RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 5

Additional Practice Questions for Chapter 5: Complex Numbers

1 Given that $(-2+3\text{i})^2+\lambda(-2+3\text{i})+\mu=0$, find the real numbers λ and μ. [6]

[$\lambda=4,\ \mu=13$]

9233/2003/01/Q2
Solution:
$(-2+3\text{i})^2+\lambda(-2+3\text{i})+\mu=0$ $(4-12\text{i}-9)+\lambda(-2+3\text{i})+\mu=0$ $(-5-2\lambda+\mu)+\text{i}(3\lambda-12)=0$ Comparing real and imaginary parts, $-5-2\lambda+\mu=0$ and $3\lambda-12=0$ Solving, $\lambda=4,\ \mu=13$

2 The complex number $x+\text{i}y$ is such that $(x+\text{i}y)^2=\text{i}$. Find the possible values of $x,\ y\in\mathbb{R}$, giving your answers in exact form. [4]

Hence find the possible values of the complex number w such that $w^2=-\text{i}$. [2]

[$x=\frac{1}{\sqrt{2}},\ y=\frac{1}{\sqrt{2}},\ \ x=-\frac{1}{\sqrt{2}},\ y=-\frac{1}{\sqrt{2}}$; $\frac{1}{\sqrt{2}}-\text{i}\frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}+\text{i}\frac{1}{\sqrt{2}}$]

9233/2002/01/Q5
Solution:
$(x+\text{i}y)^2=\text{i}$ $(x^2-y^2)+\text{i}2xy=\text{i}$ Comparing real and imaginary parts, $x^2-y^2=0 \Rightarrow x^2=y^2 \cdots(1)$ and $2xy=1 \cdots(2) \Rightarrow 4x^2y^2=1 \Rightarrow 4x^4=1 \Rightarrow x^2=\frac{1}{2}$ or $x^2=-\frac{1}{2}$ (N.A. since $x\in\mathbb{R}$) $\Rightarrow x=\pm\frac{1}{\sqrt{2}}$ (2) $\Rightarrow$ x and y must have the same sign, and (1) $\Rightarrow x=y$

$\therefore$ When $x=\frac{1}{\sqrt{2}}$, $y=\frac{1}{\sqrt{2}}$; When $x=-\frac{1}{\sqrt{2}}$, $y=-\frac{1}{\sqrt{2}}$
$w^2=-\text{i} \Rightarrow -w^2=\text{i} \Rightarrow (\text{i}w)^2=\text{i}$ $\therefore \text{i}w=x+\text{i}y \Rightarrow w=y-\text{i}x=\frac{1}{\sqrt{2}}-\text{i}\frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}+\text{i}\frac{1}{\sqrt{2}}$

3 The complex number z is given by $z=1+2a\text{i}$, where a is a non-zero real number. Given that z^3 is real, find the possible values of z. [3]

[$z=1+\sqrt{3}\,\text{i}$ or $z=1-\sqrt{3}\,\text{i}$]

TPJC Prelim 9740/2014/01/Q6bi
Solution:
$z^3=(1+2a\text{i})^3$ $=1+3(2a\text{i})+3(2a\text{i})^2+(2a\text{i})^3$ $=1-12a^2+(6a-8a^3)\text{i}$ Since z^3 is real, $6a-8a^3=0$ $2a(3-4a^2)=0$ Since $a\neq 0$, $3-4a^2=0$ $a^2=\frac{3}{4}$ $a=\pm\frac{\sqrt{3}}{2}$ $\therefore z=1+2\left(\frac{\sqrt{3}}{2}\right)\text{i}$ or $z=1+2\left(-\frac{\sqrt{3}}{2}\right)\text{i}$ $z=1+\sqrt{3}\,\text{i}$ or $z=1-\sqrt{3}\,\text{i}$

4 The complex number w is such that $w=a+\text{i}b$, where a and b are non-zero real numbers. The complex conjugate of w is denoted by w^*. Given that $\frac{w^2}{w^*}$ is purely imaginary, find the possible values of w in terms of a. [5]

[$a+\text{i}\frac{a}{\sqrt{3}}$ or $a-\text{i}\frac{a}{\sqrt{3}}$]

