2022 Preliminary Examination H2 Further Mathematics Paper 1 (9649/01) Marking Scheme

s/n	Solution
1(a)	Source: 2025 Promo HCI Qn 3 $\mathrm{f}(x,y)=\sqrt{x^2+ky}$ $\dfrac{\partial f}{\partial x}=\dfrac{1}{2}\left(x^2+ky\right)^{-\frac{1}{2}}(2x)$ $\dfrac{\partial f}{\partial y}=\dfrac{1}{2}\left(x^2+ky\right)^{-\frac{1}{2}}(k)$ At $(2,2)$, $\mathrm{f}(x,y)=\sqrt{4+2k}$ $\dfrac{\partial f}{\partial x}=2(4+2k)^{-\frac{1}{2}}$ $\dfrac{\partial f}{\partial y}=\dfrac{k}{2}(4+2k)^{-\frac{1}{2}}$ $L(x,y)=\sqrt{4+2k}+\dfrac{2}{\sqrt{4+2k}}(x-2)+\dfrac{k}{2\sqrt{4+2k}}(y-2)$
(b)	Consider $k=2$, $L(x,y)=\sqrt{8}+\dfrac{2}{\sqrt{8}}(2.1-2)+\dfrac{2}{2\sqrt{8}}(1.95-2)$ $=2\sqrt{2}+\dfrac{2}{2\sqrt{2}}(0.1)+\dfrac{1}{2\sqrt{2}}(-0.05)$ Sub $k=2,\ x=2.1,\ y=1.95$, $=2\sqrt{2}+\dfrac{1}{2\sqrt{2}}(0.15)$ $=2\sqrt{2}+\dfrac{3}{80}\sqrt{2}$ $=\dfrac{163}{80}\sqrt{2}$
2(i)	$\begin{pmatrix}1&1&a&0\\1&-2&a&0\\1&1&1&b\end{pmatrix}\xrightarrow{R_3\to -R_1+R_3}\begin{pmatrix}1&1&a&0\\1&-2&a&0\\0&0&1-a&b\end{pmatrix}$ $\xrightarrow{R_2\to -R_1+R_2}\begin{pmatrix}1&1&a&0\\0&-3&0&0\\0&0&1-a&b\end{pmatrix}$ $S=\{\underset{\sim}{v}\}\ \Leftrightarrow\ a\neq 1$ and $b\neq 0$
(ii)	S is not a subspace of $\mathbb{R}^3\ \Leftrightarrow\ a\in\mathbb{R}$ and $b\neq 0$ Possible sets of S: 1. $a\neq 1$ and $b\neq 0$ implies $S=\{\underset{\sim}{v}\}$, where $\underset{\sim}{v}\neq 0$ 2. $a=1$ and $b\neq 0$ implies $S=\varnothing$

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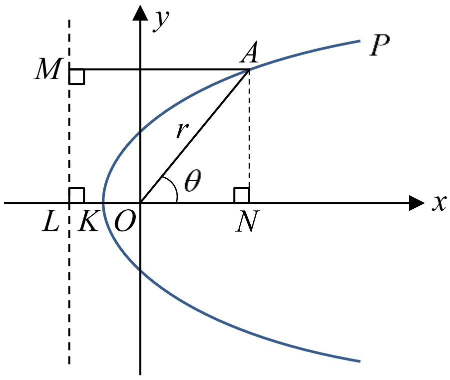
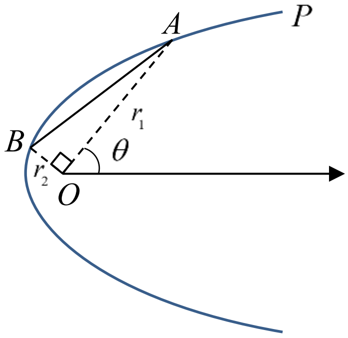
3(i)	 From diagram, ML is the directrix where $LK = KO = h$ Let e be the eccentricity of P. $\therefore\ e = \dfrac{OA}{MA} = 1$ Since $OA = r$ and $\angle AON = \theta$, $\therefore\ ON = r\cos\theta$ $\quad\therefore\ MA = LO + ON = 2h + r\cos\theta$ Hence $e = \dfrac{OA}{MA} = \dfrac{r}{2h + r\cos\theta} = 1$ $r = 2h + r\cos\theta$ $\therefore\ r = \dfrac{2h}{1-\cos\theta}$ A polar equation of P is $r = \dfrac{2h}{1-\cos\theta}$
(ii)	 Let $OA = r_1$, $OB = r_2$ Since A and B are points on P, using result from (i), $r_1 = \dfrac{2h}{1-\cos\theta}$, $\quad r_2 = \dfrac{2h}{1-\cos\left(\theta + \frac{\pi}{2}\right)}$ Area $\triangle OAB = \dfrac{1}{2} r_1 r_2$ $= \dfrac{1}{2}\left(\dfrac{2h}{1-\cos\theta}\right)\left(\dfrac{2h}{1-\cos\left(\theta+\frac{\pi}{2}\right)}\right)$ $= \dfrac{2h^2}{(1-\cos\theta)(1+\sin\theta)} \quad \ldots(1)$ Let $w = (1-\cos\theta)(1+\sin\theta)$ Since area $\triangle OAB \geq 0$, w is always positive. $\therefore$ maximum $w \Rightarrow$ minimum area $\triangle OAB$

