RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 5

Tutorial 1: Basics

Section A (Basic Questions)

1 Find the range of values of k for the line $y = x - 2k$ to intersect the curve $2y^2 - x^2 = 8$ at two distinct points.

[$\mathbb{R}$]

Solution

To find the intersection between the line and the curve, we substitute $y = x - 2k$ into $2y^2 - x^2 = 8$,

$$2(x-2k)^2 - x^2 = 8$$

$$x^2 - 8kx + 8k^2 - 8 = 0$$

For line to intersect curve at 2 distinct points, the above equation must have 2 distinct roots

$\Rightarrow$ discriminant = $64k^2 - 32(k^2 - 1) > 0$

$\Rightarrow \quad k^2 + 1 > 0$

Since $k^2 \geq 0$ for all real k, $k^2 + 1 \geq 1 > 0$ for all real k.

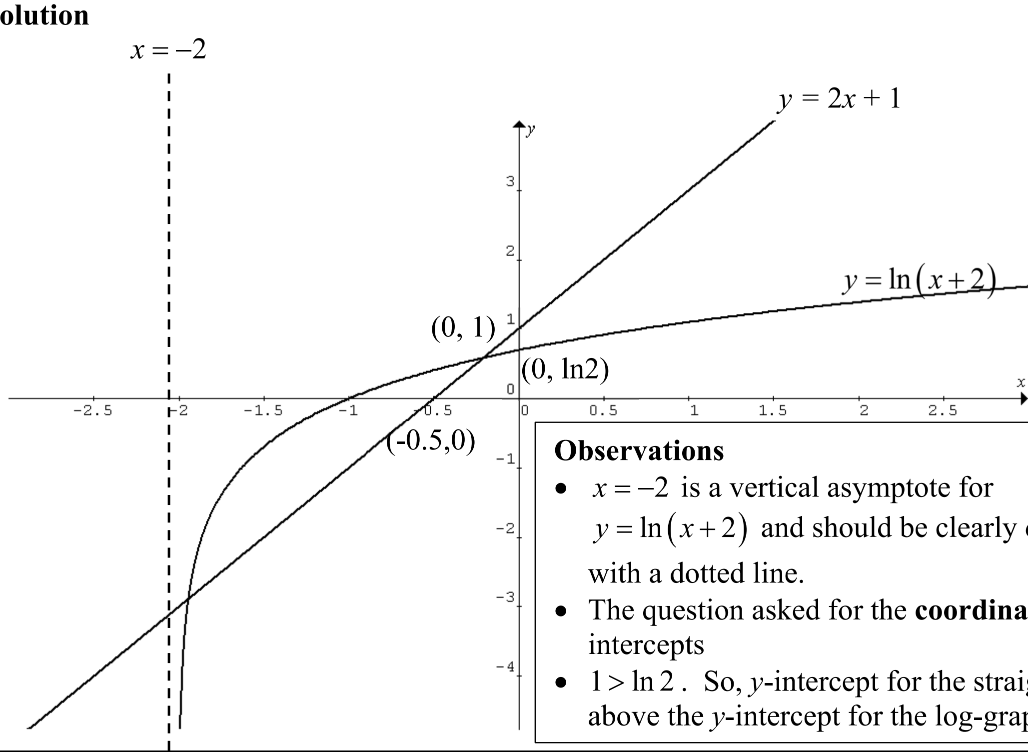
That is to say, any (real) values of k will satisfy the inequality $k^2 + 1 > 0$.

$\therefore$ Required range of values of k is $\mathbb{R}$.

2 Sketch the graph of $y = \ln(x+2)$ for $x > -2$, showing clearly the equations of any asymptotes and the coordinates of any intersections with the axes. By drawing a suitable straight line graph on the same diagram, determine the number of solutions to the equation $e^{2x+1} - x = 2$.

