check correctness;
- detect hidden assumptions;
- identify what led to success;
- find simpler or more elegant methods;
- adapt, extend, and generalise.

Key Message

Looking back turns answers into understanding.

Book link: Chapter 4: Look back, check and expand

Examining What Led to Success

After solving a problem, ask:

- What was the key move?
- Which heuristic worked?
- What structure did we notice?
- Was there a more efficient method?

Examining What Led to Success

After solving a problem, ask:

- What was the key move?
- Which heuristic worked?
- What structure did we notice?
- Was there a more efficient method?

From Homework (2)

- T-shirt problem: representation and inclusion-exclusion.
- Distribution problem: special object and recurrence.

A Simple Example

Sketch the graph:

$$|x| + |y| = 3.$$

A Simple Example

Sketch the graph:

$$|x| + |y| = 3.$$

Because of the absolute values, we split into cases according to the signs of x and y.

A Simple Example

Sketch the graph:

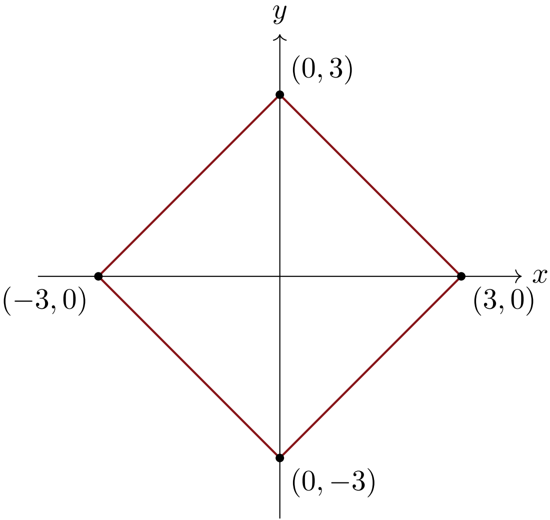
$$|x| + |y| = 3.$$

Because of the absolute values, we split into cases according to the signs of x and y.

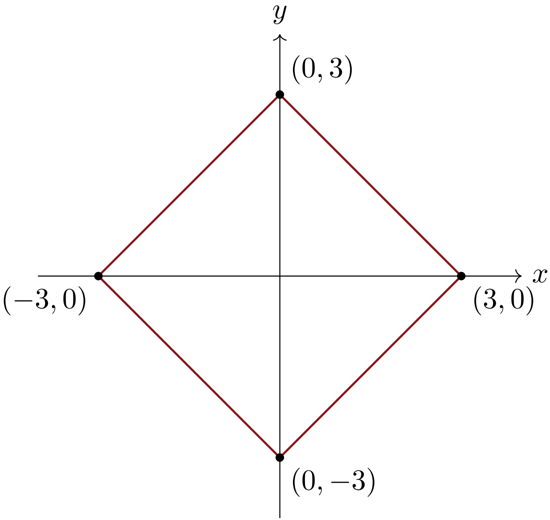
Heuristic

Case analysis guided by structure.

Graph of $|x| + |y| = 3$



Graph of $|x| + |y| = 3$



Expand the Problem

A solved problem can become a doorway to new mathematics.

Expand the Problem

A solved problem can become a doorway to new mathematics.

1. **Adapt**: change some features but preserve the core structure.
2. **Extend**: make the problem broader or harder.
3. **Generalise**: place the problem inside a wider framework.

Expand the Problem

A solved problem can become a doorway to new mathematics.

1. **Adapt**: change some features but preserve the core structure.
2. **Extend**: make the problem broader or harder.
3. **Generalise**: place the problem inside a wider framework.

Key Habit

After solving, ask: *What if?*

Book link: Chapter 4, Section 3

Expanding $|x| + |y| = 3$

Adapt:

$$|x| + |y| = C, \qquad C > 0.$$

Expanding $|x| + |y| = 3$

Adapt:

$$|x| + |y| = C, \qquad C > 0.$$

Extend:

$$|x|^p + |y|^p = 3, \qquad p > 0.$$

Expanding $|x| + |y| = 3$

Adapt:

$$|x| + |y| = C, \qquad C > 0.$$

Extend:

$$|x|^p + |y|^p = 3, \qquad p > 0.$$

Reverse the question:
Given the diamond-shaped graph, find a Cartesian equation for it.

