Sample Paper:

A) Compute all the Nash equilibria in this game

		P2		
		A	B	C
P1	A	2, 2	-1, 0	0, 1
	B	0, -1	3, 3	-1, 0
	C	1, 0	0, -1	1, 1

A1) Compute all the Nash equilibria in this game:

		P2		
		L	M	R
P1	U	3, 4	0, 0	1, 2
	C	0, 0	4, 3	-1, 1
	D	2, 1	1, -1	5, 5

B) Coke demand function $Q = 44 - 2p + q$; Pepsi demand function $Q = 44 - 2q + p$, where p denotes the unit price of Coke and q denotes the unite price of Pepsi. Both have same marginal cost 8 of producing per can of soda.

1) **From the demand function, are Coke and Pepsi complements or substitutes?**

2) **Compute the Nash equilibrium price and profits.**

3) **What will be the maximum profit for Coke and Pepsi respectively if they were to collude?**

4) **Consider a game where each of the the two firms can either price cooperatively, i.e collude or price aggressively, i.e. maximising own profit. Construct the 2x2 game matrix.**

5) **Does this game suit the features of a prisoners' dilemma? Explain.**

6) **This game is being played repeatedly for 100 games. What is the sub-game perfect equilibrium?**

7) **This game is being played repeatedly infinitely. Write down the strategy for Grim Trigger for each player.**

8) **What is the condition for Grim Trigger to be a sub-game perfect equilibrium strategy?**

9) **Consider a stick-and-carrot strategy as follow:**

Cooperative state: Play the collusive price and remain in the cooperative state if no deviation occurs, otherwise switch to punishment state

Punishment state: Price 8 and switch to cooperative state if no deviation occurs, otherwise remain in the punishment state

If both players adopt such strategy, what condition is required for it to be a sub-game perfect equilibrium?

10) **What if in the competition state, firms price 0? Rewrite the stick-and-carrot strategy, and repeat the calculations to find the condition.**

C) Consider the game below:

		P2		
		A	B	C
	A	2, 2	-2, 5	-4, -4
P1	B	5, -2	0, 0	-4, -4
	C	-4, 4	-4, 4	-5, -5

1) **By best response analysis, find the Nash equilibrium of this game.**

2) **By iterated elimination of strictly dominated strategies, find the Nash equilibrium of this game.**

3) **This game is being played repeatedly infinitely. Write a stick-and-carrot strategy where the cooperative strategy leads to a Pareto efficient outcome, the punishment leads to a Minmax payoff.**

4) **Assuming the probability that the game will end in a single round is p, what is the condition for such stick-and-carrot strategy to be a sub-game perfect equilibrium?**

D) Coke and Pepsi try to maximise profits in the soda industry. The marginal cost of Coke can either be 5 or 15, with probability p of a low cost. In the first stage, Coke can choose to price in a way that maximises low cost profit or high cost profit. In the second stage, the Pepsi choose to whether enter the market not. The marginal cost of Pepsi is 10 and Pepsi also need to incur a fixed cost of 40 to enter.

1) **The demand function of the soda industry is given by $P = 25 - Q$. Calculate the profits of Coke with high cost and low cost in a monopoly game without bluffing, and the profits of high cost Coke with bluffing.**

2) **Calculate the profits of Coke and Pepsi respectively in the duopoly game, in consideration of both scenarios where Coke is high cost and low cost.**

3) **Therefore, draw the game tree of this game and round the payoffs to whole numbers.**

4) **Is the equilibrium separating, semi-separating or pooling?**

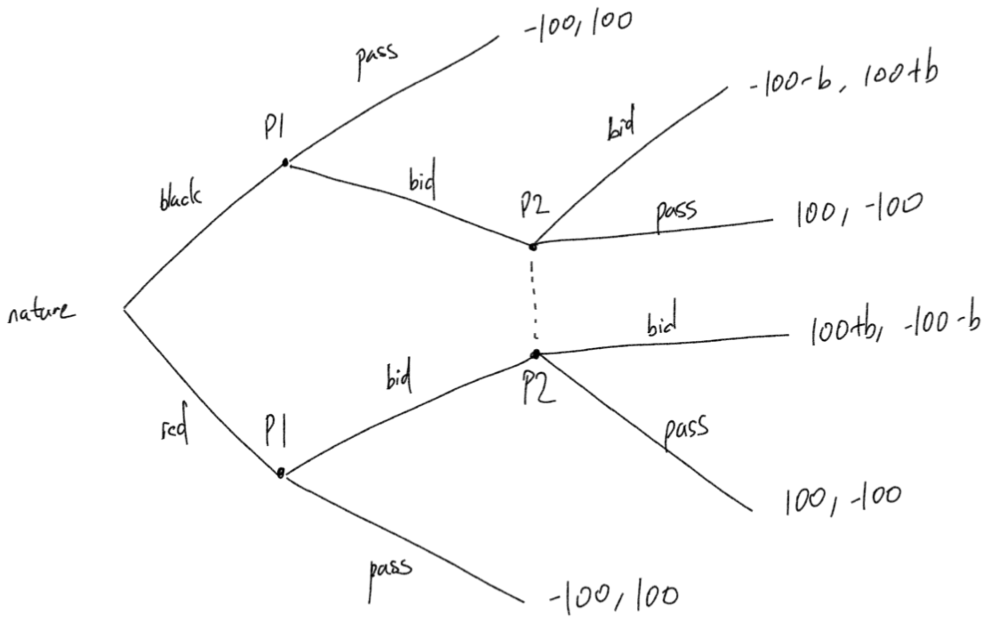
5) **Reconsider the case where Coke with low marginal cost be 10 instead of 5. Keeping everything else the same, calculate the profits of Coke with high cost and low cost in a monopoly game without bluffing, and the profits of high cost Coke with bluffing.**

6) **Recalculate the profits of Coke and Pepsi respectively in the new duopoly game, in consideration of both scenarios where Coke is high cost and low cost.**

7) **Therefore, draw the new game tree of this game and round the payoffs to whole numbers.**

8) **What are the possible equilibriums and are there any conditions, if any?**

D1) Consider the simplified poker game below:



1) **What are the possible equilibriums and are there any conditions, if any?**

2) **Express this game in normal form and calculate the payoffs**

E) A company wish to distinguish between type A and type C students from a pool of students. It is willing to pay a salary of \$160,000 and \$60,000 to type A and type C students respectively. Type A and type C students can find an alternative job with salary \$125,000 and \$30,000 respectively. Suppose the cost of taking a tough course is \$3,000 and \$14,000 to type A and type C students respectively,

1) **What is the minimum number of tough courses the company should set as a benchmark to separate type A and type C students?**

2) **Suppose the proportion of type A students is p. What will be the pooling equilibrium and what are the conditions for both types to be better off than separating equilibrium?**

3) **Is this equilibrium stable?**

F) There are a total of 2 players in a first-priced, sealed bid auction. Each player has their own private value for the object. The object is of value v to player 1. The unknown private value of a player is uniformly distributed between 0 and 1.

1) **Assuming player 2 adopts the bidding function $b(x) = \dfrac{x}{2}$ where x is the unknown private value of player 2, what is the best response for player 1?**

2) **What is the Nash equilibrium of this game?**

3) **What is the expected payment for each player in terms of their private value? What will be the expected revenue for the seller?**

G) In an all-pay auction, the common value of object to bidders is 5.

1) **Can there by any pure strategy Nash equilibrium? Why or why not?**

2) **Suppose there are 3 bidders, what will be the Nash equilibrium for mixed strategy?**

3) **Repeat the calculation for mixed strategy equilibrium for n bidders. What will be the mixed strategy equilibrium? What will be the expected revenue for the seller?**

Example A

Compute all the Nash equilibria in this game

P2

P1		A	B	C
	A	2, 2	-1, 0	0, 1
P1	B	0, -1	3, 3	-1, 0
	C	1, 0	0, -1	1, 1

<u>Answer:</u>

The Nash equilibria are (A, A), (B, B), (C, C), $(\frac{1}{2}A \oplus \frac{1}{2}C, \frac{1}{2}A \oplus \frac{1}{2}C)$,

$(\frac{2}{3}A \oplus \frac{1}{3}B, \frac{2}{3}A \oplus \frac{1}{3}B)$, $(\frac{2}{5}B \oplus \frac{3}{5}C, \frac{2}{5}B \oplus \frac{3}{5}C)$,

$(\frac{1}{2}A \oplus \frac{3}{10}B \oplus \frac{1}{5}C, \frac{1}{2}A \oplus \frac{3}{10}B \oplus \frac{1}{5}C)$

<u>Explanation:</u>

Nash equilibrium is a list of strategies, one for each player, such that all the strategies are mutual best responses to one another. Therefore, we intend to graph the best responses to find the mutual best response strategies which are the Nash equilibria, with steps as such:

Assuming P2's strategy is $(xA \oplus yB \oplus (1-x-y)C)$, we find the P1's expected payoff for each pure strategy:

$$E(A) = 2x - y$$

$$\begin{aligned} E(B) &= 3y - (1 - x - y) \\ &= x + 4y - 1 \end{aligned}$$

$$\begin{aligned} E(C) &= x + 1 - x - y \\ &= -y + 1 \end{aligned}$$

Calculating the conditions where each strategy is the best response (BR):