[2]

Solution



Observations

- $x = -2$ is a vertical asymptote for $y = \ln(x+2)$ and should be clearly drawn with a dotted line.
- The question asked for the **coordinates** of the intercepts
- $1 > \ln 2$. So, y-intercept for the straight line is above the y-intercept for the log-graph.

$e^{2x+1} - x = 2$

$e^{2x+1} = x + 2$

$2x + 1 = \ln(x+2)$

Draw $y = 2x + 1$

Since there are 2 points of intersection between the two graphs, there are 2 solutions to the equation $e^{2x+1} - x = 2$.

3 Express $2\sqrt{2}\sin\theta + 2\sqrt{2}\cos\theta$ in the form $R\sin(\theta+\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

$[4\sin\left(\theta + \frac{\pi}{4}\right)]$

Solution

$2\sqrt{2}\sin\theta + 2\sqrt{2}\cos\theta = R\sin(\theta+\alpha) = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

Comparing, we have

$R\cos\alpha = 2\sqrt{2}$ ---------(1)

$R\sin\alpha = 2\sqrt{2}$ ---------(2)

$(1)^2 + (2)^2 \qquad R^2 = 8 + 8$

$R = 4$

$\frac{(2)}{(1)} \qquad \tan\alpha = 1 \qquad \Rightarrow \qquad \alpha = \frac{\pi}{4}$

Thus $2\sqrt{2}\sin\theta + 2\sqrt{2}\cos\theta = 4\sin\left(\theta + \frac{\pi}{4}\right)$.

4 Given that $\cos A = p$ and $270^\circ \le A \le 360^\circ$, express each of the following in terms of p.

(a) $\sin A$,

(b) $\sin 2A$,

(c) $\cos\frac{A}{2}$.

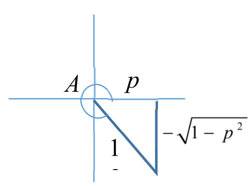
[(a) $-\sqrt{1-p^2}$ (b) $-2p\sqrt{1-p^2}$ (c) $-\sqrt{\frac{p+1}{2}}$]

Solution

(a) $\sin A = -\sqrt{1-p^2}$

(b) $\sin 2A = 2\sin A\cos A$

$= -2p\sqrt{1-p^2}$



Observation

- A lies in the 4th quadrant. So, $\cos A > 0$ while $\sin A < 0$ and $\tan A < 0$. Which also means that $p > 0$.

(c) $\cos A = 2\cos^2\frac{A}{2} - 1 \Rightarrow \cos\frac{A}{2} = \pm\sqrt{\frac{\cos A + 1}{2}}$

But $135^\circ \le \frac{A}{2} \le 180^\circ$ (i.e. $\frac{A}{2}$ lies in the 2nd quadrant)

$\therefore \cos\frac{A}{2} = -\sqrt{\frac{\cos A + 1}{2}} = -\sqrt{\frac{p+1}{2}}$.

5 $ABCD$ is a parallelogram with $\overrightarrow{AB}=\begin{pmatrix}3\\4\end{pmatrix}$. Coordinates of points A and C are $(1,1)$ and $(8,8)$ respectively.

(i) Find the coordinates of point D.

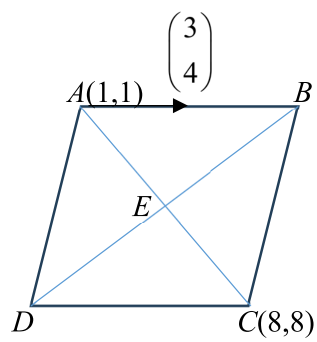
(ii) AC and BD intersect at the point E. Find the coordinates of E.

(iii) Find the length of the two sides of the parallelogram AB and BC. Deduce the geometrical relationship between A, B, C and D, and state the relationship between the two diagonals.