Expanding $|x| + |y| = 3$

Adapt:

$$|x| + |y| = C, \qquad C > 0.$$

Extend:

$$|x|^p + |y|^p = 3, \qquad p > 0.$$

Reverse the question:
Given the diamond-shaped graph, find a Cartesian equation for it.

Lesson

The original problem is only one member of a larger family.

Apply "Look Back" to Homework (2)

For the T-shirt problem:

- What if there are n colours instead of 4?
- What if the customer buys r T-shirts instead of 7?
- What if each colour must appear at least twice?

Apply “Look Back” to Homework (2)

For the T-shirt problem:

- What if there are n colours instead of 4?
- What if the customer buys r T-shirts instead of 7?
- What if each colour must appear at least twice?

For the distribution problem:

- What if there are n identical boxes?
- What if empty boxes are allowed?
- What if the boxes are distinct instead?

From Pólya to Schoenfeld

Pólya gives us a powerful framework:

$$\text{Understand} \to \text{Plan} \to \text{Execute} \to \text{Look Back.}$$

From Pólya to Schoenfeld

Pólya gives us a powerful framework:

$$\text{Understand} \rightarrow \text{Plan} \rightarrow \text{Execute} \rightarrow \text{Look Back.}$$

But a question remains:

Why do students still get stuck
even after learning heuristics?

From Pólya to Schoenfeld

Pólya gives us a powerful framework:

$$\text{Understand} \rightarrow \text{Plan} \rightarrow \text{Execute} \rightarrow \text{Look Back.}$$

But a question remains:

Why do students still get stuck
even after learning heuristics?

Schoenfeld's Contribution

Successful problem solving depends not only on strategies, but also on knowledge, control, and beliefs.

Book link: Chapter 5: Schoenfeld's Framework

Schoenfeld's Contribution

Figure: Alan Schoenfeld

Schoenfeld's Four Components

1. **Cognitive Resources**
   What mathematical knowledge do I have?
2. **Heuristics**
   What strategies can I try?
3. **Control**
   How do I monitor and regulate my thinking?
4. **Belief Systems**
   What do I believe mathematics is about?

Schoenfeld's Four Components

1. **Cognitive Resources**
   What mathematical knowledge do I have?
2. **Heuristics**
   What strategies can I try?
3. **Control**
   How do I monitor and regulate my thinking?
4. **Belief Systems**
   What do I believe mathematics is about?

Key Point

Knowing mathematics is necessary, but not sufficient.

Control: Thinking About Thinking

Control means asking:

- What exactly am I doing?
- Why am I doing this?
- Is this helping?
- Should I continue, modify, or abandon this approach?

Control: Thinking About Thinking

Control means asking:

- What exactly am I doing?
- Why am I doing this?
- Is this helping?
- Should I continue, modify, or abandon this approach?

Metacognition

Control is the ability to monitor and direct one's own thinking.

Beliefs Matter

Students behave differently depending on what they believe.

Beliefs Matter

Students behave differently depending on what they believe.

- "Mathematics is about formulas."
- "Problems should be solved quickly."
- "Being stuck means I am bad at mathematics."

Beliefs Matter

Students behave differently depending on what they believe.

- "Mathematics is about formulas."
- "Problems should be solved quickly."
- "Being stuck means I am bad at mathematics."

These beliefs can block progress.

Beliefs Matter

Students behave differently depending on what they believe.

- "Mathematics is about formulas."
- "Problems should be solved quickly."
- "Being stuck means I am bad at mathematics."

These beliefs can block progress.

Productive Belief

Mathematics is about sense-making, persistence, and structure.

The Power of Belief

A Story about George Dantzig

The Power of Belief

A Story about George Dantzig

As a graduate student, George Dantzig arrived late to a statistics class.

The Power of Belief

A Story about George Dantzig

As a graduate student, George Dantzig arrived late to a statistics class.
He saw two problems written on the board and assumed they were homework.

The Power of Belief

A Story about George Dantzig

As a graduate student, George Dantzig arrived late to a statistics class.
He saw two problems written on the board and assumed they were homework.
Although they seemed unusually difficult, he worked on them and eventually solved them.

The Power of Belief

A Story about George Dantzig

As a graduate student, George Dantzig arrived late to a statistics class.
He saw two problems written on the board and assumed they were homework.
Although they seemed unusually difficult, he worked on them and eventually solved them.

The Twist

These were not homework problems.
They were two famous unsolved problems in statistics.

What Does This Tell Us?

- Dantzig did not know the problems were "impossible"
- So he approached them as solvable

What Does This Tell Us?