A is the BR when $2x - y \geq x + 4y - 1$ and $2x - y \geq -y + 1$

$5y \leq x + 1$ $\qquad$ $x \geq \frac{1}{2}$

$y \leq \frac{1}{5}x + \frac{1}{5}$

B is the BR when $x + 4y - 1 \geq 2x - y$ and $x + 4y - 1 \geq -y + 1$

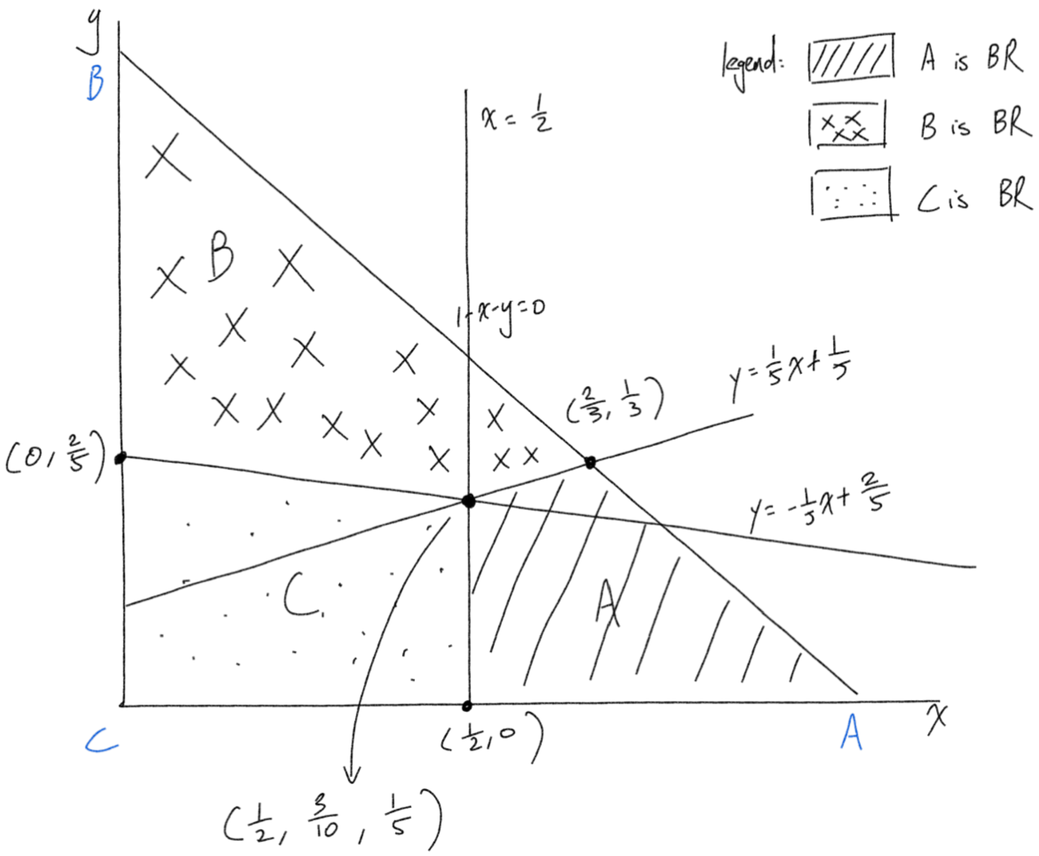
$y \geq \frac{1}{5}x + \frac{1}{5}$ $\qquad$ $5y \geq -x + 2$

$y \geq -\frac{1}{5}x + \frac{2}{5}$

C is the BR when $-y + 1 \geq 2x - y$ and $-y + 1 \geq x + 4y - 1$

$x \leq \frac{1}{2}$ $\qquad$ $y \leq -\frac{1}{5}x + \frac{2}{5}$

Combining results in a graph:



The area shaded indicates the conditions where the strategy in particular is the BR given a strategy mix of the other player (given by coordinates). Since this game is symmetric, i.e the BR diagram for P1 and P2 are the same, we can therefore identify the mutual BR strategies from the diagram and they will be the Nash equilibria: 3 pure, 4 mixed (represented by the black dots).

Example A1

Compute all the Nash equilibria in this game:

		P2		
		L	M	R
	U	3, 4	0, 0	1, 2
P1	C	0, 0	4, 3	-1, 1
	D	2, 1	1, -1	5, 5

<u>Answer:</u>

The Nash equilibria are (C, M), (D, R), (U, L), $(\frac{3}{4}C \oplus \frac{1}{4}D, \frac{2}{3}M \oplus \frac{1}{3}R)$, $(\frac{2}{3}U \oplus \frac{1}{3}D, \frac{4}{5}L \oplus \frac{1}{5}R)$, $(\frac{3}{7}U \oplus \frac{4}{7}C, \frac{4}{7}L \oplus \frac{3}{7}M)$, $(\frac{7}{18}U \oplus \frac{5}{9}C \oplus \frac{1}{18}D, \frac{6}{11}L \oplus \frac{14}{33}M \oplus \frac{1}{33}R)$

Explanation:

Nash equilibrium is a list of strategies, one for each player, such that all the strategies are mutual best responses to one another. Therefore, we intend to graph the best responses to find the mutual best response strategies which are the Nash equilibria, with steps as such:

Assuming P2's strategy is $(xA \oplus yB \oplus (1-x-y)C)$, we find P1's expected payoff for each pure strategy:

$$E(U) = 3x + 1 - x - y$$
$$= 2x - y + 1$$

$$E(C) = 4y - (1 - x - y)$$
$$= x + 5y - 1$$

$$E(D) = 2x + y + 5(1 - x - y)$$
$$= 2x + y + 5 - 5x - 5y$$
$$= -3x - 4y + 5$$

Calculating the conditions where each pure strategy is the best response (BR):

U is BR when: $2x - y + 1 \geq x + 5y - 1$ and $2x - y + 1 \geq -3x - 4y + 5$

$6y \leq x + 2$ $\qquad$ $3y \geq -5x + 4$

$y \leq \frac{1}{6}x + \frac{1}{3}$ $\qquad$ $y \geq -\frac{5}{3}x + \frac{4}{3}$

C is BR when: $x + 5y - 1 \geq 2x - y + 1$ and $x + 5y - 1 \geq -3x - 4y + 5$

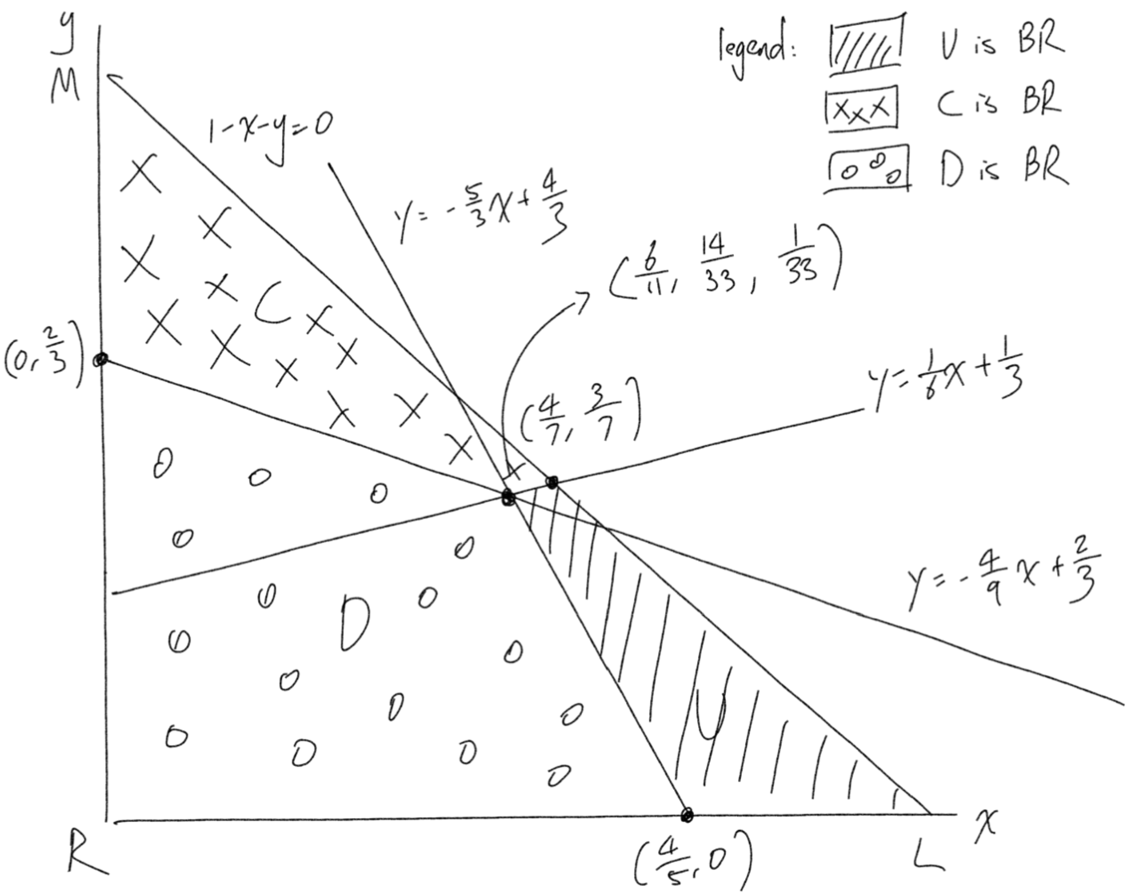
$y \geq \frac{1}{6}x + \frac{1}{3}$ $\qquad$ $9y \geq -4x + 6$

$y \geq -\frac{4}{9}x + \frac{2}{3}$

D is BR when: $-3x - 4y + 5 \geq 2x - y + 1$ and $-3x - 4y + 5 \geq x + 5y - 1$

$y \leq -\frac{5}{3}x + \frac{4}{3}$ $\qquad$ $y \leq -\frac{4}{9}x + \frac{2}{3}$

Combining results in a graph:



Repeating this process, now assuming P1's strategy is $(pA \oplus qB \oplus (1-p-q)C)$, we find P2's expected payoff for each pure strategy, and calculate the conditions where each pure strategy is the best response:

$$E(L) = 4p + 1 - p - q$$
$$= 3p - q + 1$$

$$E(M) = 3q - (1-p-q)$$
$$= p + 4q - 1$$

$$E(R) = 2p + q + 5(1-p-q)$$
$$= 2p + q + 5 - 5p - 5q$$
$$= -3p - 4q + 5$$

L is BR when $3p - q + 1 \geqslant p + 4q - 1$ and $3p - q + 1 \geqslant -3p - 4q + 5$

