MOE Advanced Level Higher 3 (H3) Math

Mathematical Proofs and Reasoning

Lecture 4

Telegram Chat

Content

Proving Existential Statements

Proving $\exists x \in D,\ P(x)$ is true. There are two approaches.

1. Constructive proof (Direct proof)
   - Find or construct a specific example for x in the set D that satisfies the statement P.
2. Non-constructive proof (Indirect proof)
   - When specific examples are not easy or not possible.
   - When the specific x satisfying P depends on more (unknown) conditions on P.

Example: Constructive

There exists a quadratic polynomial that passes through the points $(-3,2)$ and $(1,2)$.

A quadratic polynomial has the expression ax^2+bx+c for some real numbers a,b,c. Since it passes through the points $(-3,2)$ and $(1,2)$, have

$$\left\{\begin{array}{ccccccc} a(-3)^2 & + & b(-3) & + & c & = & 2 \\ a(1)^2 & + & b(1) & + & c & = & 2 \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{ccccccc} 9a & - & 3b & + & c & = & 2 \\ a & + & b & + & c & = & 2 \end{array}\right.$$

This is a simultaneous equations. Solving the system (exercise), we have $a=\frac{2}{3}-\frac{1}{3}c$, $b=\frac{4}{3}-\frac{2}{3}c$, for any choice of real number $c\neq 2$ (why?). Choosing $c=-1$, the quadratic polynomial is

$$x^2+2x-1.$$

Remark

- There are infinitely many examples, pick one to answer the existential question.

- Make sure that the chosen example satisfies the condition. Indeed,

$$\begin{cases} (-3)^2 + 2(-3) - 1 = 9 - 6 - 1 = 2 \\ (1)^2 + 2(1) - 1 = 1 + 2 - 1 = 2 \end{cases}$$

- Why $c \neq 2$?

Discussion

Generalize the previous problem:

(i) Given two points (x_1, y_1) and (x_2, y_2), is it possible to find a quadratic polynomial that passes through the points (x_1, y_1) and (x_2, y_2)?

(ii) What are the necessary and sufficient conditions on the points (x_1, y_1) and (x_2, y_2) for the existence of such a quadratic polynomial?

(iii) If a quadratic polynomial exists, is it unique? If not, explain why.

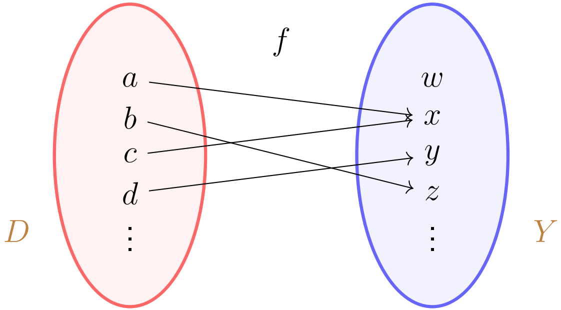
Discussion

Prove that for any 2 rational numbers p and q such that $p < q$, there is a rational number r such that $p < r < q$.

Definition

Definition

A function f from a set D to a set Y is a rule that assigns a unique value $f(x)$ in Y to each $x \in D$.



Nomenclature

Definition

A function f from a set D to a set Y is a rule that assigns a unique value $f(x)$ in Y to each $x \in D$.

(i) D is called the domain, Y is called the codomain of the function.

(ii) For every $a \in D$, there must be a unique $y \in Y$ such that $f(a) = y$. $f(a) = y$ is called the image of a.

(iii) The range $R = \{f(x) \mid x \in D\}$ of f is a subset of the codomain $R \subseteq Y$ that contains all the images of f.

(iv) If the domain of f is not explicitly stated or restricted by context, the domain is assumed to be the largest set of real x values for which the formula gives real y values. This is called the natural domain of f.

Example

1. $f(x) = x^2$, $-2 \leq x \leq 2$.
   - The domain is $[-2, 2]$.
   - The codomain is $\mathbb{R}$.
   - The range is $[0, 4]$.
2. $f(x) = \frac{1}{x}$.
   - The natural domain is $\mathbb{R} \setminus \{0\}$.
   - The codomain is $\mathbb{R}$.
   - The range is $\mathbb{R} \setminus \{0\}$.