[(i) (5,4) (ii) (4.5,4.5) (iii) 5]

Solution

(i) Since $ABCD$ is a parallelogram, $\overrightarrow{DC}=\overrightarrow{AB}=\begin{pmatrix}3\\4\end{pmatrix}$.



$$\overrightarrow{OD}=\overrightarrow{OC}-\overrightarrow{DC}=\begin{pmatrix}8\\8\end{pmatrix}-\begin{pmatrix}3\\4\end{pmatrix}=\begin{pmatrix}5\\4\end{pmatrix}$$

Coordinates of D are (5,4).

(ii) Diagonals of a parallelogram bisect each other, so point E is the midpoint of AC and BD.

$$\overrightarrow{AE}=\frac{1}{2}\overrightarrow{AC}=\frac{1}{2}\left[\begin{pmatrix}8\\8\end{pmatrix}-\begin{pmatrix}1\\1\end{pmatrix}\right]=\begin{pmatrix}3.5\\3.5\end{pmatrix}\Rightarrow\overrightarrow{OE}=\overrightarrow{OA}+\overrightarrow{AE}=\begin{pmatrix}1\\1\end{pmatrix}+\begin{pmatrix}3.5\\3.5\end{pmatrix}=\begin{pmatrix}4.5\\4.5\end{pmatrix}$$

Alternatively, $\overrightarrow{OE}=\frac{1}{2}\left(\overrightarrow{OA}+\overrightarrow{OC}\right)=\frac{1}{2}\left[\begin{pmatrix}1\\1\end{pmatrix}+\begin{pmatrix}8\\8\end{pmatrix}\right]=\begin{pmatrix}4.5\\4.5\end{pmatrix}$

Coordinates of E are (4.5,4.5).

(iii) $$\left|\overrightarrow{AB}\right|=\left|\overrightarrow{DC}\right|=\left|\begin{pmatrix}3\\4\end{pmatrix}\right|=\sqrt{3^2+4^2}=5$$

$$\left|\overrightarrow{BC}\right|=\left|\overrightarrow{AD}\right|=\left|\begin{pmatrix}5\\4\end{pmatrix}-\begin{pmatrix}1\\1\end{pmatrix}\right|=\left|\begin{pmatrix}4\\3\end{pmatrix}\right|=\sqrt{4^2+3^2}=5$$

Since $AB=DC=BC=AD$, $ABCD$ is a rhombus, and the diagonals are perpendicular to each other.

6 Solve the equation $|2x+3|=5$ using 2 different methods, namely:

(i) numerically, **(ii)** graphically.

[$x=-4$ or 1]

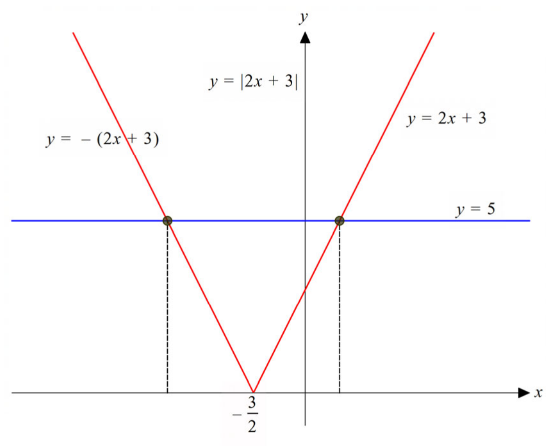
Solution

(i) $|2x+3|=5$

$2x+3=-5$ or 5

$x=-4$ or 1

(ii) To solve the equation graphically, we draw the graphs of $y=|2x+3|$ and $y=5$ on the same diagram and find the points of intersection.