9740/2015/01/Q9(a)
Solution:
$\dfrac{w^2}{w^*} = \dfrac{(a+\mathrm{i}b)^2}{a-\mathrm{i}b}\times\dfrac{a+\mathrm{i}b}{a+\mathrm{i}b} = \dfrac{(a+\mathrm{i}b)^3}{a^2+b^2}$ $= \dfrac{a^3+3\mathrm{i}a^2b-3ab^2-\mathrm{i}b^3}{a^2+b^2}$ $= \dfrac{(a^3-3ab^2)+\mathrm{i}(3a^2b-b^3)}{a^2+b^2}$ Given $\dfrac{w^2}{w^*}$ is purely imaginary, $\mathrm{Re}\left(\dfrac{w^2}{w^*}\right)=0$ $\Rightarrow \dfrac{(a^3-3ab^2)}{a^2+b^2}=0$ $\Rightarrow a(a^2-3b^2)=0$ $\Rightarrow a=0$ (rejected, since $a\neq 0$) or $b=\pm\dfrac{a}{\sqrt{3}}$ Thus, $w=a+\mathrm{i}\dfrac{a}{\sqrt{3}}$ or $w=a-\mathrm{i}\dfrac{a}{\sqrt{3}}$.

5 **(a)** Solve the simultaneous equations

$$\mathrm{i}w+z=-1-\mathrm{i},\ 2z-(1+\mathrm{i})w=\frac{40}{6-2\mathrm{i}}. \qquad [3]$$

(b) Two complex numbers w and z are such that

$$w^*=z-2\mathrm{i}, \qquad |w|^2=z+6.$$

Find w in the form $a+\mathrm{i}b$, where a and b are real and positive. [4]

[**(a)** $w=-2+2\mathrm{i}$, $z=1+\mathrm{i}$ **(b)** $w=2+2\mathrm{i}$]

Solution:	
(a)	$\mathrm{i}w+z=-1-\mathrm{i}$ ----- (1) $2z-(1+\mathrm{i})w=\dfrac{40}{6-2\mathrm{i}}$ ----- (2) 2 x (1) – (2) gives $[2\mathrm{i}+(1+\mathrm{i})]w=-2-2\mathrm{i}-\dfrac{40}{6-2\mathrm{i}}=-8-4\mathrm{i}$ $\Rightarrow w=\dfrac{-8-4\mathrm{i}}{1+3\mathrm{i}}=-2+2\mathrm{i}$ and $z=-1-\mathrm{i}-\mathrm{i}w=-1-\mathrm{i}-\mathrm{i}(-2+2\mathrm{i})=1+\mathrm{i}$

(b)	$\lvert w\rvert^2 = z+6 \Rightarrow z$ is real. Let $z=\alpha,\ \alpha\in\mathbb{R}$ Then $w^* = z-2\mathrm{i} \quad\Rightarrow\quad w^* = \alpha-2\mathrm{i} \quad\Rightarrow\quad w=\alpha+2\mathrm{i}$ Then $\lvert w\rvert^2 = \alpha^2+4=\alpha+6$ $\Rightarrow\quad \alpha^2-\alpha-2=0$ $\Rightarrow\quad \alpha=-1$ (reject since $\mathrm{Re}(w)>0$) or $\alpha=2$ $\Rightarrow w=2+2\mathrm{i}$ Alternative method: $w^* = z-2\mathrm{i} \quad\Rightarrow\quad z=w^*+2\mathrm{i}$ ----- (1) $\lvert w\rvert^2 = z+6$ ----- (2) Substitute (1) into (2) gives $\lvert w\rvert^2 = w^*+2\mathrm{i}+6$ Let $w=a+\mathrm{i}b$. We have $a^2+b^2=(a-\mathrm{i}b)+2\mathrm{i}+6=(a+6)+\mathrm{i}(2-b)$ Comparing real and imaginary parts, $2-b=0 \quad\Rightarrow\quad b=2$ $a^2+4=a+6 \quad\Rightarrow\quad a^2-a-2=0$ $\Rightarrow\quad (a-2)(a+1)=0 \quad\Rightarrow\quad a=2$ or -1(N.A. since a is positive) $\therefore\ w=2+2\mathrm{i}$

6 The complex numbers s and w satisfy the equations

$$s-w=6\mathrm{i} \qquad \text{and} \qquad sw=10.$$

Given that $\mathrm{Re}(s)>0$, solve the equations for s and w, giving all answers in the form $x+\mathrm{i}y$, where x and y are real. [4]

Hence find a solution to the following equations

$$u-v=-6 \qquad \text{and} \qquad uv=-10.$$

Give your answers for u and v in the form $x+\mathrm{i}y$, where x and y are real. [2]

[$s=1+3\mathrm{i},\quad w=1-3\mathrm{i}$; $u=-3+\mathrm{i}$ and $v=3+\mathrm{i}$]

AJC Prelim 9740/2013/01/Q11(a)
Solution:
$s-w=6\mathrm{i} \Rightarrow$ The real part of s and w are the same.