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	$\frac{\mathrm{d}w}{\mathrm{d}\theta}=(1-\cos\theta)\cos\theta+(1+\sin\theta)\sin\theta$ $=\cos\theta-\cos^2\theta+\sin\theta+\sin^2\theta$ $=\cos\theta+\sin\theta+(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)$ $=(\cos\theta+\sin\theta)(1+\sin\theta-\cos\theta)$ When $\frac{\mathrm{d}w}{\mathrm{d}\theta}=0$, $\cos\theta+\sin\theta=0$ or $1+\sin\theta-\cos\theta=0$ $\tan\theta=-1$ or $\sqrt{2}\sin\left(\theta-\frac{\pi}{4}\right)=-1$ $\therefore\ \theta=\frac{3\pi}{4}$ or $\theta=0$ (rejected) or $\theta=\frac{3\pi}{2}$ (rejected) Hence from (1), Area $\triangle OAB=\frac{2h^2}{(1-\cos\frac{3\pi}{4})(1+\sin\frac{3\pi}{4})}=\frac{2h^2}{\left(1+\frac{1}{\sqrt{2}}\right)\left(1+\frac{1}{\sqrt{2}}\right)}$ $=\frac{4h^2}{(1+\sqrt{2})^2}$ unit^2
4(a)	Using GC,

	g(x)		
	$x-x^3-4x^2+6$	$\left(\frac{6}{x}-4x\right)^{\frac{1}{2}}$	$\left(\frac{6}{4+x}\right)^{\frac{1}{2}}$
x_0	1.5	1.5	1.5
x_1	–4.875	undefined	1.0445
x_2	21.9199		1.0906
x_3	−12426		1.0857
x_4	1.9×10^{12}		1.0862
x_5	-7×10^{36}		1.0861
x_6			1.0861

(i) $g(x)=x-x^3-4x^2+6$ diverges.
Hence, it is not suitable.

(ii) $g(x)=\left(\frac{6}{x}-4x\right)^{\frac{1}{2}}$ is undefined at x_1.
Hence it is not suitable.

(iii) $g(x)=\left(\frac{6}{4+x}\right)^{\frac{1}{2}}$ converges.
Hence it is suitable.

Suppose $\alpha=1.0861$ is a root.

$x=\left(\frac{6}{4+x}\right)^{\frac{1}{2}}\ \Rightarrow\ x^3-4x^2-6=0$

Let $f(x)=x^3-4x^2-6$

s/n	Solution
	$\mathrm{f}(x)$ is continuous in $[1.08605, 1.08615]$ and $\mathrm{f}(1.08605)\mathrm{f}(1.08615) = (-0.0009806)(0.000242) < 0$ Hence $\alpha = 1.0861$ (4 d.p.)
(b)	$\mathrm{f}(-\sigma) = \dfrac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}(-1)^2}$ $\mathrm{f}(-0.5\sigma) = \dfrac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}(-0.5)^2}$ $\mathrm{f}(0) = \dfrac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}(0)^2}$ $\mathrm{f}(0.5\sigma) = \dfrac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}(0.5)^2}$ $\mathrm{f}(\sigma) = \dfrac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}(1)^2}$ By Simpson's Rule, $\int_{-\sigma}^{\sigma} \mathrm{f}(x)\,\mathrm{d}x$ $\approx \dfrac{1}{3}\left(\dfrac{\sigma}{2}\right)\left(\begin{array}{l}\dfrac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}(-1)^2} + \dfrac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}(1)^2} \\ +4\left(\dfrac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}(-0.5)^2} + \dfrac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}(0.5)^2}\right) + 2\dfrac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}(0)^2}\end{array}\right)$ $= \dfrac{1}{6}\left(\dfrac{1}{\sqrt{2\pi}}\right)\left(\mathrm{e}^{-\frac{1}{2}(-1)^2} + \mathrm{e}^{-\frac{1}{2}(1)^2} + 8\left(\mathrm{e}^{-\frac{1}{8}}\right) + 2\mathrm{e}^{0}\right)$ ≈ 0.683 (3 d.p.)
5(i)	$\text{LHS} = \cos\left(\dfrac{\pi}{10}\right) + \cos\left(\dfrac{3\pi}{10}\right) + \cos\left(\dfrac{5\pi}{10}\right) + \cos\left(\dfrac{7\pi}{10}\right) + \cos\left(\dfrac{9\pi}{10}\right)$ $= \left[\cos\left(\dfrac{9\pi}{10}\right) + \cos\left(\dfrac{\pi}{10}\right)\right] + \left[\cos\left(\dfrac{7\pi}{10}\right) + \cos\left(\dfrac{3\pi}{10}\right)\right] + \cos\left(\dfrac{5\pi}{10}\right)$ $= \left[2\cos\left(\dfrac{\pi}{2}\right)\cos\left(\dfrac{4\pi}{10}\right)\right] + \left[2\cos\left(\dfrac{\pi}{2}\right)\cos\left(\dfrac{2\pi}{10}\right)\right] + 0$ $= 0 = \text{RHS}$
(ii)	$z^{10} + (13z-1)^{10} = 0$ $(13z-1)^{10} = -z^{10}$ $\left(\dfrac{13z-1}{z}\right)^{10} = -1 \quad ...(1)$ $\left(13 - \dfrac{1}{z}\right)^{10} = \mathrm{e}^{\mathrm{i}(\pi + 2k\pi)}$ $13 - \dfrac{1}{z} = \mathrm{e}^{\mathrm{i}\left(\frac{\pi + 2k\pi}{10}\right)}, \quad k = -5, 0, \pm 1, \pm 2, \pm 3, \pm 4$ $\dfrac{1}{z} = 13 - \mathrm{e}^{\mathrm{i}\left(\frac{\pi + 2k\pi}{10}\right)}, \quad k = -5, 0, \pm 1, \pm 2, \pm 3, \pm 4 \quad ...(2)$