- Dantzig did not know the problems were “impossible”
- So he approached them as solvable

Belief Systems

What we believe about a problem affects how we engage with it.

What Does This Tell Us?

- Dantzig did not know the problems were "impossible"
- So he approached them as solvable

Belief Systems

What we believe about a problem affects how we engage with it.

- "This is too hard" $\Rightarrow$ give up
- "This is solvable" $\Rightarrow$ persist

What Does This Tell Us?

- Dantzig did not know the problems were "impossible"
- So he approached them as solvable

Belief Systems

What we believe about a problem affects how we engage with it.

- "This is too hard" $\Rightarrow$ give up
- "This is solvable" $\Rightarrow$ persist

Lesson

Beliefs shape effort, persistence, and success.

A Problem for Schoenfeld's Framework

Two runners start at the same point on a circular track, back to back, running in opposite directions.

A Problem for Schoenfeld's Framework

Two runners start at the same point on a circular track, back to back, running in opposite directions.
They subsequently meet each other 11 times.

A Problem for Schoenfeld's Framework

Two runners start at the same point on a circular track, back to back, running in opposite directions.
They subsequently meet each other 11 times.
If one runner has completed 6 rounds, how many rounds did the other complete?

Step 1: Cognitive Resources

What knowledge might help?

- circular motion;
- relative speed;
- opposite directions mean speeds add;
- a meeting occurs when their relative distance is one full lap.

Step 1: Cognitive Resources

What knowledge might help?

- circular motion;
- relative speed;
- opposite directions mean speeds add;
- a meeting occurs when their relative distance is one full lap.

Key Resource

Relative motion converts two moving runners into one moving gap.

Step 2: Heuristics

Let the other runner complete x rounds.

Step 2: Heuristics

Let the other runner complete x rounds.
Instead of tracking two runners separately, track their relative motion.

Step 2: Heuristics

Let the other runner complete x rounds.
Instead of tracking two runners separately, track their relative motion.

Key Heuristic

Track the quantity that directly controls meetings.

Step 2: Heuristics

Let the other runner complete x rounds.
Instead of tracking two runners separately, track their relative motion.

Key Heuristic

Track the quantity that directly controls meetings.

In this problem, that quantity is:

$$\text{total relative laps} = 6 + x.$$

Step 3: Control

Now we must ask carefully:

- What exactly are we counting?
- Does the starting point count as a meeting?
- When does the first meeting occur?

Step 3: Control

Now we must ask carefully:

- What exactly are we counting?
- Does the starting point count as a meeting?
- When does the first meeting occur?

Critical Interpretation

The word "subsequently" means the starting point is not counted.

Step 3: Control

Now we must ask carefully:

- What exactly are we counting?
- Does the starting point count as a meeting?
- When does the first meeting occur?

Critical Interpretation

The word "subsequently" means the starting point is not counted.

So the first counted meeting occurs after 1 full relative lap.

Step 4: Carry Out the Plan

Each full relative lap gives one meeting.

Step 4: Carry Out the Plan

Each full relative lap gives one meeting.
Since they meet 11 times after starting:

$$6 + x = 11.$$

Step 4: Carry Out the Plan

Each full relative lap gives one meeting.
Since they meet 11 times after starting:

$$6 + x = 11.$$

Therefore

$$x = 5.$$

Step 4: Carry Out the Plan

Each full relative lap gives one meeting.
Since they meet 11 times after starting:

$$6 + x = 11.$$

Therefore

$$x = 5.$$

Answer

The other runner completed

$$\boxed{5}$$

rounds.

Step 5: Looking Back

Check the answer:

$$6 + 5 = 11.$$

Step 5: Looking Back

Check the answer:

$$6 + 5 = 11.$$

This means the runners together cover 11 relative laps, producing 11 subsequent meetings.

Step 5: Looking Back

Check the answer:

$$6 + 5 = 11.$$

This means the runners together cover 11 relative laps, producing 11 subsequent meetings.

What Led to Success?

The key was not algebra.
The key was deciding what to count.

Where Schoenfeld Appears

Component	In the runner problem
Cognitive Resources	Relative motion and circular laps
Heuristics	Introduce x, track relative distance
Control	Decide whether starting point counts
Beliefs	Treat problem as sense-making, not formula-hunting

A Possible Wrong Path

A student may write:

$$(6 + x) - 1 = 11.$$

A Possible Wrong Path

A student may write:

$$(6+x)-1=11.$$

This gives

$$x=6.$$

A Possible Wrong Path

A student may write:

$$(6 + x) - 1 = 11.$$

This gives

$$x = 6.$$

But this assumes that the starting point has already been counted as a meeting.