Let $s=a+b\mathrm{i}$ and $w=a+c\mathrm{i}$, where $a>0$. $\therefore b-c=6$ --- (1) $(a+b\mathrm{i})(a+c\mathrm{i})=10$ $a^2-bc+a(b+c)\mathrm{i}=10$ $\therefore a^2-bc=10\cdots(2)$ and since $a\neq 0$, $b+c=0\cdots(3)$ Solving (1) and (3), $b=3$ and $c=-3$ Using (2), $a=\pm 1$ Since $a>0$, $s=1+3\mathrm{i}$, $w=1-3\mathrm{i}$ Alternative method: Substitute $w=\dfrac{10}{s}$ into $s-w=6\mathrm{i}$, $s-\dfrac{10}{s}=6\mathrm{i}$ $s^2-10=(6\mathrm{i})s$ $s^2-(6\mathrm{i})s-10=0$ $s=\dfrac{6\mathrm{i}\pm\sqrt{-36-4(-10)}}{2}=\pm 1+3\mathrm{i}$ Since Re(s) > 0, $s=1+3\mathrm{i}$ and $w=1-3\mathrm{i}$
Let $u=\mathrm{i}s$ and $v=\mathrm{i}w$ and we would arrive at the original pair of given equations. $\therefore u=-3+\mathrm{i}$ and $v=3+\mathrm{i}$ Alternative method: $s=\mathrm{i}v,\quad w=\mathrm{i}u \quad\Rightarrow\quad v=\dfrac{1}{\mathrm{i}}s=-\mathrm{i}s=3-\mathrm{i}$ and $u=\dfrac{1}{\mathrm{i}}w=-\mathrm{i}w=-3-\mathrm{i}$ $\therefore v=3-\mathrm{i}$ and $u=-3-\mathrm{i}$

7 **Do not use a calculator in answering this question.**

Given that $2-3\mathrm{i}$ is a root of the equation

$$z^4-10z^3+48z^2-122z+143=0,$$

solve the equation, giving your answers in exact form. [4]

[$2+3\mathrm{i}$ and $3\pm\sqrt{2}\,\mathrm{i}$]

HCI Prelim 9740/2013/01/Q11(i)
Solution:

Since all the coefficients of the polynomial are real, non-real roots will occur in conjugate pairs.
Given that $2-3\mathrm{i}$ is a root, then $2+3\mathrm{i}$ is also a root.

$$\begin{aligned}\left[z-(2+3\mathrm{i})\right]\left[z-(2-3\mathrm{i})\right] &= \left[(z-2)-3\mathrm{i}\right]\left[(z-2)+3\mathrm{i}\right] \\ &= (z-2)^2-(3\mathrm{i})^2 \\ &= z^2-4z+4+9 \\ &= z^2-4z+13\end{aligned}$$

Then, $z^4-10z^3+48z^2-122z+143=(z^2-4z+13)(z^2+az+11)$

Comparing coefficients of z^3, $-10=a-4 \quad \Rightarrow a=-6$

$\therefore z^4-10z^3+48z^2-122z+143=(z^2-4z+13)(z^2-6z+11)$

$z^2-6z+11=0$

$\therefore \qquad z=\dfrac{6\pm\sqrt{36-44}}{2}=3\pm\sqrt{2}\,\mathrm{i}$

Hence the other roots are $2+3\mathrm{i}$ and $3\pm\sqrt{2}\,\mathrm{i}$.

8 It is given that $-1+2\mathrm{i}$ satisfies the equation $z^2+(a-\mathrm{i})z+16+b\mathrm{i}=0$, where $a,b\in\mathbb{R}$.
Find the value of a and b and the other root. [5]

[$a=15$ and $b=-27$, $-14-\mathrm{i}$]

Solution:

Note: Since the coefficients are not all real, the non-real roots are not in conjugate pairs.

We substitute $-1+2\mathrm{i}$ into the given equation since it satisfies the equation:

$(-1+2\mathrm{i})^2+(a-\mathrm{i})(-1+2\mathrm{i})+16+b\mathrm{i}=0$

$1-4\mathrm{i}-4-a+\mathrm{i}+2a\mathrm{i}+2+16+b\mathrm{i}=0$

$15-a+(2a-3+b)\mathrm{i}=0$

Comparing real and imaginary parts,

$$a=15 \text{ and } b=3-2a=-27$$

$z^2+(15-\mathrm{i})z+16-27\mathrm{i}=0$

$$\begin{aligned} z &= \frac{-15+\mathrm{i}\pm\sqrt{(15-\mathrm{i})^2-4(16-27\mathrm{i})}}{2} \\ &= \frac{-15+\mathrm{i}\pm(13+3\mathrm{i})}{2} \\ &= -1+2\mathrm{i} \quad \text{or} \quad -14-\mathrm{i}\end{aligned}$$

$\therefore$ The other root is $-14-\mathrm{i}$.