$5q \leq 2p + 2$ $\qquad$ $3q \geqslant -6p + 4$

$q \leq \frac{2}{5}p + \frac{2}{5}$ $\qquad$ $q \geqslant -2p + \frac{4}{3}$

M is BR when $p + 4q - 1 \geqslant 3p - q + 1$ and $p + 4q - 1 \geqslant -3p - 4q + 5$

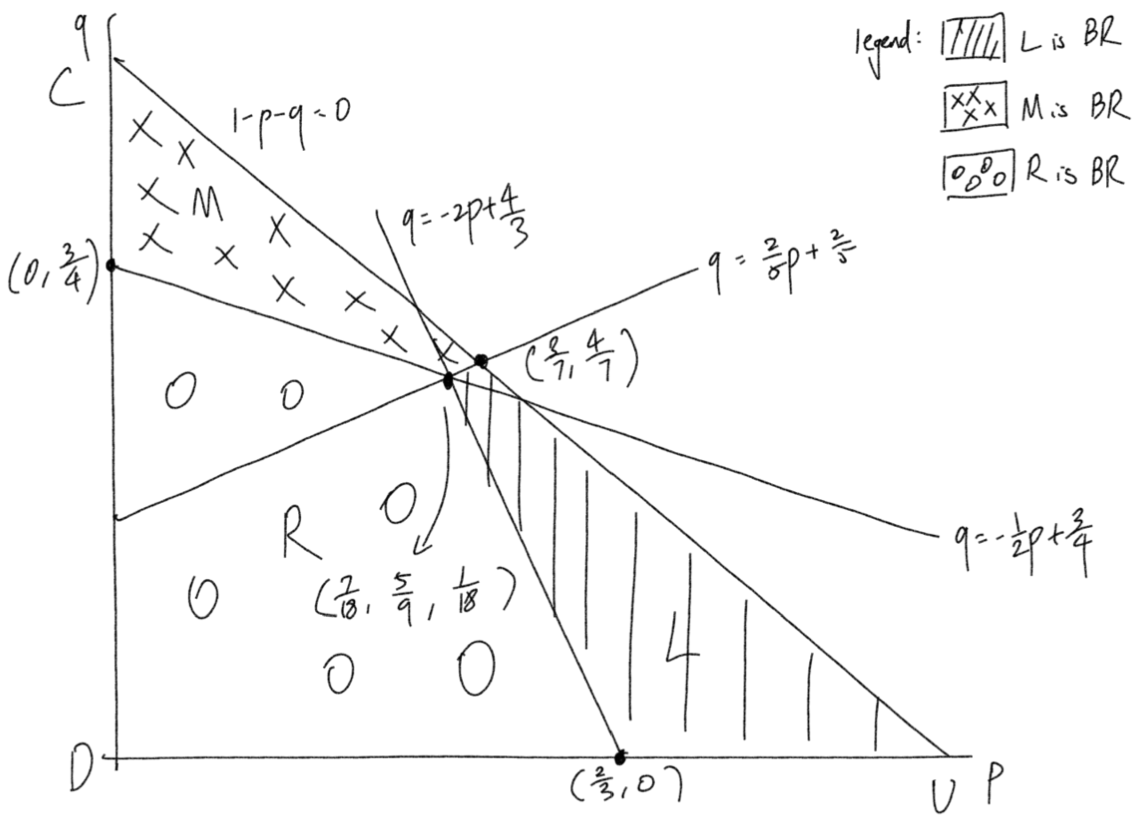
$q \geqslant \frac{2}{5}p + \frac{2}{5}$ $\qquad$ $8q \geqslant -4p + 6$

$q \geqslant -\frac{1}{2}p + \frac{3}{4}$

R is BR when $-3p - 4q + 5 \geqslant 3p - 4q - 1$ and $-3p - 4q + 5 \geqslant p + 4q - 1$

$q \leq -2p + \frac{4}{3}$ $\qquad$ $q \leq -\frac{1}{2}p + \frac{3}{4}$

Combining results in a graph:



Each shaded region in both graphs indicates the condition where the particular strategy is the BR, given the strategy mix of the other player (given by coordinates). Therefore, we can look at both graphs and find the pairs of strategies that are mutual BR and they are the Nash equilibria: 3 pure, 4 mixed (represented by the black dots).

Example B

Coke demand function $Q = 44 - 2p + q$; Pepsi demand function $Q = 44 - 2q + p$, where p denotes the unit price of Coke and q denotes the unite price of Pepsi. Both have same marginal cost 8 of producing per can of soda.

From the demand function, are Coke and Pepsi complements or substitutes?

Substitutes. From demand function, as the unit price of Pepsi (or respectively coke) increases, the demand for Coke (or respectively Pepsi) increases. Their direct relationship means that consumers will switch to lower priced Coke as a substitute whenever the price of Pepsi increases.

Alternative for complements:
Complements. From demand function, as the unit price of X (or respectively Y) increases, the demand for Y (or respectively X) decreases. Their inverse relationship means that consumers treat them as complements, where an increase in price of either X or Y can reduce the quantity demanded for both goods.

Compute the Nash equilibrium price and profits.

Answer:

The Nash equilibrium price and profits are $\begin{cases} Coke : p = 20 \\ Pepsi : q = 20 \end{cases}$ and $\begin{cases} Coke : \pi_{coke} = 288 \\ Pepsi : \pi_{pepsi} = 288 \end{cases}$ respectively.

Explanation:
Nash equilibrium is a list of strategies, one for each player, such that all strategies are mutual best responses to one another. Therefore, we first compute the best response (BR) functions for each player, and then solving them simultaneously to find the mutual BR which is the Nash equilibrium, with steps as such:

Profit = total number of units x profit margin per unit i.e. profit = quantity x (price - cost). Hence profit of Coke:

$\pi_{coke} = (p - 8)(44 - 2p + q)$

For BR function, we maximise the profit for a given price of Pepsi using calculus:

F.O.C: $\pi'_{coke} = (p - 8)(-2) + (44 - 2p + q) = 0$

$-2p + 16 + 44 - 2p + q = 0$

$4p = 60 + q$

$\therefore p = \frac{1}{4}q + 15$ is the BR function for Coke (1)

Since the game is symmetric, the calculations will be the same:

$\therefore q = \frac{1}{4}p + 15$ is the BR function for Pepsi (2)

Solving simultaneously, substitute (1) into (2),

$q = \frac{1}{4}(\frac{1}{4}q + 15) + 15$

$q = \frac{1}{16}q + \frac{15}{4} + 15$

$\frac{15}{16}q = \frac{75}{4}$

$\therefore q = 20$

Similarly, $p = 20$

Substituting into profit functions:

$$\begin{aligned}\pi_{coke} = \pi_{pepsi} &= (20 - 8)(44 - 2 \times 20 + 20) \\ &= (12)(24) \\ &= 288\end{aligned}$$

What will be the maximum profit for Coke and Pepsi respectively if they were to collude?

Answer:

The maximum profit for $\begin{cases} \pi_{coke} = 324 \\ \pi_{pepsi} = 324 \end{cases}$

Explanation:

We first write out the joint profit function:

$\pi = (p - 8)(44 - 2p + q) + (q - 8)(44 - 2q + p)$

For maximisation, we find the partial derivatives using calculus:

Differentiating with respect to p,

$(p - 8)(-2) + (44 - 2p + q) + (q - 8) = 0$

$-2p + 16 + 44 - 2p + q + q - 8 = 0$

$-4p + 2q + 52 = 0$

$\Rightarrow -2p + q + 26 = 0$

Since game is symmetric, joint profit is maximised at same price $p = q$,
i.e. $p = q = 26$

$$\begin{aligned}\therefore \pi_{coke} = \pi_{pepsi} &= (26 - 8)(44 - 2 \times 26 + 26) \\ &= (18)(18) \\ &= 324\end{aligned}$$

Consider a game where each of the the two firms can either price cooperatively, i.e collude or price aggressively, i.e. maximising own profit. Construct the 2x2 game matrix.

<u>Answer:</u>

		Pepsi	
		C	D
Coke	C	324, 324	243, $\frac{729}{2}$
	D	$\frac{729}{2}$, 243	288, 288

legend: C – cooperate (collude and price 26)
D – defect (maximise own profit)

<u>Explanation:</u>
We have already found the payoffs when both players cooperate or defect. To find the payoff of defection when other player cooperate (price 26):

From earlier results, BR: price $\frac{1}{4}(26) + 15 = \frac{43}{2}$

Profit for defection player while other player cooperate:

$$\pi = (\frac{43}{2} - 8)(44 - 2 \times \frac{43}{2} + 26)$$
$$= (\frac{27}{2})(\frac{88 - 86 + 52}{2})$$
$$= (\frac{27}{2})(27)$$
$$= \frac{729}{2}$$

Profit for cooperative player when other player defect:

$$\pi = (26 - 8)(44 - 26 \times 2 + \frac{43}{2})$$
$$= (18)(\frac{88 - 104 + 43}{2})$$
$$= (9)(27)$$
$$= 243$$

Does this game suit the features of a prisoners' dilemma? Explain.

The game is a prisoners' dilemma. The payoff for defector when other player cooperate > payoff for mutual cooperation > payoff for mutual defection > payoff for cooperator when other player defects, i.e. $\frac{729}{2} > 324 > 288 > 243$ satisfies the conditions for a prisoners' dilemma game.

This game is being played repeatedly for 100 games. What is the sub-game perfect equilibrium?

Answer:
The sub-game perfect equilibrium will be (D,D) for every single round.

Explanation:
In the last round, both players will choose D since it is their dominant strategy, i.e. D will always get a higher payoff than C, regardless of the other player's choice ($\frac{729}{2} > 324$ and $288 > 243$). Knowing that defection will surely occur in the last round, both players will choose D in the 2nd last round as well. By such rollback analysis, therefore the only sub-game perfect equilibrium is for them to defect all the way.

This game is being played repeatedly infinitely. Write down the strategy for Grim Trigger for each player.