Example

3. $f(x) = \sqrt{x}$.
    - The natural domain is $[0, \infty)$.
    - The codomain is $\mathbb{R}$.
    - The range is $[0, \infty)$.
4. $f(x) = \sqrt{1 - x^2}$.
    - The natural domain is $[-1, 1]$.
    - The codomain is $\mathbb{R}$.
    - The range is $[0, 1]$.

Algebraic Operations on Functions

Let f be a function with domain D_f, and g a function with domain D_g. Define new functions as follows.

Addition: $(f+g)(x) = f(x) + g(x)$. Domain of $f+g$ is $D_f \cap D_g$.

Subtraction: $(f-g)(x) = f(x) - g(x)$. Domain of $f-g$ is $D_f \cap D_g$.

Multiplication: $(fg)(x) = f(x)g(x)$. Domain of fg is $D_f \cap D_g$.

Division: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$. Domain of $\frac{f}{g}$ is $D_f \cap \{x \in D_g \mid g(x) \neq 0\}$.

Composition

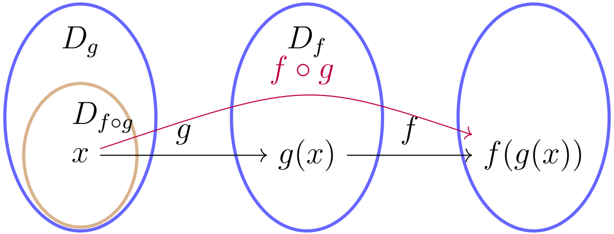
Definition

Suppose f and g are functions with domains D_f and D_g, respectively. The composite function $f \circ g$ ("f composed with g") is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is the set of x in D_g for which $g(x)$ lies in D_f,

$$D_{f \circ g} = \{x \in D_g \mid g(x) \in D_f\}.$$



Shifting

https://www.geogebra.org/m/jqzvk84g

1. Shifting: Let $g(x) = x + k$ for some constant $k \in \mathbb{R}$. Then for any function $f(x)$,
   - Vertical shift: $(g \circ f)(x) = f(x) + k$.

     Shifts the graph of f up by k units if $k > 0$.

     Shifts the graph of f down by $|k|$ units if $k < 0$.
   - Horizontal shift: $(f \circ g)(x) = f(x + k)$.

     Shifts the graph of f left by k units if $k > 0$.

     Shifts the graph of f right by $|k|$ units if $k < 0$.

Scaling

https://www.geogebra.org/m/jqzvk84g

2. Scaling: Let $g(x) = cx$ for some positive real number $c > 0$. Then for any function $f(x)$
   - Vertical scaling: $(g \circ f)(x) = cf(x)$.

     Stretches the graph of f vertically by a factor of c if $c > 1$.

     Compresses the graph of f vertically by a factor of $\frac{1}{c}$ units if $0 < c < 1$.
   - Horizontal scaling: $(f \circ g)(x) = f(cx)$.

     Compresses the graph of f horizontally by a factor of c if $c > 1$.

     Stretches the graph of f horizontally by a factor of $\frac{1}{c}$ units if $0 < c < 1$.

Reflection

https://www.geogebra.org/m/jqzvk84g

3. Reflection: Let $g(x) = -x$. Then for any function $f(x)$,

   Reflection along the x-axis: $(g \circ f)(x) = -f(x)$.

   Reflection along the y-axis: $(f \circ g)(x) = f(-x)$.

Linear function

Definition

A function of the form $f(x) = mx + c$, for some $m, c \in \mathbb{R}$, is called a linear function.

(i) m is called the gradient and c is called the y-intercept.

(ii) If $m = 0$, $f(x) = c$ is called a constant function.

(iii) if $m = 1$ and $c = 0$, $f(x) = x$ is called the identity function.

Polynomial

Definition

A function p is a polynomial if it has the expression

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a nonnegative integer and $a_n, a_{n-1}, ..., a_1, a_0 \in \mathbb{R}$ are real constants.

(i) All polynomials have natural domain $\mathbb{R} = (-\infty, \infty)$.

(ii) The integer n is called the degree of p, denoted as deg(p)=n.

(iii) The constant $a_n, a_{n-1}, ..., a_1, a_0 \in \mathbb{R}$ are called coefficients of the polynomial.

(iv) Linear functions are degree 1 polynomial, with $a_1 = m$ and $a_0 = c$. Degree 2 polynomials are called quadratic functions, degree 3 polynomials are called cubic functions, degree 4 polynomials are called quartic functions,... .