9 **Do not use a calculator in answering this question.**

(a) Verify that $-1+5\mathrm{i}$ is a root of the equation $w^2+(-1-8\mathrm{i})w+(-17+7\mathrm{i})=0$. Hence, or otherwise, find the second root of the equation in cartesian form, $p+\mathrm{i}q$, showing your working. [5]

(b) The equation $z^3-5z^2+16z+k=0$, where k is a real constant, has a root $z=1+a\mathrm{i}$, where a is a positive real constant. Find the values of a and k, showing your working. [5]

[**(a)** $2+3\mathrm{i}$ **(b)** $a=3$, $k=-30$]

<table>
<tr><td colspan="2">9740/2016/01/Q7</td></tr>
<tr><td colspan="2">Solution:</td></tr>
<tr><td>(a)</td><td>
$w^2+(-1-8\mathrm{i})w+(-17+7\mathrm{i})=0$ −−−−−−−−(1)
Substitute $-1+5\mathrm{i}$ into the LHS of (1),
$\text{LHS}=(-1+5\mathrm{i})^2+(-1-8\mathrm{i})(-1+5\mathrm{i})+(-17+7\mathrm{i})$
$=(1-10\mathrm{i}-25)+(1+3\mathrm{i}+40)+(-17+7\mathrm{i})$
$=0$ (verified)
$\therefore$ $-1+5\mathrm{i}$ is a root of the equation.

Let w be the 2nd root of the equation.
Use the result that “sum of roots” = $-\dfrac{b}{a}$ where a and b are the coefficients in $az^2+bz+c=0$
Sum of roots $=w+\left(-1+5\mathrm{i}\right)=-\left(-1-8\mathrm{i}\right)$
$\therefore w=\left(1+8\mathrm{i}\right)-\left(-1+5\mathrm{i}\right)=2+3\mathrm{i}$
Alternative method 1:
$w^2+(-1-8\mathrm{i})w+(-17+7\mathrm{i})$
$=[w-(-1+5\mathrm{i})][w-(p+q\mathrm{i})]$
Comparing the coefficients of w,
$-1-8\mathrm{i}=-(p+q\mathrm{i})-(-1+5\mathrm{i})$
$-1-8\mathrm{i}=(1-p)+(-q-5)\mathrm{i}$
Comparing real and imaginary parts,
$-1=1-p\Rightarrow p=2$
$-8=-q-5\Rightarrow q=3$
$\therefore$ the second root of the equation is $2+3\mathrm{i}$.
</td></tr>
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	Alternative method 2: $w^2+(-1-8\text{i})w+(-17+7\text{i})$ $=[w-(-1+5\text{i})][w-k]$ where k is a complex no. $=w^2-(-1+5\text{i}-k)w+(-1+5\text{i})k$ Comparing coefficient of w, $-1+5\text{i}-k=-(-1-8\text{i})$ $k=1+8\text{i}+1-5\text{i}=2+3\text{i}$
(b)	$z^3-5z^2+16z+k=0------(2)$ Substitute $1+a\text{i}$ into (2). $(1+a\text{i})^3-5(1+a\text{i})^2+16(1+a\text{i})+k=0$ $\left(1+3a\text{i}-3a^2-a^3\text{i}\right)-5(1+2a\text{i}-a^2)+16(1+a\text{i})+k=0$ $(k+12+2a^2)+\text{i}(9a-a^3)=0$ Comparing the real and imaginary parts, $k^2+12+2a^2=0----------(3)$ and $9a-a^3=0-------(4)$ From (4), $a(9-a^2)=0$ $a=0\,(\text{rej}), a=-3\,(\text{rej}), a=3\,(\text{since } a>0)$. Substitute $a=3$ into (3), $k=-12-2(3)^2=-30$

10 **Do not use a calculator in answering this question.**

(a) Find the roots of the equation $z^2(1-\text{i})-2z+(5+5\text{i})=0$, giving your answers in cartesian form $a+\text{i}b$. [3]

(b) **(i)** Given that $\omega=1-\text{i}$, find ω^2, ω^3 and ω^4 in cartesian form. Given also that

$$\omega^4+p\omega^3+39\omega^2+q\omega+58=0,$$

where p and q are real, find p and q. [4]