Answer:
The Grim Trigger strategy for both player is:
Cooperative state: Play C (price 26) and switch to competition state if deviation occurs
Competition state: Play D (price 20) and remain there

What is the condition for Grim Trigger to be a sub-game perfect equilibrium strategy?

Answer:
Grim Trigger is a sub-game perfect equilibrium strategy if and only if the discount factor $\delta \geq \frac{81}{153}$

Explanation:
In the sub-game perfect equilibrium, no player can deviate from their strategy, given the strategy of other players, to get a higher payoff. Therefore, we apply the one shot deviation principle and find the conditions where there can be no profitable deviation from Grim Trigger strategy, as such:

We first calculate $V(C)$ and $V(D)$:
$V(C) = 324 + \delta V(C)$, where δ is the discount factor
$$\Rightarrow V(C) = \frac{324}{1-\delta}$$

$$V(D) = 288 + \delta V(D)$$
$$\Rightarrow V(D) = \frac{288}{1-\delta}$$

We consider two separate mutually exclusive cases:

Case 1: cooperative state:

Hence one shot deviation is unprofitable when
$$V(C) \geq \frac{729}{2} + \delta V(D)$$

i.e $\frac{324}{1-\delta} \geq \frac{729}{2} + \frac{288\delta}{1-\delta}$
$648 \geq 729(1-\delta) + 576\delta$
$(729-576)\delta \geq 729-648$
$153\delta \geq 81$
$\therefore \delta \geq \frac{81}{153}$

Case 2: competition state:
One shot deviation will not be profitable since it will not alter the other player's strategy afterwards, and will result in lower payoff at the moment of deviation since the the best response to D is D.

Combining cases, we can see that Grim Trigger is sub-game perfect equilibrium if and only if $\delta \geq \frac{81}{153}$

Consider a stick-and-carrot strategy as follow:

Cooperative state: Play the collusive price and remain in the cooperative state if no deviation occurs, otherwise switch to punishment state

Punishment state: Price 8 and switch to cooperative state if no deviation occurs, otherwise remain in the punishment state

If both players adopt such strategy, what condition is required for it to be a sub-game perfect equilibrium?

<u>Answer:</u>
Stick-and-carrot is the sub-game perfect equilibrium if and only if $\delta \geq \frac{1}{2}$ where δ is the discount factor

<u>Explanation:</u>
In the sub-game perfect equilibrium, no player can gain their payoff by switching to another strategy, given the strategy of the other players. Therefore, we apply one-shot deviation principle and find the conditions where there cannot be any profitable deviation, as such:

From previous results,
$\Rightarrow V(C) = \frac{324}{1-\delta}$
Single round deviation payoff from cooperation $= \frac{729}{2}$

When both players are in the punishment state, since they are both pricing at 8 which is equal to their marginal cost, they both earn 0 profit.

Now to calculate $V(D)$:
$V(D) = 0 + \delta V(C)$
$\quad\quad = \delta V(C)$

Now we consider two mutual exclusive cases:

Case 1: cooperative state:

Deviation is unprofitable when:

$$V(C) \geq \frac{729}{2} + \delta V(D)$$

$$V(C) \geq \frac{729}{2} + \delta^2 V(C)$$

$$324 + 324\delta + \delta^2 V(C) \geq \frac{729}{2} + \delta^2 V(C)$$

$$324\delta \geq \frac{729}{2} - 324$$

$$324\delta \geq \frac{729 - 648}{2}$$

$$\delta \geq \frac{81}{648}$$

$$\therefore \delta \geq \frac{1}{8}$$

Case 2: punishment state:

Best one shot deviation price when other player price 8:

$$\frac{1}{4}(8) + 15 = 17$$

Profit for such deviation:

$$\begin{aligned}\pi &= (17 - 8)(44 - 2 \times 17 + 8) \\ &= (9)(52 - 34) \\ &= (9)(18) \\ &= 162\end{aligned}$$

Deviation is unprofitable when:

$$V(D) \geq 162 + \delta V(D)$$

$$(1 - \delta)V(D) \geq 162$$

$$(1 - \delta)\delta V(C) \geq 162$$

$$\because V(C) = \frac{324}{1 - \delta} \quad \Rightarrow 324\delta \geq 162$$

$$\therefore \delta \geq \frac{162}{324}$$

i.e $\delta \geq \frac{1}{2}$

Combining cases, therefore stick-and-carrot is the sub-game perfect equilibrium if and only if $\delta \geq \frac{1}{2}$

What if in the competition state, firms price 0? Rewrite the stick-and-carrot strategy, and repeat the calculations to find the condition.

Answer:
The stick-and carrot strategy for both players will be:
Cooperation state: price 26 and remain in cooperation state if no deviation occurs, otherwise switch to punishment state
Punishment state: price 0 and switch to cooperation state if no deviation occurs, otherwise remain in punishment state
Stick-and-carrot is the sub-game perfect equilibrium if and only if $\delta \geq \frac{225}{338}$ where δ is the discount factor

Explanation:
When both players price 0, the associated profits are:

$$\begin{aligned} profit &= (0-8)(44-2\times 0+0) \\ &= (-8)(44) \\ &= -352 \end{aligned}$$

When the other player price 0, the best response is to price:

$$\frac{1}{4}(0)+15=15$$

Profit for such a deviation:

$$\begin{aligned} \pi &= (15-8)(44-2\times 15+0) \\ &= (7)(14) \\ &= 98 \end{aligned}$$

From previous results,

$$\Rightarrow V(C)=\frac{324}{1-\delta}$$

We have as found $V(D)=-352+\delta V(C)$

We reconsider case 1 and case 2.
Case 1: cooperation state:
Deviation is unprofitable when:

$$V(C)\geq \frac{729}{2}+\delta V(D)$$

$$324+324\delta+\delta^2 V(C)\geq \frac{729}{2}+\delta[-352+\delta V(C)]$$

$$324+324\delta \geq \frac{729}{2}-352\delta$$

$$676\delta \geq \frac{729}{2}-324$$

$$\delta \geq (\frac{81}{2})(\frac{1}{676})$$

$$\therefore \delta \geq \frac{81}{1352}$$

Case 2: punishment state:
Deviation is unprofitable when:

$V(D) \geq 98 + \delta V(D)$

$(1-\delta)[-352 + \delta V(C)] \geq 98, \quad \because V(D) = -352 + \delta V(C)$

$-352 + 352\delta + (1-\delta)[\delta V(C)] \geq 98$

$(352 + 324)\delta \geq 98 + 352, \quad \because V(C) = \dfrac{324}{1-\delta}$

$676\delta \geq 450$

$\delta \geq \dfrac{450}{676}$

$\therefore \delta \geq \dfrac{225}{338}$

Combing cases, stick-and-carrot is the sub-game perfect equilibrium if and only if $\delta \geq \dfrac{225}{338}$

Example C

Consider the game below:

		P2		
		A	B	C
	A	2, 2	-2, 5	-4, -4
P1	B	5, -2	0, 0	-4, -4
	C	-4, 4	-4, 4	-5, -5

By best response analysis, find the Nash equilibrium of this game.

<u>Answer:</u>
The Nash equilibrium of this game is (B,B).

<u>Explanation:</u>
Nash equilibrium is a list of strategies, one for each player, where all the strategies are mutual best responses of one another. We first find the best response strategy of a player to each strategy of the other player. The strategy profile with mutual best responses is therefore the Nash equilibrium.

The payoff linked to the best response strategies are being circled. Therefore we see that only (B,B) is a mutual best response strategy profile.

<u>Alternative definition for Nash equilibrium:</u>
Nash equilibrium is a list of strategies, one for each player, such that no player can deviate from their strategy to obtain a higher payoff, given the strategy of the other players.

By iterated elimination of strictly dominated strategies, find the Nash equilibrium of this game.

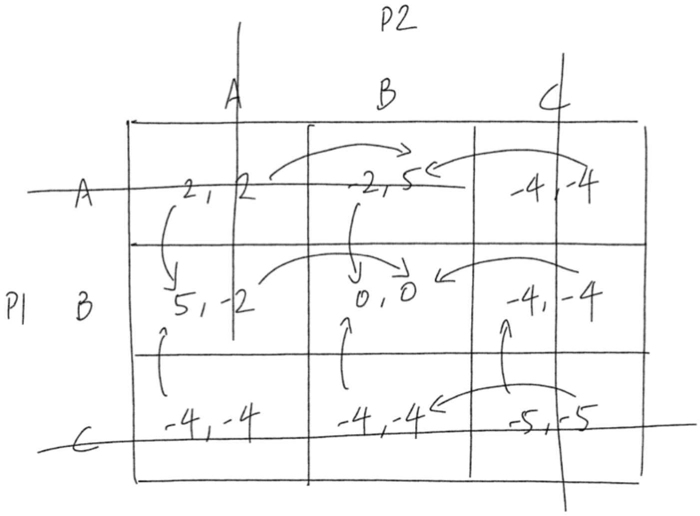
<u>Answer:</u>
The Nash equilibrium of this game is (B,B).

<u>Explanation:</u>
A strategy is strictly dominated if there exists another strategy that always gives a higher payoff than that strategy, regardless of the strategy of the other players. A rational player will not choose a dominated strategy.

In the game matrix, we see that C for both players is strictly dominated by A ($5 > -4$, $0 > -4$, $-4 > -5$). Therefore we can eliminate C. After C is eliminated, we find now that A for both players is strictly dominated by B ($5 > 2$, $0 > -2$). Therefore we can eliminate A.

Now B is the only strategy left for both players, i.e (B,B) is the Nash equilibrium.

Diagrammatic working:



This game is being played repeatedly infinitely. Write a stick-and-carrot strategy where the cooperative strategy leads to a Pareto efficient outcome, the punishment leads to a Minmax payoff.