s/n	Solution
	Hence $\frac{1}{z_1}=13-e^{i\left(\frac{\pi}{10}\right)}$, $\frac{1}{z_1^*}=13-e^{i\left(-\frac{\pi}{10}\right)}$ $\frac{1}{z_2}=13-e^{i\left(\frac{3\pi}{10}\right)}$, $\frac{1}{z_2^*}=13-e^{i\left(-\frac{3\pi}{10}\right)}$ $\frac{1}{z_3}=13-e^{i\left(\frac{5\pi}{10}\right)}$, $\frac{1}{z_3^*}=13-e^{i\left(-\frac{5\pi}{10}\right)}$ $\frac{1}{z_4}=13-e^{i\left(\frac{7\pi}{10}\right)}$, $\frac{1}{z_4^*}=13-e^{i\left(-\frac{7\pi}{10}\right)}$ $\frac{1}{z_5}=13-e^{i\left(\frac{9\pi}{10}\right)}$, $\frac{1}{z_5^*}=13-e^{i\left(-\frac{9\pi}{10}\right)}$ $\frac{1}{z_1 z_1^*}=\frac{1}{z_1}.\frac{1}{z_1^*}=[13-e^{i\left(\frac{\pi}{10}\right)}][13-e^{i\left(-\frac{\pi}{10}\right)}]$ $=169-13[e^{i\left(\frac{\pi}{10}\right)}+e^{i\left(-\frac{\pi}{10}\right)}]+1$ $=170-26\cos\left(\frac{\pi}{10}\right)$ Similarly, $\frac{1}{z_2 z_2^*}=170-26\cos\left(\frac{3\pi}{10}\right)$ $\frac{1}{z_3 z_3^*}=170-26\cos\left(\frac{5\pi}{10}\right)$ $\frac{1}{z_4 z_4^*}=170-26\cos\left(\frac{7\pi}{10}\right)$ $\frac{1}{z_5 z_5^*}=170-26\cos\left(\frac{9\pi}{10}\right)$ Hence $\frac{1}{z_1 z_1^*}+\frac{1}{z_2 z_2^*}+\frac{1}{z_3 z_3^*}+\frac{1}{z_4 z_4^*}+\frac{1}{z_5 z_5^*}$ $=5(170)-26\left[\cos\left(\frac{\pi}{10}\right)+\cos\left(\frac{3\pi}{10}\right)+\cos\left(\frac{5\pi}{10}\right)+\cos\left(\frac{7\pi}{10}\right)+\cos\left(\frac{9\pi}{10}\right)\right]$ $=850$
6(i)	$I_n=\int_0^t x^n(1+x^2)^{\frac{1}{2}}\,dx$ $=\int_0^t x^{n-1}x(1+x^2)^{\frac{1}{2}}\,dx$ $=\left[x^{n-1}(\frac{1}{3})(1+x^2)^{\frac{3}{2}}\right]_0^t-\int_0^t(n-1)x^{n-2}(\frac{1}{3})(1+x^2)^{\frac{3}{2}}\,dx$ $=\frac{1}{3}t^{n-1}(1+t^2)^{\frac{3}{2}}-\frac{n-1}{3}\int_0^t x^{n-2}(1+x^2)(1+x^2)^{\frac{1}{2}}\,dx$ $=\frac{1}{3}t^{n-1}(1+t^2)^{\frac{3}{2}}-\frac{n-1}{3}\int_0^t x^{n-2}(1+x^2)^{\frac{1}{2}}\,dx-\frac{n-1}{3}\int_0^t x^n(1+x^2)^{\frac{1}{2}}\,dx$ $=\frac{1}{3}t^{n-1}(1+t^2)^{\frac{3}{2}}-\frac{n-1}{3}I_{n-2}-\frac{n-1}{3}I_n$