(ii) Using the values of p and q in part **(b)(i)**, express $\omega^4+p\omega^3+39\omega^2+q\omega+58$ as the product of two quadratic factors. [3]

[**(a)** $-1+2\text{i}$ or $2-\text{i}$ **(bi)** $p=-6,\ q=-66$ **(bii)** $\left(\omega^2-2\omega+2\right)\left(\omega^2-4\omega+29\right)$]

9758/2017/01/Q8	
Solution:	
(a)	$z^2(1-\text{i})-2z+(5+5\text{i})=0$

	$z=\dfrac{2\pm\sqrt{(-2)^2-4(1-\mathrm{i})(5+5\mathrm{i})}}{2(1-\mathrm{i})}$ $=\dfrac{2\pm\sqrt{4-4(5)(1-\mathrm{i})(1+\mathrm{i})}}{2(1-\mathrm{i})}$ $=\dfrac{1\pm\sqrt{1-5\left(1-\mathrm{i}^2\right)}}{1-\mathrm{i}}$ $=\dfrac{1\pm\sqrt{-9}}{1-\mathrm{i}}$ $=\dfrac{1\pm 3\mathrm{i}}{1-\mathrm{i}}$ $z=\dfrac{1+3\mathrm{i}}{1-\mathrm{i}}\times\dfrac{1+\mathrm{i}}{1+\mathrm{i}}$ $=\dfrac{1}{2}\left(1+\mathrm{i}+3\mathrm{i}+3\mathrm{i}^2\right)$ $=-1+2\mathrm{i}$ or $z=\dfrac{1-3\mathrm{i}}{1-\mathrm{i}}\times\dfrac{1+\mathrm{i}}{1+\mathrm{i}}$ $=\dfrac{1}{2}\left(1+\mathrm{i}-3\mathrm{i}-3\mathrm{i}^2\right)$ $=2-\mathrm{i}$
(b) (i)	$\omega=1-\mathrm{i}$ $\omega^2=(1-\mathrm{i})^2=1-2\mathrm{i}+\mathrm{i}^2=-2\mathrm{i}$ $\omega^3=\omega\left(\omega^2\right)=(1-\mathrm{i})(-2\mathrm{i})=-2\mathrm{i}+2\mathrm{i}^2=-2-2\mathrm{i}$ $\omega^4=\left(\omega^2\right)^2=(-2\mathrm{i})^2=4\mathrm{i}^2=-4$ $\omega^4+p\omega^3+39\omega^2+q\omega+58=0$ $-4+p(-2-2\mathrm{i})+39(-2\mathrm{i})+q(1-\mathrm{i})+58=0$ $(54-2p+q)+(-2p-78-q)\mathrm{i}=0$ Comparing the real and imaginary parts, $54-2p+q=0$ …… (1) $-2p-78-q=0$ …… (2) Solving (1) and (2), $-4p-24=0$ $p=-6$ $q=-2p-78$ $=-2(-6)-78$ $=-66$
(b) (ii)	Since p and q are both real, then all the coefficients of the polynomial are real, so non-real roots will occur in conjugate pairs. Given $\omega=1-\mathrm{i}$ is a root, then $\omega=1+\mathrm{i}$ is also a root. One of the quadratic factor is

	$(\omega-(1-\mathrm{i}))(\omega-(1+\mathrm{i}))=(\omega-1+\mathrm{i})(\omega-1-\mathrm{i})$ $=(\omega-1)^2-\mathrm{i}^2$ $=\omega^2-2\omega+2$ Thus, $\omega^4+p\omega^3+39\omega^2+q\omega+58=(\omega^2-2\omega+2)(\omega^2+a\omega+29)$ By comparing coefficients of ω^2: $39=-2a+2+29$ $-2a=8$ $a=-4$ So, $\omega^4+p\omega^3+39\omega^2+q\omega+58=(\omega^2-2\omega+2)(\omega^2-4\omega+29)$

11 A graphic calculator is **not** to be used in answering this question.

(i) It is given that $z_1=1+\sqrt{3}\mathrm{i}$. Find the value of $z_1^{\,3}$, showing clearly how you obtain your answer. [3]

(ii) Given that $1+\sqrt{3}\mathrm{i}$ is a root of the equation

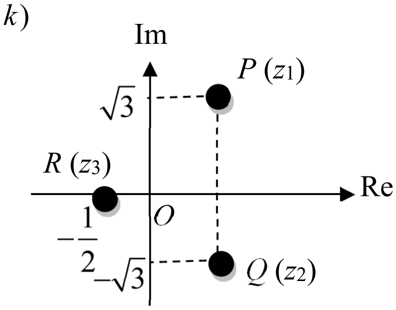
$$2z^3+az^2+bz+4=0,$$

Find the values of the real numbers a and b. [4]