<u>Answer:</u>
The stick-and-carrot strategy for both players is:
Cooperation state: play A and remain in cooperation state if no deviation occurs, otherwise switch to punishment state
Punishment state: play C and switch to cooperation state if no deviation occurs, otherwise remain in punishment state

<u>Explanation:</u>
In the Pareto efficient outcome, no player can be better off (get higher payoff) without getting the other player worse off (lower payoff). Therefore, the Pareto efficient outcome of this game is (A,A) with corresponding strategy A.

The Minmax payoff is the minimum payoff a player can earn, by simply best responding to the other player's strategies. Note that this game is symmetric. Therefore, the Minmax payoff in this game is -4 (out of -4, 0, 5), with corresponding strategy C.

Assuming the probability that the game will end in a single round is p, what is the condition for such stick-and-carrot strategy to be a sub-game perfect equilibrium?

<u>Answer:</u>

The condition for stick-and-carrot to be a sub-game perfect equilibrium is $p\delta \geq \frac{3}{7}$ where δ is the discount factor

<u>Explanation:</u>

In the sub-game perfect equilibrium, no player can deviate from their strategy to get a higher payoff, given the strategy of the other players. Therefore, we find the conditions where any deviation is unprofitable in both cooperation and punishment state.

First we calculate $V(Coop)$ and $V(pun)$

$V(coop) = 2 + p\delta V(coop)$, where δ is the discount factor

$\Rightarrow V(coop) = \dfrac{2}{1 - p\delta}$

$\Rightarrow V(pun) = -5 + p\delta V(coop)$, $\because$ B is the best response to A

We consider two mutually exclusive cases:

Case 1: cooperative state:

Deviation is unprofitable when:

$V(coop) \geq 5 + p\delta V(pun)$

$V(coop) \geq 5 + p\delta(-5 + p\delta V(coop))$

$2 + 2p\delta + (p\delta)^2 V(coop) \geq 5 - 5p\delta + (p\delta)^2 V(coop)$

$7p\delta \geq 3$

$\therefore p\delta \geq \dfrac{3}{7}$

Case 2: punishment state:

Deviation is unprofitable when:

$V(pun) \geq -4 + p\delta V(pun)$, $\because$ A or B is the best response to C with payoff -4

$(1 - p\delta)V(pun) \geq -4$

$(1 - p\delta)(-5 + p\delta V(coop)) \geq -4$

$-5(1 - p\delta) + (1 - p\delta)(p\delta V(coop)) \geq -4$

$-5 + 5p\delta + 2p\delta \geq -4, \quad \because V(coop) = \dfrac{2}{1 - p\delta}$

$\therefore p\delta \geq \dfrac{1}{7}$

Combining results, therefore stick-and-carrot is the sub-game perfect equilibrium if and only if $p\delta \geq \dfrac{3}{7}$

Example D

Coke and Pepsi try to maximise profits in the soda industry. The marginal cost of Coke can either be 5 or 15, with probability p of a low cost. In the first stage, Coke can choose to price in a way that maximises low cost profit or high cost profit. In the second stage, the Pepsi choose to whether enter the market not. The marginal cost of Pepsi is 10 and Pepsi also need to incur a fixed cost of 40 to enter.

The demand function of the soda industry is given by $P = 25 - Q$. Calculate the profits of Coke with high cost and low cost in a monopoly game without bluffing, and the profits of high cost Coke with bluffing.

<u>Answer:</u>

The profits of Coke is $\begin{cases} high \quad cost : \pi_{coke} = 25 \\ low \quad cost : \pi_{coke} = 100 \\ bluff : \pi_{coke} = 0 \end{cases}$

<u>Explanation:</u>

Profit = Total number of units x profit margin per unit. i.e. profit = quantity x(price-marginal cost)

$$\begin{aligned} \therefore \pi_{coke} &= Q(P - MC) \\ &= (25 - P)(P - MC) \end{aligned}$$

For profit maximisation, we use differentiation:

F.O.C: $\pi'_{coke} = (25 - P) - (P - MC) = 0$

$\therefore P = \dfrac{25 + MC}{2}$ maximises profit

Associated profit:

$$\begin{aligned} \pi_{coke} &= (25 - \frac{25 + MC}{2})(\frac{25 + MC}{2} - MC) \\ &= (\frac{25 - MC}{2})^2 \end{aligned}$$

Now we consider the different cases:

Case 1: cost 5, price 15:

$$\begin{aligned} \pi_{coke} &= (\frac{25 - 5}{2})^2 \\ &= 100 \end{aligned}$$

Case 2: cost 15, price 20:

$$\begin{aligned} \pi_{coke} &= (\frac{25 - 15}{2})^2 \\ &= 25 \end{aligned}$$

Case 3: cost 15, price 15:

Since price = marginal cost, the profit will be 0

Calculate the profits of Coke and Pepsi respectively in the duopoly game, in consideration of both scenarios where Coke is high cost and low cost.

Answer:

For high cost Coke, profits: $\begin{cases} \pi_{coke} = \frac{25}{9} \\ \pi_{pepsi} = \frac{400}{9} \end{cases}$; for low cost coke, profits: $\begin{cases} \pi_{coke} = \frac{625}{9} \\ \pi_{pepsi} = \frac{100}{9} \end{cases}$

Explanation:

Let MC be the marginal cost of Coke; q_1, q_2 be the quantity of Coke, Pepsi respectively.

From earlier,

$$\pi_{coke} = q_1(P - MC)$$
$$= (25 - q_1 - q_2 - MC)(q_1)$$

For $MC = 5$, profit is maximised when:

F.O.C.: $\pi'_{coke} = (20 - q_1 - q_2) - q_1 = 0$

$\Rightarrow q_1 = -\dfrac{1}{2}q_2 + 10$ (1)

For $MC = 15$, profit is maximised when:

F.O.C.: $\pi'_{coke} = (10 - q_1 - q_2) - q_1 = 0$

$\Rightarrow q_1 = -\dfrac{1}{2}q_2 + 5$ (2)

Similarly,

$$\pi_{pepsi} = q_2(P - 10)$$
$$= (15 - q_1 - q_2)(q_2)$$

Profit is maximised when:

F.O.C.: $\pi'_{pepsi} = (15 - q_1 - q_2) - q_2 = 0$

$\Rightarrow q_2 = \dfrac{15 - q_1}{2}$ (3)

Solving for Nash equilibrium profit from best response functions:

Case 1: $MC = 5$

Substitute (1) into (3),

$$q_2 = \frac{15 - (-\frac{1}{2}q_2 + 10)}{2}$$

$$2q_2 = 5 + \frac{1}{2}q_2$$

$$\therefore q_2 = \frac{10}{3}$$

$$\therefore q_1 = -\frac{1}{2}(\frac{10}{3}) + 10$$
$$= \frac{25}{3}$$

$$\therefore \pi_{pepsi} = (15 - \frac{10}{3} - \frac{25}{3})(\frac{10}{3})$$
$$= (\frac{10}{3})(\frac{10}{3})$$
$$= \frac{100}{9} \approx 11$$
$$\therefore \pi_{coke} = (20 - \frac{25}{3} - \frac{10}{3})(\frac{25}{3})$$
$$= (\frac{25}{3})(\frac{25}{3})$$
$$= \frac{625}{9} \approx 69$$

Case 2: $MC = 15$
Substitute (2) into (3):
$$q_2 = \frac{15 - (-\frac{1}{2}q_2 + 5)}{2}$$
$$2q_2 = 10 + \frac{1}{2}q_2$$
$$\therefore q_2 = \frac{20}{3}$$
$$\therefore q_1 = -\frac{1}{2}(\frac{20}{3}) + 5$$
$$= \frac{5}{3}$$

$$\therefore \pi_{pepsi} = (15 - \frac{20}{3} - \frac{5}{3})(\frac{20}{3})$$
$$= (\frac{20}{3})(\frac{20}{3})$$
$$= \frac{400}{9} \approx 45$$
$$\therefore \pi_{coke} = (10 - \frac{20}{3} - \frac{5}{3})(\frac{5}{3})$$
$$= (\frac{5}{3})(\frac{5}{3})$$
$$= \frac{25}{9} \approx 3$$