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	$\therefore\ \frac{n+2}{3} I_n = \frac{1}{3} t^{n-1}(1+t^2)^{\frac{3}{2}} - \frac{n-1}{3} I_{n-2}$ $\therefore\ (n+2)I_n = t^{n-1}(1+t^2)^{\frac{3}{2}} - (n-1)I_{n-2}$ (shown)
(ii)	$\frac{\mathrm{d}}{\mathrm{d}x}\ln(x+\sqrt{1+x^2}) = \frac{1+\frac{1}{2}\frac{(2x)}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}}$ $\int_0^{\frac{4}{3}} \sqrt{1+x^2}\,\mathrm{d}x = \left[x\sqrt{1+x^2}\right]_0^{\frac{4}{3}} - \int_0^{\frac{4}{3}} \frac{x^2}{\sqrt{1+x^2}}\,\mathrm{d}x$ $= \frac{20}{9} + \int_0^{\frac{4}{3}} \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x - \int_0^{\frac{4}{3}} \frac{1+x^2}{\sqrt{1+x^2}}\,\mathrm{d}x$ $= \frac{20}{9} + \int_0^{\frac{4}{3}} \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x - \int_0^{\frac{4}{3}} \sqrt{1+x^2}\,\mathrm{d}x$ $2\int_0^{\frac{4}{3}} \sqrt{1+x^2}\,\mathrm{d}x = \frac{20}{9} + \left[\ln(x+\sqrt{1+x^2})\right]_0^{\frac{4}{3}}$ $= \frac{20}{9} + \ln\left(\frac{4}{3}+\frac{5}{3}\right)$ $\therefore\ \int_0^{\frac{4}{3}} \sqrt{1+x^2}\,\mathrm{d}x = \frac{10}{9} + \frac{1}{2}\ln 3$ where $m = \frac{10}{9},\ n = \frac{1}{2}$
(iii)	$y = \frac{1}{\cos 2\theta + 1} = \frac{1}{2\cos^2\theta} = \frac{1}{2}\sec^2\theta = \frac{1}{2}(1+x^2)$ Surface area of revolution about x-axis $= \int_0^{\frac{4}{3}} 2\pi y\left[1+(\frac{\mathrm{d}y}{\mathrm{d}x})^2\right]^{\frac{1}{2}}\mathrm{d}x$ $= \pi\int_0^{\frac{4}{3}} (1+x^2)(1+x^2)^{\frac{1}{2}}\,\mathrm{d}x$ $= \pi\left[\int_0^{\frac{4}{3}} (1+x^2)^{\frac{1}{2}}\,\mathrm{d}x + \int_0^{\frac{4}{3}} x^2(1+x^2)^{\frac{1}{2}}\,\mathrm{d}x\right]$ $= \pi\left[\int_0^{\frac{4}{3}} (1+x^2)^{\frac{1}{2}}\,\mathrm{d}x + \frac{1}{4}\left[\frac{4}{3}\left[1+\left(\frac{4}{3}\right)^2\right]^{\frac{3}{2}} - \int_0^{\frac{4}{3}} (1+x^2)^{\frac{1}{2}}\,\mathrm{d}x\right]\right]$ $= \pi\left[\frac{125}{81} + \frac{3}{4}\int_0^{\frac{4}{3}} (1+x^2)^{\frac{1}{2}}\,\mathrm{d}x\right]$ $= \pi\left[\frac{125}{81} + \frac{3}{4}\left(\frac{10}{9} + \frac{1}{2}\ln 3\right)\right]$ $= \pi\left[\frac{385}{162} + \frac{3}{8}\ln 3\right]$
7(i)	$\frac{\mathrm{d}^2 r}{\mathrm{d}\theta^2} + 4r = 5\sin 3\theta \quad \ldots(1)$ Characteristic equation is $m^2 + 4 = 0$ $\therefore\ m = \pm 2\mathrm{i}$ Hence complementary function is $r = A\cos 2\theta + B\sin 2\theta,\ A, B \in \mathbb{R}$. Let a particular integral be $r = \lambda\sin 3\theta + \mu\cos 3\theta,\ \lambda, \mu \in \mathbb{R}$.