(iii) For these values of a and b, solve the equation in part **(ii)**, and show all the roots on an Argand diagram. [4]

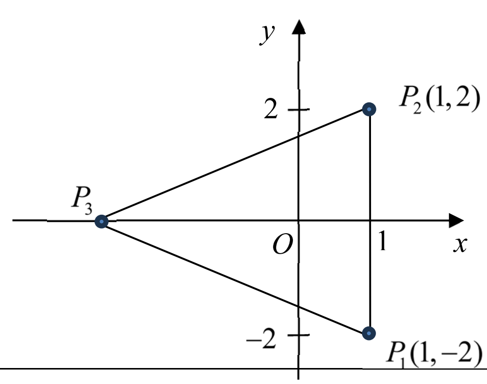
[**(i)** -8 **(ii)** $a=-3$, $b=6$ **(iii)** $z_1=1+\sqrt{3}\mathrm{i}$, $z_2=1-\sqrt{3}\mathrm{i}$ and $z_3=-\frac{1}{2}$]

9740/2008/01/Q8	
Solution:	
(i)	$z_1=1+\sqrt{3}\mathrm{i}$ $z_1^{\,3}=(1+\sqrt{3}\mathrm{i})^3=(1+\sqrt{3}\mathrm{i})^2(1+\sqrt{3}\mathrm{i})$ $=(1+2\sqrt{3}\mathrm{i}-3)(1+\sqrt{3}\mathrm{i})=(-2+2\sqrt{3}\mathrm{i})(1+\sqrt{3}\mathrm{i})=-2-6+\mathrm{i}\left(-2\sqrt{3}+2\sqrt{3}\right)=-8$
(ii)	Since $1+\sqrt{3}\mathrm{i}$ is a root of $2z^3+az^2+bz+4=0$, $2(1+\sqrt{3}\mathrm{i})^3+a(1+\sqrt{3}\mathrm{i})^2+b(1+\sqrt{3}\mathrm{i})+4=0$ $2(-8)+a(-2+2\sqrt{3}\mathrm{i})+b(1+\sqrt{3}\mathrm{i})+4=0$ $(-16-2a+b+4)+\mathrm{i}(2\sqrt{3}a+\sqrt{3}b)=0$ Comparing real and imaginary parts, $2\sqrt{3}a+\sqrt{3}b=0 \Rightarrow b=-2a$ $-12-2a+b=0 \Rightarrow 4a=-12 \Rightarrow a=-3$ and $b=6$

(iii)	Since the coefficients of the polynomial are real, non-real roots will occur in conjugate pairs. Given $1+\sqrt{3}\text{i}$ is a root, $1-\sqrt{3}\text{i}$ is also a root. Let $2z^3-3z^2+6z+4=\left[z-(1+\sqrt{3}\text{i})\right]\left[z-(1-\sqrt{3}\text{i})\right](2z+k)$ $=(z^2-2z+4)(2z+k)$ Comparing coefficient of constant, $k=1$ The 3 roots of $2z^3-3z^2+6z+4=0$ are $z_1=1+\sqrt{3}\text{i}$, $z_2=1-\sqrt{3}\text{i}$ and $2z+1=0 \Rightarrow z_3=-\frac{1}{2}$ Let P, Q, R be the points representing the roots z_1, z_2, z_3 respectively. 

12 Given that $z_1=1-2\text{i}$ is a root of the quadratic equation $z^2+az+b=0$ where $a,\ b\in\mathbb{R}$, find the values of a and b. Mark on an Argand diagram the points P_1 and P_2, which represent the roots z_1 and z_2 of the quadratic equation $z^2+az+b=0$. A third point P_3 is on the real axis such that P_1, P_2 and P_3 form a triangle which encloses the origin with an area of 8 square units. Find the complex number represented by the point P_3. [5]

[$a=-2,\ b=5$; $z_3=-3$]

TPJC Prelim 9740/2008/01/Q6 (modified)
Solution:
$z^2+az+b=0$ Since all the coefficients of the polynomial are real, complex roots will occur in conjugate pairs. Given $z_1=1-2\text{i}$ is a root, $z_2=1+2\text{i}$ is also a root. $\left[z-(1+2\text{i})\right]\left[z-(1-2\text{i})\right]=0$ $z^2-2z+5=0$ $\therefore\ a=-2,\ b=5$