Rational Functions

Definition

A rational function is a quotient of polynomials,

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials.

- (i) The natural domain of a rational function is the set of all reals x for which $q(x) \neq 0$,

$$D_{\frac{p}{q}} = \{x \mid q(x) \neq 0\}.$$

- (ii) Polynomials are rational function with $q(x) = 1$.

Algebraic Functions

Definition

Any function constructed from polynomials using the following algebraic operations

1. addition,
2. subtraction,
3. multiplication,
4. division,
5. taking roots,

lies within the class of algebraic functions.

Rational functions are algebraic functions.

Example

$$\sqrt[3]{1-x^2}, \qquad \frac{1+\sqrt{x}}{1-\sqrt{x}}, \qquad \ldots$$

Transcendental Functions

Functions that are not algebraic are known as transcendental functions.

1. Trigonometric functions
$$\sin(x), \quad \cos(x), \quad \tan(x), \quad \ldots$$
2. Inverse trigonometric functions
$$\sin^{-1}(x), \quad \cos^{-1}(x), \quad \tan^{-1}(x), \quad \ldots$$
3. Exponential functions
$$a^x, \quad e^{2x}, \quad 10^{-x}, \quad \ldots$$
4. Logarithmic functions
$$\log_a(x), \quad \ln(x), \quad \lg(x), \quad \ldots$$
5. Hyperbolic functions
$$\sinh(x), \quad \cosh(x), \quad \tanh(x), \quad \ldots$$

Continuity

Definition

A function $f : D \to Y$ is said to be continuous at a point c in its domain D if

$$\lim_{x \to c} f(x) = f(c).$$

(i) Heuristically, it means as x gets nearer to c, $f(x)$ gets nearer to $f(c)$.

(ii) Or, the graph of $y = f(x)$ near c is continuous, having no breaks.

Example

https://www.geogebra.org/m/zyek2psg

Which of the functions above are continuous at $x = 0$?

1. $f(x)$ and $g(x)$ are continuous at $x = 0$.
2. $h(x)$ is not continuous at $x = 0$.

Properties of Continuity

Theorem

Suppose $f(x)$ and $g(x)$ are continuous at the point c. Then the following functions are continuous at c.

1. $(f+g)(x)$;
2. $(f-g)(x)$;
3. $(af)(x)$, *for any real number* a;
4. $(fg)(x)$;
5. $\left(\dfrac{f}{g}\right)(x)$, *provided* $g(c) \neq 0$;
6. $\left(\sqrt[n]{f}\right)(x)$ *for any positive integer* $n > 0$, *provided* $f(c) \geq 0$.

Proof omitted.

Continuous Functions

Definition

A function $f : D \to Y$ is said to be continuous if it is continuous at every point c in its domain;

$$\lim_{x \to c} f(x) = f(c) \qquad \forall c \in D.$$

Example

The function $f(x) = x$ is continuous on $\mathbb{R}$.

Corollary

All algebraic functions are continuous.

Proof.

This follows from the fact that $f(x) = x$ is continuous and that all algebraic functions are constructed from $f(x) = x$ by addition, subtraction, scaling, multiplication, division, and taking roots. $\square$

Nonconstructive Proof

A monk ascends a mountain along a path on the first day and descends the same path the next day. The monk starts at sunrise each day and reaches his destination (either the top or the bottom) by sunset. He may walk at varying speeds (or even take rest) throughout the day.

There exists at least one point on the path where the monk is at the same location at the same time of day on both the ascent and the descent.

- Let $A(t)$ denote the monk's position during the ascent, starting at the bottom at $t = 0$ and reaching the top at $t = T$.
- Let $D(t)$ denote the monk's position during the descent, starting at the top at $t = 0$ and reaching the bottom at $t = T$.
- Let $f(t) = A(t) - D(t)$. Then $f(0) = A(0) - D(0) < 0$ and $f(T) = A(T) - D(T) > 0$.
- Since f is continuous, we can conclude (using IVT, see later) that there is a $t_0 \in [0, T]$ such that $f(t_0) = 0$, that is, $A(t_0) = D(t_0)$.