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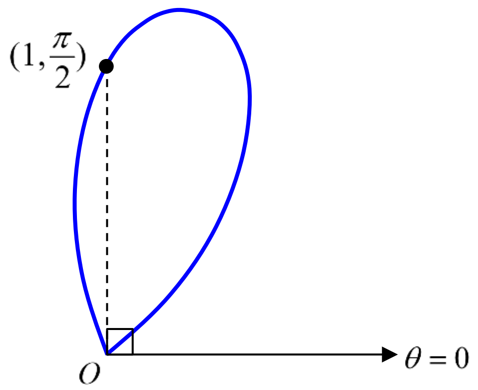
	$\therefore \dfrac{\mathrm{d}r}{\mathrm{d}\theta} = 3\lambda\cos 3\theta - 3\mu\sin 3\theta \quad ...(2)$ $\dfrac{\mathrm{d}^2r}{\mathrm{d}\theta^2} = -9\lambda\sin 3\theta - 9\mu\cos 3\theta \quad ...(3)$ Substitute (2), (3) into (1): $(-9\lambda\sin 3\theta - 9\mu\cos 3\theta) + 4(\lambda\sin 3\theta + \mu\cos 3\theta) = 5\sin 3\theta$ Comparing coefficients: $-9\lambda + 4\lambda = 5$ $-9\mu + 4\mu = 0$ $\therefore\ \lambda = -1,\ \mu = 0$ Hence a particular integral is $r = -\sin 3\theta$ $\therefore\ r = A\cos 2\theta + B\sin 2\theta - \sin 3\theta$ $\dfrac{\mathrm{d}r}{\mathrm{d}\theta} = 2A\sin 2\theta + 2B\cos 2\theta + 3\cos 3\theta$ When $\theta = \dfrac{\pi}{2}$, $r = 1$ and $\dfrac{\mathrm{d}r}{\mathrm{d}\theta} = -2$. $\therefore\ 1 = A\cos\pi + B\sin\pi - \sin\dfrac{3\pi}{2} \Rightarrow A = 0$ $-2 = 2A\sin\pi + 2B\cos\pi + 3\cos\dfrac{3\pi}{2} \Rightarrow B = 1$ Hence $r = \sin 2\theta - \sin 3\theta$
(ii)	
(iii)	Area $= \frac{1}{2}\int_{\frac{\pi}{5}}^{\frac{3\pi}{5}} (\sin 2\theta - \sin 3\theta)^2\,\mathrm{d}\theta$ $= \frac{1}{2}\int_{\frac{\pi}{5}}^{\frac{3\pi}{5}} \sin^2 2\theta - 2\sin 2\theta\sin 3\theta + \sin^2 3\theta\,\mathrm{d}\theta$ $= \frac{1}{2}\int_{\frac{\pi}{5}}^{\frac{3\pi}{5}} \left(\frac{1-\cos 4\theta}{2}\right) + (\cos 5\theta - \cos\theta) + \left(\frac{1-\cos 6\theta}{2}\right)\mathrm{d}\theta$ $= \frac{1}{4}\int_{\frac{\pi}{5}}^{\frac{3\pi}{5}} 2 - \cos 4\theta + 2\cos 5\theta - 2\cos\theta - \cos 6\theta\,\mathrm{d}\theta$ $= \frac{1}{4}\left[2\theta - \frac{\sin 4\theta}{4} + \frac{2\sin 5\theta}{5} - 2\sin\theta - \frac{\sin 6\theta}{6}\right]_{\frac{\pi}{5}}^{\frac{3\pi}{5}}$ $= \frac{1}{4}\left[\begin{array}{l}\left(\frac{6\pi}{5} - \frac{\sin\frac{12\pi}{5}}{4} + 0 - 2\sin\frac{3\pi}{5} - \frac{\sin\frac{18\pi}{5}}{6}\right) \\ -\left(\frac{2\pi}{5} - \frac{\sin\frac{4\pi}{5}}{4} + 0 - 2\sin\frac{\pi}{5} - \frac{\sin\frac{6\pi}{5}}{6}\right)\end{array}\right]$