****Faster way to calculate profit:*

$\pi_{coke} = (25 - q_1 - q_2 - MC)(q_1)$

F.O.C.: $\pi'_{coke} = (25 - q_1 - q_2 - MC) - q_1 = 0$

$\Rightarrow q_1 = 25 - q_1 - q_2 - MC$

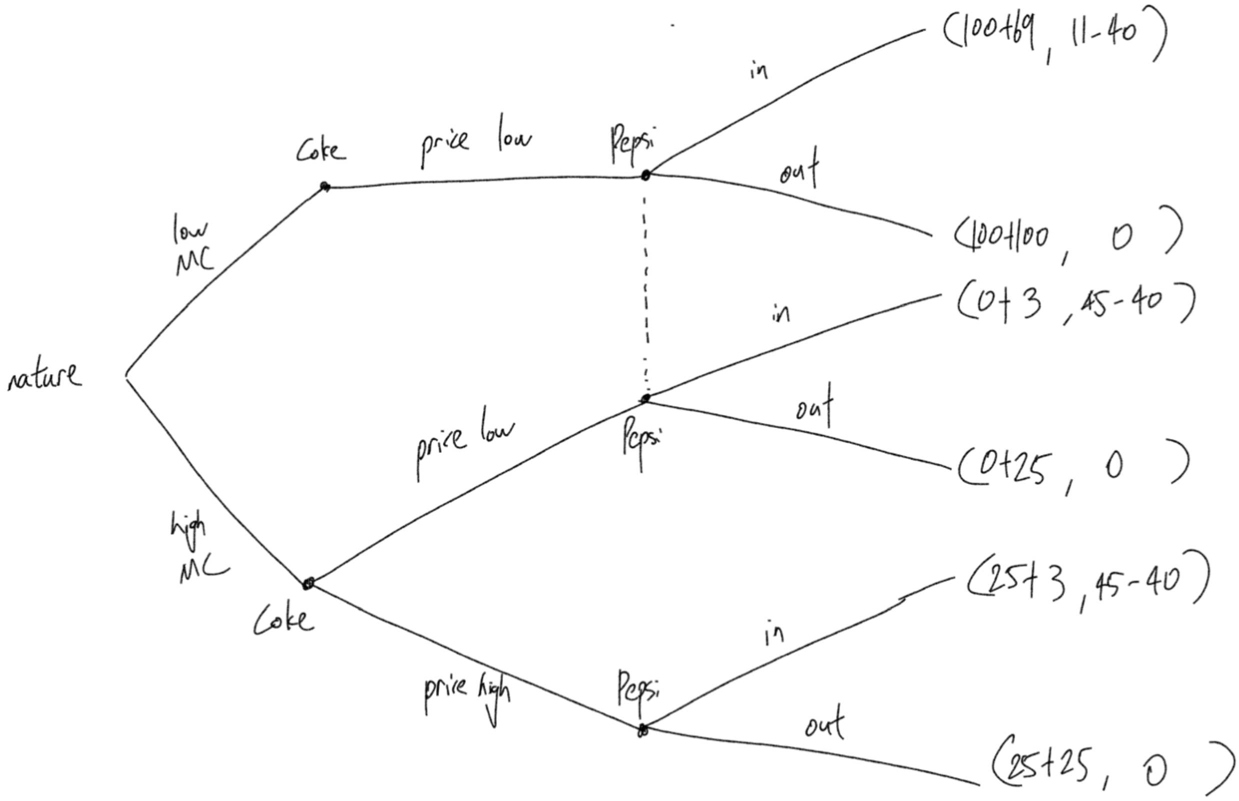
$\therefore \pi_{coke} = q_1^2$

Similarly, $\pi_{pepsi} = q_2^2$

Therefore, draw the game tree of this game and round the payoffs to whole numbers.

We assume rational low MC Coke will not price high (since price low is already profit-maximising and deviation will be unprofitable) and we exclude that strategy in our tree

Game tree:



Is the equilibrium separating, semi-separating or pooling?

<u>Answer:</u>
The equilibrium (price low, price high), (out, in) is separating

<u>Explanation:</u>
In the equilibrium, no player can deviate from their strategy to get a higher payoff, given the strategy of the other player. Hence deviation is unprofitable in any equilibrium.

We consider two mutually exclusive cases:

Case 1: Coke is of low MC:
The only strategy for Coke is to price low as explained earlier.

Case 2: Coke is of high MC:
From the game tree, price high is the dominant strategy for coke since any of possible payoff from pricing high (50 or 28) is greater than any of possible payoff from pricing low (25 or 3), i.e. min(50,28) > max (25,3). Therefore, rational Coke of high MC will price high. Pepsi will hence choose to be in since 5 > 0.

Similarly, Pepsi will choose out upon seeing a low price Coke since 0 > -29.

Since Coke will price low if and only if its low cost; price high if and only if its high cost, the strategy of Coke fully reveals full information of Coke's cost and hence it is separating.

Reconsider the case where Coke with low marginal cost be 10 instead of 5. Keeping everything else the same, calculate the profits of Coke with high cost and low cost in a monopoly game without bluffing, and the profits of high cost Coke with bluffing.

Answer:

The profits of Coke is $\begin{cases} high \quad cost : \pi_{coke} = 25 \\ low \quad cost : \pi_{coke} = \frac{225}{4} \\ bluff : \pi_{coke} = \frac{75}{4} \end{cases}$

Explanation:
From earlier, $\pi_{coke} = (\frac{25 - MC}{2})^2$

Now we reconsider the different cases:

Case 1: cost 10, price 17.5:

$$\pi_{coke} = (\frac{25 - 10}{2})^2$$
$$= \frac{225}{4} \approx 56$$

Case 2: cost 15, price 20:
$\pi_{coke} = 25$ as found earlier

Case 3 (bluff): cost 15, price 17.5:

$$\pi_{coke} = (25 - \frac{35}{2})(\frac{35}{2} - 15)$$
$$= (\frac{15}{2})(\frac{5}{2})$$
$$= \frac{75}{4} \approx 19$$

Recalculate the profits of Coke and Pepsi respectively in the new duopoly game, in consideration of both scenarios where Coke is high cost and low cost.

Answer:

For high cost Coke, profits: $\begin{cases} \pi_{coke} = \frac{25}{9} \\ \pi_{pepsi} = \frac{400}{9} \end{cases}$; for low cost coke, profits: $\begin{cases} \pi_{coke} = 25 \\ \pi_{pepsi} = 25 \end{cases}$

Explanation:

Let MC be the marginal cost of Coke; q_1, q_2 be the quantity of Coke, Pepsi respectively.

From earlier,

$\pi_{coke} = (25 - q_1 - q_2 - MC)(q_1)$

Profit is maximised when:

F.O.C.: $\pi'_{coke} = (25 - q_1 - q_2 - MC) - q_1 = 0$

$\Rightarrow q_1 = \dfrac{25 - MC - q_2}{2}$

For $MC = 10$, profit is maximised when:

$q_1 = \dfrac{15 - q_2}{2}$ (1)

For $MC = 15$, profit is maximised when:

$q_1 = \dfrac{10 - q_2}{2}$ (2)

As found earlier, Pepsi's profits is maximised when:

$q_2 = \dfrac{15 - q_1}{2}$ (3)

Solving for Nash equilibrium profit from best response functions:

Case 1: $MC = 10$

Substitute (1) into (3),

$q_2 = \dfrac{15 - (\frac{15 - q_2}{2})}{2}$

$2q_2 = \dfrac{15 + q_2}{2}$

$\therefore q_2 = 5$

$\therefore q_1 = \dfrac{15 - 5}{2}$

$= 5$

$\therefore \pi_{coke} = (15 - 5 - 5)(5)$

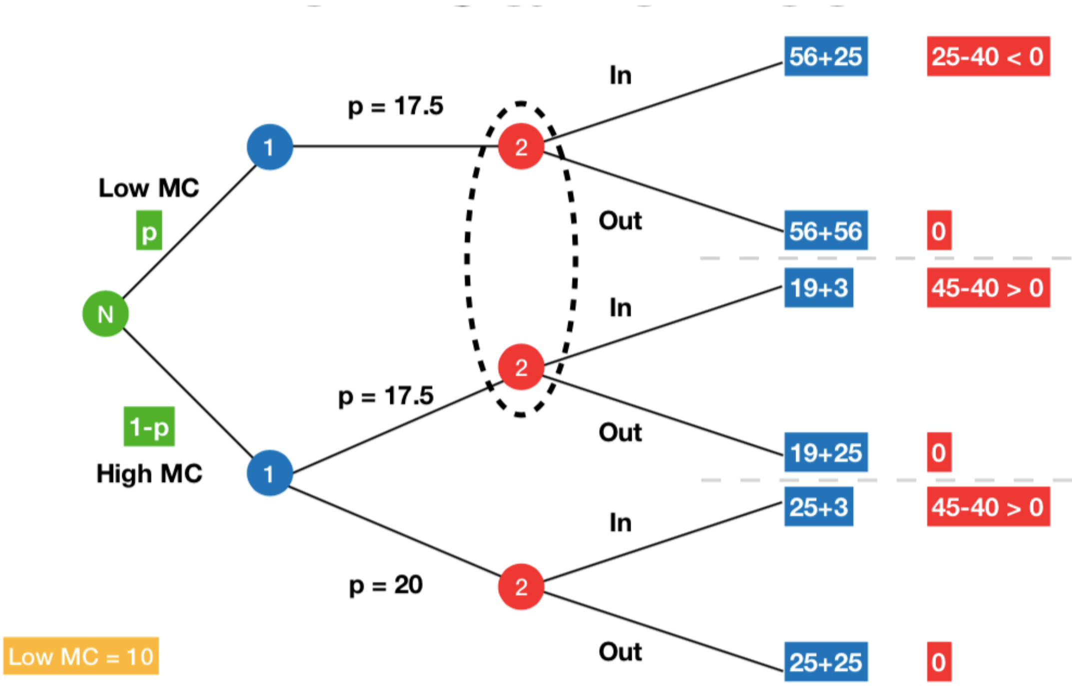
$= 25$

$\therefore \pi_{pepsi} = 25$

Case 2: $MC = 15$
The associated profits will be the same as calculated previously.

Therefore, draw the new game tree of this game and round the payoffs to whole numbers.

We assume rational low MC Coke will not price high (since price low is already profit-maximising).
Game tree:



What are the possible equilibriums and are there any conditions, if any?

Answer:
There is no separating equilibrium

There exists a pooling equilibrium: {always low price, (out, in)} if and only if $p \geq \frac{1}{4}$

There exists a semi-separating equilibrium: {(low price, $\frac{3p}{1-p}$low price$\oplus\frac{1-4p}{1-p}$high price), ($\frac{8}{11}$in$\oplus\frac{3}{11}$out, in)} if and only if $0 < p < \frac{1}{4}$

Explanation:
In an equilibrium, no player can deviate from their strategy to get a higher payoff, given the strategy of the other player. Hence deviation is unprofitable in any equilibrium.

We consider the three equilibriums separately and verify if they exist:

Hypothesis 1: Separating equilibrium
In the separating equilibrium (where Coke's strategy reveals full information of its cost), high MC Coke will always price high, where Pepsi will therefore always choose in since 5 > 0. Similarly, low MC Coke will always price low, where Pepsi will therefore choose out since 0 > -15.

High MC coke can make a profitable deviation by pricing low and get a payoff of 34 (since Pepsi will be out), which is greater than the payoff 28 of pricing high. Hence equilibrium is not possible.

Hypothesis 2: Pooling equilibrium
In the pooling equilibrium, Coke will always price low regardless of its cost (Coke's strategy reveals no information of its cost).

Check if pooling equilibrium exists when Pepsi chooses (in):
Such equilibrium cannot exist since a high MC Coke can deviate profitably by pricing high and gain payoff 28 > 21.