Since P_1 and P_2 represent z_1 and z_2 which are conjugates of each other and P_3 is on the real axis, $P_1P_2 \perp$ real axis.
Base of triangle $P_1P_2P_3 = P_1P_2 = 4$
Height of triangle $P_1P_2P_3 = P_3O + 1$
Now, area of triangle $P_1P_2P_3 = \frac{1}{2}(4)(P_3O+1) = 8 \quad \Rightarrow \; P_3O = 3$
$\therefore z_3 = -3$

13 The complex number z and w are given by and $z = -3 + 2\mathrm{i}$ and $w = 5 + 4\mathrm{i}$. In an Argand diagram, the points Z and W represent the complex numbers z and w respectively, and Z' represents z^*, the complex conjugate of z. The point P is such that $ZWZ'P$ is a parallelogram. Find the complex number p represented by P.

[$-11-4\mathrm{i}$]

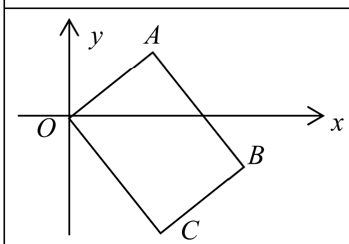
Solution:

Since $ZWZ'P$ is a parallelogram,
$\overrightarrow{Z'P} = \overrightarrow{WZ}$
This corresponds to
$p - z^* = z - w$
$p = z + z^* - w = -11 - 4\mathrm{i}$

14 The complex number $7+5\mathrm{i}$ is represented by the point A in an Argand diagram with origin O. Given that $OABC$ is a rectangle described in a clockwise sense with $OC = 2OA$, find the complex numbers represented by the points B and C in the form $x + \mathrm{i}y$, where x and y are real.

[$b = 17-9\mathrm{i}$, $c = 10-14\mathrm{i}$]

Solution:



Rotate $2\overrightarrow{OA}$ about O by $\frac{3\pi}{2}$ anticlockwise to obtain $\overrightarrow{OC}$.
This corresponds to $c = 2\mathrm{i}^3 a$
Hence $c = 2\mathrm{i}^3 a = 2\mathrm{i}^3(7+5\mathrm{i}) = 10-14\mathrm{i}$.

Now, $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{OC}$
This corresponds to $b = a + c = (7+5\mathrm{i}) + (10-14\mathrm{i}) = 17-9\mathrm{i}$.

15 The complex numbers a, b, c and d are represented by the points A, B, C and D respectively on the Argand diagram such that $ABCD$ is a square described in the anti-clockwise sense. It is given that $a = 2\mathrm{i}$ and $d = \left(\sqrt{3}+1\right)+\left(\sqrt{3}+1\right)\mathrm{i}$.

(i) Find the complex numbers b and c, leaving your answers in the form $x + \mathrm{i}y$, where x and y are real numbers to be determined exactly.

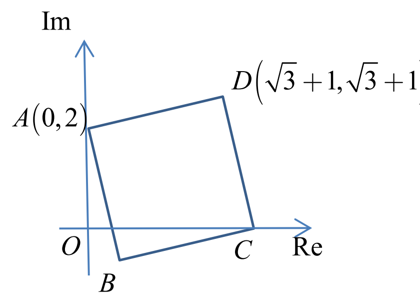
(ii) The points A, B, C and D lie on the circle of radius k, centred at the point M representing the complex number m. State the values of m and k.

[**(i)** $b=\left(\sqrt{3}-1\right)+\left(1-\sqrt{3}\right)\mathrm{i}$, $c=2\sqrt{3}$ **(ii)** $m=\sqrt{3}+\mathrm{i}$ and $k=2$ units]

Solution:

(i)

Rotate $\overrightarrow{AB}$ about A by $\dfrac{\pi}{2}$ anticlockwise to obtain $\overrightarrow{AD}$.



This corresponds to

$$\mathrm{i}(b-a)=d-a$$
$$\mathrm{i}b-\mathrm{i}a=\left(\sqrt{3}+1\right)+\left(\sqrt{3}+1\right)\mathrm{i}-2\mathrm{i}$$
$$\mathrm{i}b+2=\left(\sqrt{3}+1\right)+\left(\sqrt{3}-1\right)\mathrm{i}$$
$$\mathrm{i}b=\left(\sqrt{3}-1\right)+\left(\sqrt{3}-1\right)\mathrm{i}$$
$$b=\left(\sqrt{3}-1\right)+\left(1-\sqrt{3}\right)\mathrm{i}$$

Since $ABCD$ is a square, $\overrightarrow{BC}=\overrightarrow{AD}$.