s/n	Solution
	$=\frac{1}{4}\left[\begin{matrix}\left(\frac{4\pi}{5}-\frac{1}{4}\sin\frac{2\pi}{5}-2\sin\frac{2\pi}{5}+\frac{1}{6}\sin\frac{2\pi}{5}\right)\\-\left(-\frac{1}{4}\sin\frac{\pi}{5}-2\sin\frac{\pi}{5}+\frac{1}{6}\sin\frac{\pi}{5}\right)\end{matrix}\right]$ $=\frac{1}{4}\left[\frac{4\pi}{5}-\frac{25}{12}\sin\frac{2\pi}{5}+\frac{25}{12}\sin\frac{\pi}{5}\right]$ $=\frac{\pi}{5}+\frac{25}{48}\left[\sin\frac{\pi}{5}-\sin\frac{2\pi}{5}\right]$ unit^2 where $a=\frac{1}{5}$, $b=\frac{25}{48}$
8(i)	$R_n=\frac{3}{2}R_{n-1}-\frac{1}{2}W_{n-1} \Rightarrow W_{n-1}=3R_{n-1}-2R_n \quad \ldots(1)$ Substitute (1) into $W_n=\frac{3}{4}R_{n-1}+\frac{1}{4}W_{n-1}$: $W_n=\frac{3}{4}R_{n-1}+\frac{1}{4}(3R_{n-1}-2R_n)=\frac{3}{2}R_{n-1}-\frac{1}{2}R_n$ Hence $W_{n-1}=\frac{3}{2}R_{n-2}-\frac{1}{2}R_{n-1} \quad \ldots(2)$ Substitute (2) into $R_n=\frac{3}{2}R_{n-1}-\frac{1}{2}W_{n-1}$: $\therefore\ R_n=\frac{3}{2}R_{n-1}-\frac{1}{2}\left(\frac{3}{2}R_{n-2}-\frac{1}{2}R_{n-1}\right)$ $=\frac{7}{4}R_{n-1}-\frac{3}{4}R_{n-2}$
(ii)	The original equations can be written in matrices as follow. $\begin{pmatrix}R_n\\W_n\end{pmatrix}=\begin{pmatrix}\frac{3}{2}&-\frac{1}{2}\\\frac{3}{4}&\frac{1}{4}\end{pmatrix}\begin{pmatrix}R_{n-1}\\W_{n-1}\end{pmatrix}$ Hence $\begin{pmatrix}R_n\\W_n\end{pmatrix}=\begin{pmatrix}\frac{3}{2}&-\frac{1}{2}\\\frac{3}{4}&\frac{1}{4}\end{pmatrix}^n\begin{pmatrix}R_0\\W_0\end{pmatrix}$ Characteristic equation for $\begin{pmatrix}\frac{3}{2}&-\frac{1}{2}\\\frac{3}{4}&\frac{1}{4}\end{pmatrix}$ is $\begin{vmatrix}\lambda-\frac{3}{2}&\frac{1}{2}\\-\frac{3}{4}&\lambda-\frac{1}{4}\end{vmatrix}=0$ $\left(\lambda-\frac{3}{2}\right)\left(\lambda-\frac{1}{4}\right)-\left(\frac{1}{2}\right)\left(-\frac{3}{4}\right)=0$ $\lambda^2-\frac{7}{4}\lambda+\frac{3}{4}=0$ $(\lambda-1)(\lambda-\frac{3}{4})=0$ $\therefore\ \lambda=1$ or $\lambda=\frac{3}{4}$ For $\lambda=1$, solving $\begin{pmatrix}-\frac{1}{2}&\frac{1}{2}\\-\frac{3}{4}&\frac{3}{4}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ by GC, $\begin{pmatrix}x\\y\end{pmatrix}=k_1\begin{pmatrix}1\\1\end{pmatrix}$ $\therefore$ corresponding eigenvector is $\begin{pmatrix}1\\1\end{pmatrix}$ For $\lambda=\frac{3}{4}$, solving $\begin{pmatrix}-\frac{3}{4}&\frac{1}{2}\\-\frac{3}{4}&\frac{1}{2}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ by GC, $\begin{pmatrix}x\\y\end{pmatrix}=k_2\begin{pmatrix}2\\3\end{pmatrix}$