Check if pooling equilibrium exists when Pepsi chooses (out):
High MC Coke will not have incentive to deviate since 44 > 28 (high price can only come from high MC Coke, so Pepsi will choose in since 5 > 0)

Now from Pepsi's perspective:
Pepsi's expected payoff for (in):

$$E(in) = p(-15) + (1-p)(5)$$
$$= -20p + 5$$

Pepsi will not have incentive to deviate if and only if $E(out) \geq E(in)$

$$\therefore 0 \geq -20p + 5$$

i.e. the condition $p \geq \frac{1}{4}$ must be met for pooling equilibrium to exist

Hypothesis 3: Semi-separating equilibrium
In the semi-separating equilibrium, high MC Coke will mix between pricing low and high, and Pepsi will mix between in and out (Coke's strategy partially reveals information of its cost) . We use the opponent's indifference property in mixed strategy equilibrium:

Suppose high MC Coke choose price low q of the time and Pepsi choose in x of the time:

From Pepsi's perspective:

$$P(high \quad MC | low \quad price) = \frac{(1-p)q}{(1-p)q + p}$$

$$P(low \quad MC | low \quad price) = \frac{q}{(1-p)q + p}$$

Since Pepsi is indifferent between in and out,

$$E(in) = E(out)$$

$$\frac{(1-p)q}{(1-p)q + p}(5) + \frac{p}{(1-p)q + p}(-15) = 0$$

$$5q - 5pq - 15p = 0$$

$$q(1-p) = 3p$$

$$\therefore q = \frac{3p}{1-p} \quad \Rightarrow 1 - q = \frac{1-4p}{1-p}$$

$\because q \in (0,1) \quad \Rightarrow 0 < p < \frac{1}{4}$ must be satisfied

Since high MC Coke is indifferent between high price and low price,

$$E(price \quad low) = E(price \quad high)$$

$$22x + 44(1 - x) = 28$$

$$44 - 22x = 28$$

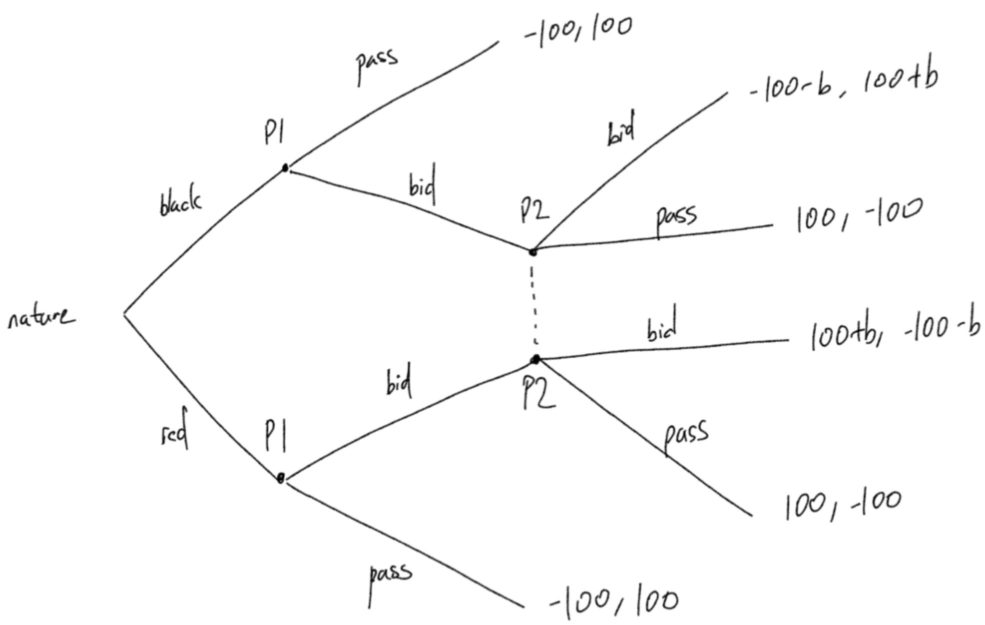
$$\therefore x = \frac{16}{22}$$

$$= \frac{8}{11} \quad \Rightarrow 1 - x = \frac{3}{11}$$

Therefore semi-separating equilibrium can exist if the above conditions are met.

Example D1

Consider the simplified poker game below:



What are the possible equilibriums and are there any conditions, if any?

<u>Answer:</u>
There is no separating equilibrium
There is no pooling equilibrium

There exists a separating equilibrium: { ($\frac{b}{b+200}$bid⊕$\frac{200}{b+200}$pass, bid), $\frac{200}{b+200}$ bid⊕ $\frac{b}{b+200}$pass}

<u>Explanation:</u>
In the equilibrium, no player can deviate from their strategy to gain a higher payoff, given the strategy of the other player. Hence deviation is unprofitable in any equilibrium.

Note that bid strictly dominates pass for P1 with red card, i.e. min(100, 100+b) > -100. Therefore, we can eliminate pass as a strategy for P1 with red card.

We consider the different equilibriums:

Hypothesis 1: Separating equilibrium
In the separating equilibrium, P1 with black card always pass; P1 with red card always bid (P1's strategy reveals full information of its card). This means that P2 will always pass upon seeing a bid since -100 > -100-b

Therefore, P1 with black card can make a profitable deviation by bidding (given P2 will pass anyways) as it can yield a payoff 100 > -100. Hence there is no separating equilibrium.

Hypothesis 2: Pooling equilibrium

In the pooling equilibrium, P1 will bid regardless of the colour of the card (P1's strategy reveals no information of its card)

We calculate the expected payoff for P2 for bid and pass:

$$\begin{aligned} E(bid) &= \frac{1}{2}(-100-b) + \frac{1}{2}(100+b) \\ &= 0 \end{aligned}$$

$$E(pass) = -100$$

Since $E(bid) > E(pass)$, P2 will choose to always bid.

However, P1 with black card can make a profitable deviation by passing (knowing that P2 will surely bid) as it can yield a payoff -100 > -100-b. Hence there is no pooling equilibrium.

Hypothesis 3: Semi-separating equilibrium

In the semi-separating equilibrium, black card P1 will mix between bid and pass; P2 will mix between bid and pass (P1's strategy partially reveals information of its card). Using the opponent's indifference property in mixed strategy equilibrium:

Suppose P1 with black card bid q of the time and P2 bid x of the time:

From P2's perspective:

$$\begin{aligned} P(red\,|\,bid) &= \frac{\frac{1}{2}}{\frac{1}{2}q + \frac{1}{2}} \\ &= \frac{1}{q+1} \end{aligned}$$

$$P(black\,|\,bid) = \frac{q}{q+1}$$

P2 is indifferent between bid and pass:

$$\frac{1}{q+1}(-100-b) + \frac{q}{q+1}(100+b) = -100$$

$$-100 - b + 100q + bq = -100q - 100$$

$$200q + bq = b$$

$$\therefore q = \frac{b}{b+200} \quad \Rightarrow \quad 1-q = \frac{200}{b+200}$$

P1 with black card is indifferent between bid and pass:

$$(-100-b)x + 100(1-x) = -100$$

$$-100x - bx + 100 - 100x = -100$$

$$bx + 200x = 200$$

$$\therefore x = \frac{200}{b+200} \quad \Rightarrow 1-x = \frac{b}{b+200}$$

Therefore if these conditions are met, there exists a semi-separating equilibrium.

Express this game in normal form and calculate the payoffs

		P2	
		bid	pass
P1	bid, bid	$\frac{1}{2}(-100-b)+\frac{1}{2}(100+b)=0$, $\frac{1}{2}(100+b)+\frac{1}{2}(-100-b)=0$	100, -100
	pass, bid	$\frac{1}{2}(-100)+\frac{1}{2}(100+b)=\frac{1}{2}b$, $\frac{1}{2}(100)+\frac{1}{2}(-100-b)=-\frac{1}{2}b$	$\frac{1}{2}(-100)+\frac{1}{2}(100)=0$, 0

Example E

A company wish to distinguish between type A and type C students from a pool of students. It is willing to pay a salary of \$160,000 and \$60,000 to type A and type C students respectively. Type A and type C students can find an alternative job with salary \$125,000 and \$30,000 respectively. Suppose the cost of taking a tough course is \$3,000 and \$14,000 to type A and type C students respectively, what is the minimum number of tough courses the company should set as a benchmark to separate type A and type C students?

Answer:
The minimum number of tough courses is 8

Explanation:
To fully separate type A and type C students, we need to ensure:

1. Participation constraint: both types are willing to participate in this screening process and not find a job elsewhere
2. Incentive compatibility: types will not have the incentive to fake their identity

In order to achieve separating equilibrium by self selection

Let n represent the number of tough courses:

Participation constraint to be met:

Type C student will prefer screening by taking 0 tough courses since $60000 \geq 30000$.

Type A student will prefer screening by taking n tough courses when:

$$160000 - 3000n \geq 125000$$

$$3000n \leq 35000$$

$$n \leq \frac{35}{3}$$

$\therefore n \leq 11$ rounded off to whole numbers

Incentive compatibility to be met:
Type C students will not have to incentive to pretend to be type A students:

$$60000 \geq 160000 - 14000n$$

$$14000n \geq 100000$$

$$n \geq \frac{50}{7}$$

$\therefore n \geq 8$ rounded off to whole numbers
Type A students will not have to incentive to pretend to be type C students:

$$160000 - 3000n \geq 60000$$

$$3000n \leq 100000$$

$$n \leq \frac{100}{3}$$

$\therefore n \leq 33$ rounded off to whole numbers

Combining inequalities, therefore the condition for all these constraints to be met is $8 \leq n \leq 11$

Therefore the minimum number of tough courses is 8

Suppose the proportion of type A students is p. What will be the pooling equilibrium and what are the conditions for both types to be better off than separating equilibrium?

Answer:

The pooling equilibrium is a common salary $100000p + 60000$ to be paid to both types of students.

