1 The integral I_n, where n is a non-negative integer, is defined by $I_n = \int_0^{\frac{1}{2}} (1-2x)^n e^x \, dx$.

(a) Show that, for $n \geq 1$,

$$I_n = -1 + 2nI_{n-1}.$$ [3]

(b) Find the exact values of

(i) I_3, [3]

(ii) $\int_0^{\frac{1}{2}} (1-2x)^4 e^x \, dx$. [2]

(c) Without using a graphing calculator, deduce that $\lim_{n \to \infty} I_n = 0$. [3]

(d) Hence, find the exact value of $\frac{1}{2} + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{6}\right) + \ldots$. [3]

2 **(a)** For positive integers m, r and s,

(i) Show that $\binom{2m}{m} < 2^{2m}$ by considering $(1+x)^{2m}$. [3]

(ii) With $r < s$, $\mathrm{P}_{r,s}$ is defined as follows:

$\mathrm{P}_{r,s}$ is the product of all prime numbers greater than r and less than or equal to s, if there are such primes; if there are no such primes then $\mathrm{P}_{r,s} = 1$.

Show that, for any positive integer m, $\mathrm{P}_{m+1,2m}$ divides $\binom{2m}{m}$. [3]

(b) For positive numbers a, b, c, x, y and z,

(i) Show that $a+b+c \leq 2\left(\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b}\right)$, using the Cauchy-Schwarz Inequality. [3]

(ii) Hence, given that $xyz = 8$, find the minimum value of

$$\frac{yz}{x^2(y+z)} + \frac{zx}{y^2(z+x)} + \frac{xy}{z^2(x+y)}.$$

State the necessary condition for this value to occur. [5]

3 The nth term of a Fibonacci sequence, f_n is defined by the following recurrence relation

$$f_{n+1} = f_n + f_{n-1} \quad \text{and} \quad f_1 = f_2 = 1.$$

Let $r_n = \dfrac{f_{n+1}}{f_n}$ be the sequence of ratios of consecutive Fibonacci numbers.

(a) Prove that $f_n f_{n+2} - f_{n+1}^2 = (-1)^{n+1}$ for all positive integer n. [4]

(b) Prove that $r_2, r_4, r_6, r_8, \ldots$ and $r_1, r_3, r_5, r_7, \ldots$ are decreasing and increasing bounded sequences respectively. [4]

(c) Prove that the sequence r_n converges and find its exact limit. [6]

4 Let $S = \{1, 2, 3, \ldots, 500\}$. Find

(a) the number of 2-element subsets of S ; [1]

(b) the number of 2-element subsets $\{x, y\}$ of S such that the product xy is a multiple of 3 ; [4]

(c) the number of 2-element subsets $\{x, y\}$ of S such that $(x+y)$ is a multiple of 3. [5]

5 On the Argand diagram, the points A, B, C and D represent the complex numbers a, b, c and d respectively such that $ABCD$ is a quadrilateral.

(a) Show algebraically that for any complex numbers z and w,

$$|z+w| \leq |z| + |w|$$

and state the necessary condition for equality to hold in terms of $\arg(z)$ and $\arg(w)$. [3]

(b) Prove that $|a-b||c-d| + |b-c||a-d| \geq |a-c||b-d|$. [3]

(c) When $|a-b||c-d| + |b-c||a-d| = |a-c||b-d|$, prove that the sum of the angles BAD and BCD is π. [5]

Use the information in the mathematical text to answer Question 6. You should read the whole mathematical text before you start answering the questions.

Combinatorial Proof of Fibonomial Identity

The Fibonacci series is defined by

$$F_1 = F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n \text{ for } n \geq 1.$$

Fibonomial coefficients are defined like binomial coefficients, with integers replaced by their respective Fibonacci numbers. Specifically, for integers $n \geq k \geq 1$,

$$\binom{n}{k}_F = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_1 F_2 \cdots F_k}$$

since $F_n = F_{k+1}F_{n-k} + F_k F_{n-k-1}$ with $F_n = \dfrac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}$ where $\varphi = \dfrac{1+\sqrt{5}}{2}$. Using Pascal's Identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, we have the following analog:

Fibonomial Identity

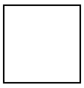
For integer $n \geq 2$,

$$\binom{n}{k}_F = F_{k+1}\binom{n-1}{k}_F + F_{n-k-1}\binom{n-1}{k-1}_F. \qquad (1)$$

Combinatorial Interpretation (Refer to Diagram 1 for Example)

A lattice path from (0, 0) to (a, b) is the shortest path on a $b \times a$ grid where we may only move along the grid lines horizontally or vertically. There may be more than one lattice path from (0, 0) to (a, b).

The diagram below shows an illustration of a unit square and a domino, a rectangular block consisting of two unit squares. Unit squares and dominos are used for tiling rows above a lattice path and columns below the lattice path.