A Possible Wrong Path

A student may write:

$$(6 + x) - 1 = 11.$$

This gives

$$x = 6.$$

But this assumes that the starting point has already been counted as a meeting.

Control Question

Does this interpretation match the phrase "subsequently they met"?

A Possible Wrong Path

A student may write:

$$(6 + x) - 1 = 11.$$

This gives

$$x = 6.$$

But this assumes that the starting point has already been counted as a meeting.

Control Question

Does this interpretation match the phrase "subsequently they met"?

No. Hence the -1 should not be used.

The Schoenfeld Lesson

This problem is not difficult because of computation.

The Schoenfeld Lesson

This problem is not difficult because of computation.
It is difficult because of:

- interpretation;
- representation;
- monitoring;
- deciding what counts.

The Schoenfeld Lesson

This problem is not difficult because of computation.
It is difficult because of:

- interpretation;
- representation;
- monitoring;
- deciding what counts.

Problem solving is not just doing.
It is thinking about doing.

Homework (3)

From Chapter 4, attempt:

1. H3 2018 Question 1: telescoping sum problem
2. H3 2018 Question 6: pigeonhole principle problem
3. H3 2017 Question 6: circular arrangements and bijection

Homework (3)

From Chapter 4, attempt:

1. H3 2018 Question 1: telescoping sum problem
2. H3 2018 Question 6: pigeonhole principle problem
3. H3 2017 Question 6: circular arrangements and bijection

Instruction

For each problem, include a short paragraph explaining:

- which heuristics you used;
- where you had to exercise control;
- one possible adaptation, extension, or generalisation.

Book link: Chapter 4, Exercises

Summary: Lecture 3

Today we learned that:

- counting problems require structure;
- looking back is not optional;
- solved problems can be adapted, extended, and generalised;
- Schoenfeld helps us see why problem solving succeeds or fails.

Summary: Lecture 3

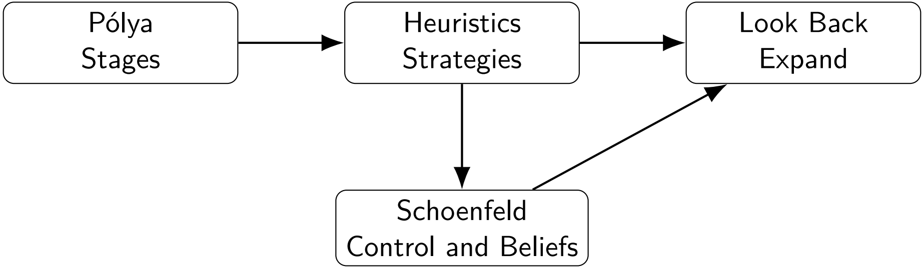
Today we learned that:

- counting problems require structure;
- looking back is not optional;
- solved problems can be adapted, extended, and generalised;
- Schoenfeld helps us see why problem solving succeeds or fails.

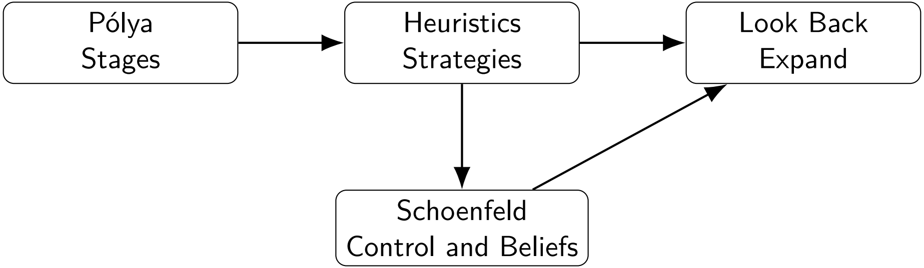
Big Idea

A solution is not the end of thinking.
It is the beginning of deeper understanding.

Unified View of Problem Solving



Unified View of Problem Solving



We are learning not only to solve problems,
but to become better problem solvers.

References

- G. Pólya, *How to Solve It*, Princeton University Press, 1957.
- A. H. Schoenfeld, *Mathematical Problem Solving*, Academic Press, 1985.
- I. Lakatos, *Proofs and Refutations*, Cambridge University Press, 1976.
- W. K. Ho, *The Art of Problem Solving*, H3 Mathematics Notes, 2026.