s/n	Solution
	$\therefore$ corresponding eigenvector is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ Hence $\begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{4} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}^{-1}$ $\therefore \begin{pmatrix} R_n \\ W_n \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}^n \begin{pmatrix} R_0 \\ W_0 \end{pmatrix}$ $= \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{4} \end{pmatrix}^n \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}^{-1}\begin{pmatrix} R_0 \\ W_0 \end{pmatrix}$ $= \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (\frac{3}{4})^n \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}^{-1}\begin{pmatrix} R_0 \\ W_0 \end{pmatrix}$ $= \mathrm{PDP}^{-1}\begin{pmatrix} R_0 \\ W_0 \end{pmatrix}$ where $\mathrm{P} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, $\mathrm{D} = \begin{pmatrix} 1 & 0 \\ 0 & (\frac{3}{4})^n \end{pmatrix}$ (shown)
(iii)	From (ii), $\begin{pmatrix} R_n \\ W_n \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (\frac{3}{4})^n \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}^{-1}\begin{pmatrix} R_0 \\ W_0 \end{pmatrix}$ $= \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (\frac{3}{4})^n \end{pmatrix}\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} R_0 \\ W_0 \end{pmatrix}$ $= \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ -(\frac{3}{4})^n & (\frac{3}{4})^n \end{pmatrix}\begin{pmatrix} R_0 \\ W_0 \end{pmatrix}$ $= \begin{pmatrix} 3-2(\frac{3}{4})^n & -2+2(\frac{3}{4})^n \\ 3-3(\frac{3}{4})^n & -2+3(\frac{3}{4})^n \end{pmatrix}\begin{pmatrix} R_0 \\ W_0 \end{pmatrix}$ Hence $R_n = 3R_0 - 2(\frac{3}{4})^n R_0 - 2W_0 + 2(\frac{3}{4})^n W_0$ $= (2W_0 - 2R_0)(\frac{3}{4})^n + (3R_0 - 2W_0)$ $W_n = 3R_0 - 3(\frac{3}{4})^n R_0 - 2W_0 + 3(\frac{3}{4})^n W_0$ $= (3W_0 - 3R_0)(\frac{3}{4})^n + (3R_0 - 2W_0)$
(iv)	From (iii), $R_n = (2W_0 - 2R_0)(\frac{3}{4})^n + (3R_0 - 2W_0)$ $W_n = (3W_0 - 3R_0)(\frac{3}{4})^n + (3R_0 - 2W_0)$ As $n \to \infty$, both $R_n \to 3R_0 - 2W_0$ and $W_n \to 3R_0 - 2W_0$. Since equilibrium is non-zero, $3R_0 - 2W_0 > 0 \Rightarrow R_0 > \frac{2}{3}W_0 \quad \ldots(*)$ Also, as both R_n and W_n are decreasing, $(2W_0 - 2R_0)(\frac{3}{4})^n \to 0^+$ and $(3W_0 - 3R_0)(\frac{3}{4})^n \to 0^+$ Hence $R_0 < W_0 \quad \ldots(**)$ Combining (*) and (**): $\therefore \frac{2}{3}W_0 < R_0 < W_0$

9(i)	Using $u_{n+2} = 2u_{n+1} - 2u_n$, $u_n = 2u_{n-1} - 2u_{n-2}$ $= 2(2u_{n-2} - 2u_{n-3}) - 2u_{n-2}$ $= 2u_{n-2} - 4u_{n-3}$ $= 2(2u_{n-3} - 2u_{n-4}) - 4u_{n-3}$ $= -4u_{n-4}$ Hence $u_{n+4} = -4u_n$ (shown)
(ii)	From (i), $u_{n+4} = -4u_n$ $\sum_{r=1}^{4N} u_r = u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + u_7 + u_8$ $+ u_9 + u_{10} + u_{11} + u_{12} + \ldots + u_{4N-3} + u_{4N-2} + u_{4N-1} + u_{4N}$ $= (u_1 + u_2 + u_3 + u_4) + (-4)(u_1 + u_2 + u_3 + u_4)$ $+ (-4)^2(u_1 + u_2 + u_3 + u_4) + \ldots + (-4)^{N-1}(u_1 + u_2 + u_3 + u_4)$ $\therefore \sum_{r=1}^{4N} u_r$ is sum of a GP with first term $(u_1 + u_2 + u_3 + u_4)$, common ratio -4 and N terms. $u_1 + u_2 + u_3 + u_4 = a + ka + (2ka - 2a) + [2(2ka - 2a) - 2ka]$ $= 5ka - 5a$ Hence $\sum_{r=1}^{4N} u_r = \frac{(5ka-5a)[1-(-4)^N]}{1-(-4)} = (ka - a)[1 - (-4)^N]$ **Alternative Method**

$$\sum_{r=1}^{4N} u_r = u_1 + u_2 + \sum_{r=3}^{4N} u_r$$

$$= a + ka + \sum_{r=1}^{4N-2} u_{r+2}$$

$$= a + ka + \sum_{r=1}^{4N-2} \left(2u_{r+1} - 2u_r\right)$$

$$= a + ka + 2\begin{pmatrix} \cancel{u_2} & - & u_1 \\ \cancel{u_3} & - & \cancel{u_2} \\ & \vdots & \\ \cancel{u_{4N-2}} & - & \cancel{u_{4N-3}} \\ u_{4N-1} & - & \cancel{u_{4N-2}} \end{pmatrix}$$

$$= a + ka + 2\left(u_{4N-1} - u_1\right)$$

$$= a + ka - 2a + 2u_{4N-1}$$

$$= ka - a + 2\left((-4)^{N-1} u_3\right)$$

$$= ka - a + 2(-4)^{N-1}\left(2ka - 2a\right)$$

$$= ka - a + 4(-4)^{N-1}\left(ka - a\right)$$

$$= \left(ka - a\right) - (-4)^N\left(ka - a\right)$$

$$= \left(ka - a\right)\left(1 - (-4)^N\right)$$

(iii)