A unit square

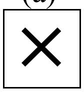
A domino

(a rectangular block consisting of two unit squares)

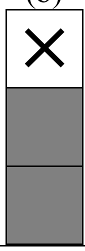
For $a, b \geq 1$, $\binom{a+b}{a}_F$ counts the number of ways of tiling for valid lattice paths from (0, 0) to (a, b). A lattice path from (0, 0) to (a, b) is first drawn, then each row above the lattice path is tiled, followed by each column below the lattice path.

There is no restriction for row tiling, but each column tiling is not allowed to start with a unit square (refer to examples (a) and (b)). Lattice paths are invalid if column tiling starts with a unit square.

(a)



(b)



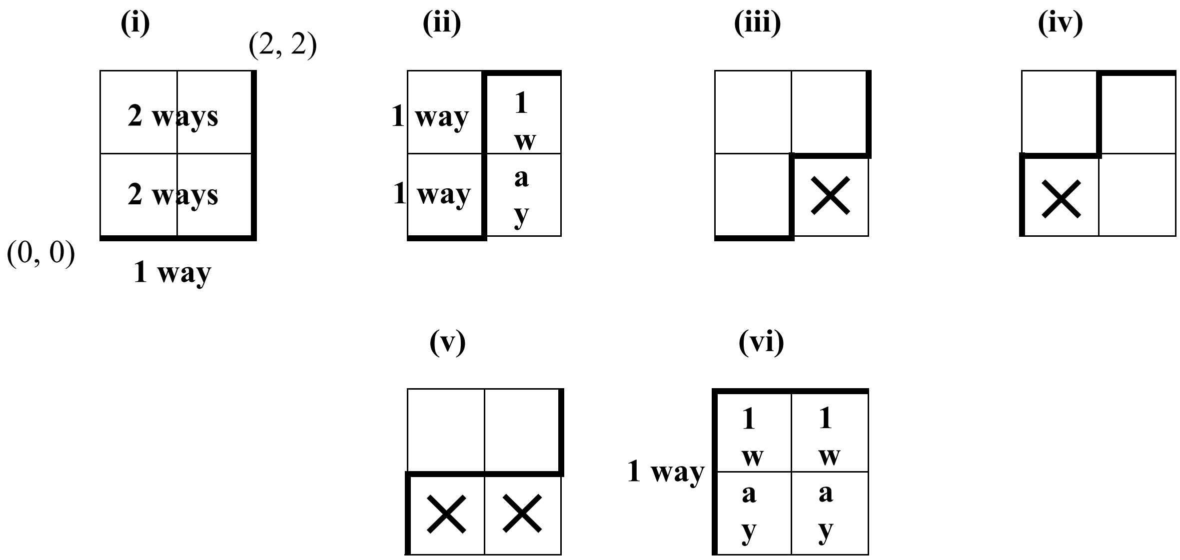
For combinatorial proof of Fibonacci Identity, rewrite equation (1) as follows,

For integer $k, m \geq 1$,

$$\binom{k+m}{k}_F = F_{k+1}\binom{k+m-1}{k}_F + F_{m-1}\binom{k+m-1}{k-1}_F$$

<u>Diagram 1: Illustration of possible lattice paths from (0, 0) to (2,2) and the number of ways of tiling</u>

All possible lattice paths from (0, 0) to (2, 2) are shown below, and there are $\binom{4}{2} = 6$ ways of tiling for valid lattice paths from (0, 0) to (2, 2).



Note: lattice paths **(iii)**, **(iv)** and **(v)** are invalid since each column tiling is not allowed to start with a unit square.

For lattice path **(i)**, there are
- 2 ways to tile both the first and second rows (using two unit squares or one domino),
- no column below the lattice path to tile, so 1 way with empty tiling.

For lattice path **(ii)**, there are
- 1 way to tile both the first and second row (using one unit square),
- 1 way to tile the column (using one domino).

For lattice path **(vi)**, there are
- no row above the lattice path to tile, so 1 way with empty tiling,
- 1 way to tile both the first and second columns (using one domino).

$\therefore$ number of ways of tiling for valid lattice paths $= (2\times2\times1)+(1\times1\times1)+(1\times1\times1) = 6$

$$= \frac{F_4F_3}{F_1F_2} = \binom{4}{2}_F$$

6 **(a)** Prove that $F_n = F_{k+1}F_{n-k} + F_kF_{n-k-1}$ for integers $n \ge k \ge 1$. [5]

(b) Prove that for integers $n \ge k \ge 1$, $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. [2]

(c) Draw the valid lattice paths with tiling for $\binom{5}{3}_F$ and explain why $\binom{5}{3}_F = 15$. [4]

(d) Let T_r be the number of ways to tile a row of r squares with unit squares and dominos for integer $r \ge 1$. Show that $T_r = F_{r+1}$. [3]

(e) Given that the number of ways to tile a column of t squares with unit squares and dominos, with the restriction that the column tilings are not allowed to start with a unit square, is F_{t-1}. Use part **(d)** to prove that

$$\binom{k+m}{k}_F = F_{k+1}\binom{k+m-1}{k}_F + F_{m-1}\binom{k+m-1}{k-1}_F \quad \text{for integers } k, m \ge 1.$$

[3]

End of Paper