Intermediate Value Theorem I

Definition

Let $a < b$ be real numbers. The closed interval $[a, b]$ is the set containing all real numbers x such that $a \leq x \leq b$,

$$[a, b] = \{x \mid a \leq x \leq b\}.$$

Theorem (Intermediate Value Theorem)

Let f be continuous on $[a, b]$, and assume that $f(a) < 0 < f(b)$. Then there is a $c \in [a, b]$ such that $f(c) = 0$.

Proof omitted.

Intermediate Value Theorem II

The Intermediate Value Theorem exemplifies a nonconstructive proof of an existential statement. We will use it to establish a variation of the theorem, which is another example of nonconstructive proof of an existential statement.

Corollary

Let f be continuous on $[a, b]$, and assume that $f(a) < f(b)$. Then for any k such that $f(a) < k < f(b)$, there is a $c \in [a, b]$ such that $f(c) = k$.

Proof.

Consider the function $g(x) = f(x) - k$. Since f is continuous, so is $g(x)$. Now

$$g(a) = f(a) - k < 0, \quad \text{and} \quad g(b) = f(b) - k > 0.$$

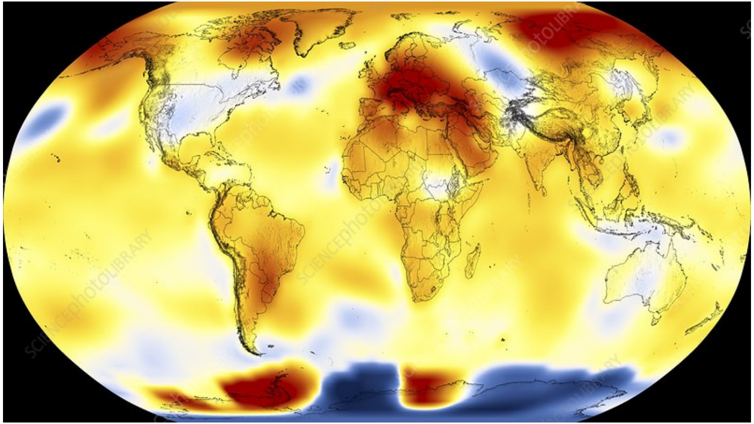
So, by the intermediate value theorem, there is a $c \in [a, b]$ such that $0 = g(c) = f(c) - k$, or $f(c = k)$. $\square$

Discussion

Prove that all odd degree polynomials has a root.

Application

There are always two points on opposite sides of the Earth with the exact same temperature.



https://youtu.be/5Px6fajpSio

Brouwer Fixed-point Theorem

Theorem

Let $f : [a, b] \to [a, b]$ be a continuous function. Then there is a $c \in [a, b]$ such that $f(c) = c$.

Proof.

Consider the function $g(x) = x - f(x)$; it is continuous. Since the range of $f(x)$ is in $[a, b]$, we have $f(a) \geq a$ and $f(b) \leq b$. Hence,

$$g(a) = a - f(a) \leq 0 \quad \text{and} \quad g(b) = b - f(b) \geq 0.$$

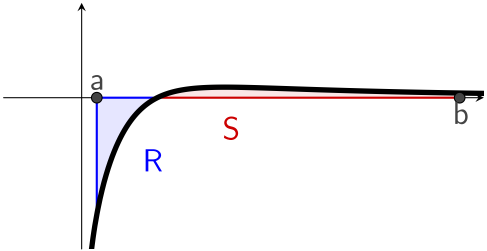
If either $f(a) = a$ or $f(b) = b$, we are done. Otherwise,

$$g(a) = a - f(a) < 0 \quad \text{and} \quad g(b) = b - f(b) > 0.$$

So, by the intermediate value theorem, there is a $c \in [a, b]$ such that $0 = g(c) = c - f(c)$, or $f(c) = c$. $\square$

Example

(Adapted from Y2021 Q1). Fix $0 < a < 1$. Let R be the region between the x-axis and the curve $y = \dfrac{\ln x}{1+x^2}$ for $a \leq x \leq 1$.



Use the substitution $x = \dfrac{1}{t}$ if necessary, is it possible to find a $b > 1$ such that the area of R is equal to the area of S, where S be the region between the x-axis and the curve $y = \dfrac{\ln x}{1+x^2}$ for $x \geq b$. If so, find b in terms of a.