From the graphs, we can see that they intersect at 2 points.

$$y=|2x+3|$$
$$=\begin{cases} 2x+3, & \text{if } 2x+3\geq 0 \\ -(2x+3), & \text{if } 2x+3<0 \end{cases}$$
$$=\begin{cases} 2x+3, & \text{if } x\geq -\frac{3}{2} \\ -(2x+3), & \text{if } x<-\frac{3}{2} \end{cases}$$

So, we consider $-(2x+3)=5$ or $2x+3=5$

$x=-4$ or $x=1$

7 Given that $|x|=4$, find the possible values of $|1-3x|$.

[11, 13]

Solution

$|x|=4 \quad \Rightarrow \quad x=\pm 4$

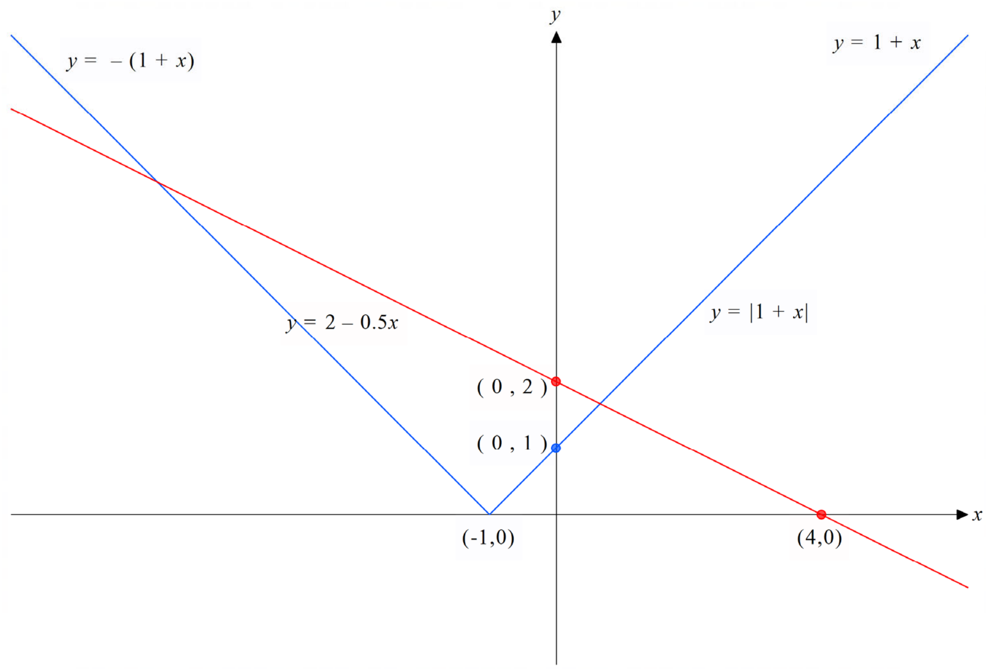
When $x=4$, $|1-3x|=|1-3(4)|=|-11|=11$

When $x=-4$, $|1-3x|=|1-3(-4)|=|13|=13$

The possible values of $|1-3x|$ is 11 and 13.

8 Draw the graphs of $y=|x+1|$ and $y=2-\frac{1}{2}x$ on the same diagram. Hence, solve the equation $|x+1|=2-\frac{1}{2}x$.

Solution



From the diagram, the 2 graphs intersect at 2 points - 1 point on the right of $(-1,0)$ and 1 point on the left of $(-1,0)$.

$$|x+1|=\begin{cases} x+1, & \text{if } x+1\geq 0 \\ -(x+1), & \text{if } x+1<0 \end{cases}$$
$$=\begin{cases} x+1, & \text{if } x\geq -1 \\ -(x+1), & \text{if } x<-1 \end{cases}$$

For the point of intersection on the right of $(-1,0)$, $x+1=2-\frac{1}{2}x$

$$\frac{3}{2}x=1$$
$$x=\frac{2}{3}$$

For the point of intersection on the left of $(-1,0)$, $-(x+1)=2-\frac{1}{2}x$

$$\frac{1}{2}x=-3$$
$$x=-6$$

The solution to the equation $|x+1|=2-\frac{1}{2}x$ is $x=\frac{2}{3}$ or -6.