This corresponds to $c-b=d-a$

$$c-\left[\left(\sqrt{3}-1\right)+\left(1-\sqrt{3}\right)\mathrm{i}\right]=\left(\sqrt{3}+1\right)+\left(\sqrt{3}-1\right)\mathrm{i}$$
$$c=2\sqrt{3}$$

<u>Alternative method</u> :

Midpoint of BD is $\left(\dfrac{\sqrt{3}-1+\sqrt{3}+1}{2},\dfrac{1-\sqrt{3}+\sqrt{3}+1}{2}\right)=\left(\sqrt{3},1\right)$

Let $c=x+\mathrm{i}y$

Since the diagonals of a square bisect other, midpoint of AC is $\left(\sqrt{3},1\right)$.

$$\left(\frac{x}{2},\frac{y+2}{2}\right)=\left(\sqrt{3},1\right)$$
$$\Rightarrow x=2\sqrt{3},\ y=0$$
$$\therefore c=2\sqrt{3}$$

(ii)	The mid-point M of AC is represented by the complex number $\frac{2\mathrm{i}+2\sqrt{3}}{2}=\sqrt{3}+\mathrm{i}$. Since the points A, B, C and D are equidistant from this mid-point, the points lie on the circle centred at M. The radius of the circle is the length of the line segment MA which is 2 units. So $m=\sqrt{3}+\mathrm{i}$ and $k=2$ units.

16 The complex numbers z_1 and z_2 are given by $z_1=-1+\mathrm{i}a$ and $z_2=\sqrt{2}-\sqrt{2}\,\mathrm{i}$ where a is a real number. Given that $\mathrm{Im}\left(\frac{z_1}{z_2}\right)=\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$, show that $a=\sqrt{3}$. [2]

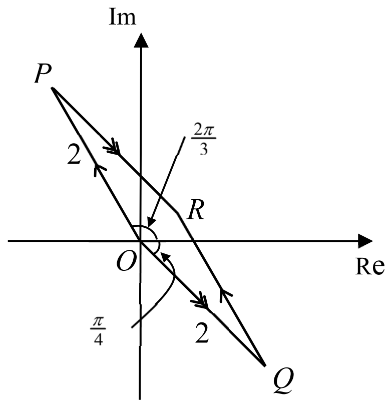
State the modulus and argument of each of z_1 and z_2. [2]

Sketch an Argand diagram with origin O, showing the points P, Q and R representing the complex numbers z_1, z_2 and (z_1+z_2) respectively. Give a geometrical description of $OPRQ$. [2]

Deduce from your diagram that $\tan\frac{5\pi}{24}=\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-1}$. [3]

[$|z_1|=2$, $\arg(z_1)=\frac{2\pi}{3}$; $|z_2|=2$, $\arg(z_2)=-\frac{\pi}{4}$; rhombus]

Solution:
$\frac{z_1}{z_2}=\frac{-1+\mathrm{i}a}{\sqrt{2}-\sqrt{2}\,\mathrm{i}}\times\frac{\sqrt{2}+\sqrt{2}\,\mathrm{i}}{\sqrt{2}+\sqrt{2}\,\mathrm{i}}=\frac{(-\sqrt{2}-a\sqrt{2})+\mathrm{i}(a\sqrt{2}-\sqrt{2})}{2+2}$ $\mathrm{Im}\left(\frac{z_1}{z_2}\right)=\frac{1}{4}(a\sqrt{2}-\sqrt{2})=\frac{1}{4}(\sqrt{6}-\sqrt{2})$ $\Rightarrow a=\sqrt{3}$ (Shown)
$\lvert z_1\rvert=2$, $\arg(z_1)=\frac{2\pi}{3}$; $\lvert z_2\rvert=2$, $\arg(z_2)=-\frac{\pi}{4}$



OPRQ is a rhombus.

$\angle POQ = \frac{2\pi}{3} + \frac{\pi}{4} = \frac{11\pi}{12}$ and $\angle ROQ = \frac{1}{2}\angle POQ = \frac{11\pi}{24}$

$\therefore \arg\left(z_1 + z_2\right) = \angle ROQ - \frac{\pi}{4} = \frac{5\pi}{24}$

Since $z_1 + z_2 = \sqrt{2} - 1 + (\sqrt{3} - \sqrt{2})\mathrm{i}$,

from the diagram, $\tan\frac{5\pi}{24} = \frac{\mathrm{Im}\,(z_1 + z_2)}{\mathrm{Re}\,(z_1 + z_2)} = \frac{\sqrt{3} - \sqrt{2}}{\sqrt{2} - 1}$ (deduced)