Example

Note that area of R is $\left|\int_a^1 \frac{\ln x}{1+x^2}dx\right|$ and area of S is $\left|\int_1^b \frac{\ln x}{1+x^2}dx\right|$. Let $x=\frac{1}{t}$. When $x=1$, $t=1$; when $x=a$, $t=\frac{1}{a}$. Now $\frac{1}{1+x^2}=\frac{t^2}{t^2+1}$, and $\frac{dx}{dt}=-\frac{1}{t^2}$. So, by substitution,

$$\begin{aligned}\left|\int_a^1 \frac{\ln x}{1+x^2}dx\right| &= \left|-\int_{\frac{1}{a}}^1 \ln\left(\frac{1}{t}\right)\frac{t^2}{1+t^2}\frac{1}{t^2}dt\right| = \left|\int_1^{\frac{1}{a}} \ln\left(\frac{1}{t}\right)\frac{t^2}{1+t^2}\frac{1}{t^2}dt\right| \\ &= \left|-\int_1^{\frac{1}{a}} \ln t\frac{1}{1+t^2}dt\right| = \left|\int_1^{\frac{1}{a}} \frac{\ln x}{1+x^2}dx\right|\end{aligned}$$

By changing of the (dummy) variable from t to x, we see that the last integral is the area of S if $b=\frac{1}{a}$. Indeed, see `https://www.geogebra.org/m/v4xx2gas`.

Pigeon Hole Principle

Theorem

Suppose $k > 0$ objects are placed in $n > 0$ boxes. Then

- *if $k < n$, some boxes has no objects in it;*
- *if $k > n$, some boxes has more than one objects in it.*

Proof.

We will prove the contrapositive of both statements.

- If every box has at least 1 object, then there are at least n objects. Hence, $k \geq n$.
- If all boxes have at most 1 object, then there are at most n objects. Hence, $k \leq n$.

□

Example: A hairy problem

A hairy problem.

It is known that the maximum number of hairs on a human head is less than 200,000. Prove that there are at least two people in Singapore with exactly the same number of hairs on their heads.

Let the number of hairs on a person's head be the "boxes" and the number of people be the objects. There are approximately 6 million people in Singapore. Since $k = 6,000,000 > n = 200,000$. So, some people must be in the same "box", that is, some people must have the same number of hairs.

In particular, there are more than 1 bald person in Singapore.

Example

Fix a positive integer n and let $S = \{1, 2, ..., 2n-1\}$. Prove that any subset of S containing $n+1$ numbers must contain a pair of numbers whose sum is $2n$.

Let A be any subset of S containing $n+1$ numbers. Consider the sets

$$\{1, 2n-1\}, \quad \{2, 2n-2\}, \quad \dots \quad , \quad \{n-1, n+1\}, \{n\};$$

there are a total of n sets, with $n-1$ sets containing pairs of numbers whose sum is $2n$. By pigeon hole principle, A must contain at least one pair of numbers. Otherwise, if A only contains one number from each of the pairs, then A can at most have n numbers, a contradiction.

Discussion

Two people are said to be acquaintances if they have met before, and they are said to be strangers if they have never met before. Prove that in any group of 6 people, there is either a group of 3 people who are mutual acquaintances (i.e., any two in this group of 3 have met before) or there is a group of 3 mutual strangers (i.e., no two in this group of 3 have met before).

Floor and Ceiling Function

Definition

- The floor function, denoted as $\lfloor x \rfloor$ returns the largest integer n smaller than or equal to x,

$$\lfloor x \rfloor = n, \quad \text{where} \quad n \leq x < n+1.$$

- The ceiling function, denoted as $\lceil x \rceil$ returns the smallest integer n larger than or equal to x,

$$\lceil x \rceil = n, \quad \text{where} \quad n-1 < x \leq n.$$

Example

$\lfloor 3.142 \rfloor = 3$, $\lfloor 10 \rfloor = 10$, $\lfloor 9.999 \rfloor = 9$ $\lfloor 29.5 \rfloor = 29$, $\lfloor \sqrt{2} \rfloor = 1$,

$\lceil \sqrt{2} \rceil = 2$, $\lceil 5 \rceil = 5$, $\lceil 1.0001 \rceil = 1$, $\lceil 9.99999 \rceil = 10$.

Challenge

Suppose $k > 0$ objects are placed in $n > 0$ boxes. If $k > n$, then at least one box contains at least $\left\lceil \frac{k}{n} \right\rceil$ objects.