Both types will be better of if and only if $p > \dfrac{100 - 3n}{100}$

Explanation:

In the pooling equilibrium, type A and type C students are paid the same amount of salary regardless of their identity. Since the proportion of type A students is p, the common salary will be the expected value for hiring a randomly selected student:

$$160000p + 60000(1 - p) = 100000p + 60000$$

Type C students are better off in the pooling equilibrium since as long as $p > 0$ (which must be true), as they can earn a higher salary than before

Type A students are better off if they see an increase in salary:

$$100000p + 60000 > 160000 - 3000n$$

$$p > \frac{100000 - 3000n}{100000}$$

$\therefore p > \dfrac{100 - 3n}{100}$ is the condition to be met

Is this equilibrium stable?

Answer:

The equilibrium is unstable

Explanation:

Type A students will have incentive to deviate from this equilibrium to obtain a higher salary. Suppose a type A student takes just one tough course and asks for an additional $3001 of salary, the signal will be credible since no type C student is willing to gain $3001 of salary and lose $14000 in terms of cost. With the value of type A student at $160000, the company will be willing to accept this offer, and type A student will be even better off with additional $1 in surplus

Since there is credible incentive for type A students to deviate, the equilibrium is unstable

Further explanation:

When sufficient type A students do the same thing, type C students will also have the incentive to take just one tough course and pretend to be type A, incentivising type A student to take even more tough courses. Therefore, this process will continue until it reaches the separating equilibrium.

Example F

There are a total of 2 players in a first-priced, sealed bid auction. Each player has their own private value for the object. The object is of value v to player 1. The unknown private value of a player is uniformly distributed between 0 and 1.

Assuming player 2 adopts the bidding function $b(x) = \frac{x}{2}$ where x is the unknown private value of player 2, what is the best response for player 1?

<u>Answer:</u>

The best response for player 1 is to bid $\frac{v}{2}$

<u>Explanation:</u>

Since the unknown private value of player 2 (X) is uniformly distributed, i.e. $X \sim U(0,1)$

$\therefore P(X \leq x) = x$

⇒c.d.f of x: $F(x) = x$; p.d.f of x: $f(x) = 1$

Assuming player 1 bid b for the object, player 1 will win the object when:

$b > \frac{x}{2}$, i.e. $x < 2b$

Therefore, the probability of player 1 winning the object will be:

$F(2b) = 2b$, where $b \leq \frac{1}{2}$

We calculate the expected consumer surplus (CS) for player 1:

$E(CS_1) = 2b(v - b)$

To maximise the expected consumer surplus i.e. the best response, we use calculus:

F.O.C.: $E(CS_1)' = -2b + 2v - 2b = 0$

$\therefore b = \frac{v}{2}$

$E(CS_1)'' = -4 < 0$ shows that $b = \frac{v}{2}$ is indeed the maxima

Therefore, player 1 should bid $\frac{v}{2}$

What is the Nash equilibrium of this game?

<u>Answer:</u>

The Nash equilibrium of this game will be $(\frac{v}{2}, \frac{x}{2})$ where both players bid half of their known private value.

<u>Explanation:</u>

Nash equilibrium is a list of strategies, one for each player, such that all the strategies are mutual best responses to one another. In this case, we found that the best response for player 2's $\frac{x}{2}$ is $\frac{v}{2}$.

Therefore, by the same calculation, the best response for player 1's $\frac{v}{2}$ is $\frac{x}{2}$. Since the pair of strategies are mutual best responses, they form the Nash equilibrium.

What is the expected payment for each player in terms of their private value? What will be the expected revenue for the seller?

Answer:

The expected payment for a player is $\frac{v^2}{2}$ where v is the known private value for that player

The expected revenue for the seller is $\frac{1}{3}$

Explanation:

A player who bids $\frac{v}{2}$ wins the auction from another player who bids $\frac{x}{2}$ when $\frac{v}{2} > \frac{x}{2}$, i.e. $v > x$. Associated probability is $F(v)$

The payment from the ith bidder (let it be p_i) is:

$$
\begin{aligned}
pay_i &= F(v)\frac{v}{2} \\
&= v(\frac{v}{2}) \\
&= \frac{v^2}{2}
\end{aligned}
$$

$\therefore$ The expected payment from the ith bidder is:

$$
\begin{aligned}
E(pay_i) &= \int_0^1 \frac{v^2}{2} f(v) \quad dv \\
&= [\frac{v^3}{6}]_0^1 \\
&= \frac{1}{6}
\end{aligned}
$$

Since the total expected revenue is the sum of the expected payment of all the individual 2 bidders,

$$
\begin{aligned}
\therefore E(rev) &= 2(\frac{1}{6}) \\
&= \frac{1}{3}
\end{aligned}
$$

Example G

In an all-pay auction, the common value of object to bidders is 5. Can there by any pure strategy Nash equilibrium? Why or why not?

Answer:
There is no pure strategy Nash equilibrium

Explanation:
Nash equilibrium is a list of strategies, one for each player, such that no player can get a higher payoff by deviating from their strategy, given the strategy of the other players, i.e. there cannot be any profitable deviation.

We consider two mutually exclusive cases which covers all possibilities:

Case 1: there exist at least two bidders who bid different values:
This means that among the two, there will be at least one loser who will pay the amount that is equal to their bid. This means that their payoff is negative, of which they can make a profitable deviation by bidding 0.

Case 2: all bidders bid the same value:
Assuming each bidder bids b for $0 \leq b \leq 5$ and the total number of bidders is n, the expected consumer surplus i.e. payoff for each player will be:

$$E(CS) = \frac{5}{n} - b$$

If a single bidder deviates to just bid an increment δ to win the price, the expected consumer surplus for that player will be:

$$E(CS) = 5 - b - \delta$$

Therefore, deviation is possible if:

$$5 - b - \delta > \frac{5}{n} - b$$

$$5 - \delta > \frac{5}{n}$$

i.e. $\delta < 5 - \dfrac{5}{n}$ which there surely exist a small increment value of δ where the inequality satisfies.

Combining the possible cases, since profitable deviation is always possible, there cannot exist a Nash equilibrium.

Suppose there are 3 bidders, what will be the Nash equilibrium for mixed strategy?

Answer:

The Nash equilibrium is when each bidder adopts the same bidding c.d.f: $F(x) = \sqrt{\frac{x}{5}}$ for $x \in (0{,}5)$, where $F(x)$ is the probability that a bid less than or equal to x is chosen

Explanation:
We apply the opponent's indifference property in the mixed strategy equilibrium, where any deviation by a single player alone will yield the same payoff and cannot be profitable

We look for a symmetric mixed strategy equilibrium. Let $F(x)$ where $x \in (a, b)$ be the bidding c.d.f for each player. $\therefore F(a) = 0$ and $F(b) = 1$

The probability for a player to win the auction is for its bid to be greater than all the other 2 players, $\therefore$ such probability is $[F(x)]^2$

Using the opponent's indifference property,
$E(CS) = 5[F(x)]^2 - x = c$ for some constant c since the expected payoff for any bid is the same for $x \in (a, b)$
$\Rightarrow -a = c$ and $5 - b = c$

Since $c \geq 0$ (otherwise a player can make a profitable deviation by bidding 0 which means it cannot be an equilibrium), the only condition for a is that $a = 0$
$\Rightarrow b = 5$ and $c = 0$

$\because 5[F(x)]^2 - x = 0$
$\therefore F(x) = \sqrt{\frac{x}{5}}$ is the bidding c.d.f for all players in the Nash equilibrium

Repeat the calculation for mixed strategy equilibrium for n bidders. What will be the mixed strategy equilibrium? What will be the expected revenue for the seller?

Answer:
The mixed strategy equilibrium is where each bidder adopts the same bidding c.d.f: $F(x) = (\frac{x}{5})^{\frac{1}{n-1}}$ for $x \in (0{,}5)$, where $F(x)$ is the probability that a bid less than or equal to x is chosen

The expected revenue for the seller will be 5

Explanation:
We apply the opponent's indifference property in the mixed strategy equilibrium, where any deviation by a single player will yield the same payoff and cannot be profitable

We look for a symmetric mixed strategy equilibrium. Let the bidding c.d.f. be $F(x)$ for $x \in (0{,}5)$.
$\therefore F(a) = 0$ and $F(b) = 1$

To win the auction, the bid needs to be greater than all the other $n-1$ bidders, therefore such probability will be $[F(x)]^{n-1}$

Using opponent's indifference property,
$E(CS) = 5[F(x)]^{n-1} - x = c$ for some constant c since any payoff for $x \in [a, b]$ is the same
$\Rightarrow -a = c$ and $5 - b = c$
Since $c \geq 0$ (otherwise a player can make a profitable deviation by bidding 0 and it will not be an equilibrium), the only condition for a is $a = 0$
$\Rightarrow b = 5$ and $c = 0$

$\because 5[F(x)]^{n-1} - x = 0$
$\therefore F(x) = (\frac{x}{5})^{\frac{1}{n-1}}$ is the bidding c.d.f for all players in the Nash equilibrium

The expected payment from a single bidder with bid X:

$$
\begin{aligned}
E(X) &= \int_0^5 xf(x) \quad dx \\
&= [xF(x)]_0^5 - \int_0^5 F(x) \quad dx \\
&= 5 - \int_0^5 (\frac{x}{5})^{\frac{1}{n-1}} \quad dx \\
&= 5 - \frac{n-1}{n}(\frac{1}{5})^{\frac{1}{n-1}}[x^{\frac{n}{n-1}}]_0^5 \\
&= 5 - \frac{n-1}{n}(\frac{1}{5})^{\frac{1}{n-1}}(5)^{\frac{n}{n-1}} \\
&= 5 - \frac{5(n-1)}{n} \\
&= \frac{5}{n}
\end{aligned}
$$

Since the total expected revenue is the sum of the expected bidding value of all the individual n bidders,

$$
\begin{aligned}
\therefore E(X_1 + X_2 + \ldots + X_n) &= n(\frac{5}{n}) \\
&= 5
\end{aligned}
$$