$u_{n+2} = 2u_{n+1} - 2u_n \Rightarrow u_{n+2} - 2u_{n+1} + 2u_n = 0$

Characteristic equation is $m^2 - 2m + 2 = 0$

$\therefore\ m = \frac{2 \pm \sqrt{4-4(2)}}{2} = 1 \pm \mathrm{i} = \sqrt{2}\left(\cos\frac{\pi}{4} \pm \mathrm{i}\sin\frac{\pi}{4}\right)$

Hence general solution is

$u_n = (\sqrt{2})^n\left(A\cos\frac{n\pi}{4} + B\sin\frac{n\pi}{4}\right)$, where $p = \sqrt{2}$, $q = \frac{\pi}{4}$

When $n = 1$, $u_1 = A + B = a$

When $n = 2$, $u_2 = 2B = ka$

$\therefore\ B = \frac{ka}{2}$, $A = a - \frac{ka}{2} = a(1 - \frac{k}{2})$

Hence $u_n = (\sqrt{2})^n\left[a(1-\frac{k}{2})\cos\frac{n\pi}{4} + \frac{ka}{2}\sin\frac{n\pi}{4}\right]$

$$\sum_{r=1}^{20} p^r(A\cos qr + B\sin qr) = (ka-a)[1-(-4)^5]$$

$$= 1025(ka-a)$$

10(i)

Let $f(x,y) = 0.2x - 0.01xy$,

$g(x,y) = -0.05y + 0.004xy$

i	t_i	x_i	y_i	$f(x_i, y_i)$	$g(x_i, y_i)$
0	0	10	15	0.5	–0.15
1	0.5	$10 + 0.5(0.5)$ $= 10.25$	$15 + 0.5(-0.15)$ $= 14.925$	0.5201875	–0.134325

2	1	10.25 $+0.5(0.5201875)$ $=10.51$ (2 d.p.)	14.925 $+0.5(-0.134325)$ $=14.86$ (2 d.p.)		

After 1 month, share price of company X is \$10.51 and company Y's share price is \$14.86.

(ii)

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{-0.05y+0.004xy}{0.2x-0.01xy}=\frac{y(-0.05+0.004x)}{x(0.2-0.01y)}$$

$$\int \frac{0.2-0.01y}{y}\,dy=\int \frac{-0.05+0.004x}{x}\,dx$$

$0.2\ln y-0.01y=-0.05\ln x+0.004x+C$

$y^{0.2}\,e^{-0.01y}=Kx^{-0.05}\,e^{0.004x}$, where $K=e^{C}$

Initially, $x=10$ and $y=15$.

$\therefore\ 15^{0.2}\,e^{-0.01(15)}=K(10^{-0.05})\,e^{0.004(10)}$

$K=1.59478558$

$=1.59$ (3 s.f.)

Hence $y^{0.2}\,e^{-0.01y}=1.59x^{-0.05}\,e^{0.004x}$ (shown)

where $g=0.2$, $h=-0.01$, $K=1.59$, $m=-0.05$, $n=0.004$

(iii)

At points A and C, $\frac{dy}{dx}=0$.

$$\therefore\ \frac{y(-0.05+0.004x)}{x(0.2-0.01y)}=0$$

$y(-0.05+0.004x)=0$

$\therefore\ x=12.5$

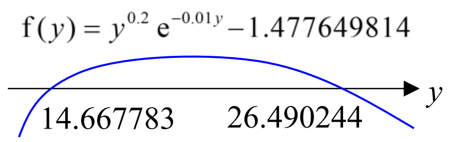
When $x=12.5$,

$y^{0.2}\,e^{-0.01y}=1.59478558(12.5)^{-0.05}\,e^{0.004(12.5)}$

$=1.477649814$

Using GC,

$y=14.67$ or $y=26.49$



Hence

$A(12.5, 26.49)$

$C(12.5, 14.67)$

At points B and D, $\frac{dx}{dy}=0$.

$$\therefore\ \frac{x(0.2-0.01y)}{y(-0.05+0.004x)}=0$$

$x(0.2-0.01y)=0$

$\therefore\ y=20$

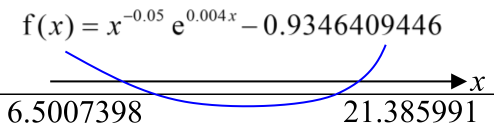
When $y=20$,

$20^{0.2}\,e^{-0.01(20)}=1.59478558x^{-0.05}\,e^{0.004x}$

$\therefore\ x^{-0.05}\,e^{0.004x}=0.9346409446$

Using GC,

$x=6.50$ or $x=21.39$



Hence

	$B(21.39, 20)$ $D(6.50, 20)$
(iv)	The values of x and y varies with each other in a cyclical manner with the values of x varying between \$6.50 and \$21.39, and the values of y varying between \$14.67 and \$26.49.