1 Show that $p > 0$ for the inequality $px^2 + 1 > 2px - p$ for all real values of x. [3]

$px^2 + 1 > 2px - p \Rightarrow px^2 - 2px + (1 + p) > 0$

Since the quadratic function is positive, then

$D < 0$ and coeff of $x^2 > 0$

$\Rightarrow (-2p)^2 - 4p(1 + p) < 0$ and $p > 0$

$\Rightarrow 4p^2 - 4p - 4p^2 < 0$

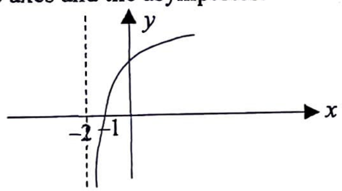
$\Rightarrow -4p < 0$

$\Rightarrow 4p > 0$

$\Rightarrow p > 0$

$\therefore \underline{p > 0}$

2 **(i)** Sketch the graph of $y = \ln(x + 2)$ for $x > -2$, indicating clearly the point where the graph cuts the axes and the asymptotes. [2]



(ii) In order to solve the equation $(x + 2) = e^{x-3}$, a graph with a suitable straight line is drawn on the same set of axes as the graph of $y = \ln(x + 2)$. Find the equation of this straight line. [2]

$(x + 2) = e^{x-3}$

$\ln(x + 2) = x - 3$

$\therefore$ The line to be drawn is $\underline{y = x - 3}$.

3 Express $\dfrac{x^2 - 4x + 2}{x^2 - 3x + 2}$ in partial fractions. [4]

$\dfrac{x^2 - 4x + 2}{x^2 - 3x + 2} = \dfrac{(x^2 - 3x + 2) - x}{x^2 - 3x + 2}$

$= 1 + \dfrac{-x}{(x-2)(x-1)}$

Let $\dfrac{-x}{(x-2)(x-1)} = \dfrac{A}{x-2} + \dfrac{B}{x-1}$

Multiplying by $(x - 2)(x - 1)$,

$-x = A(x - 1) + B(x - 2)$

When $x = 2$, $A = -2$

When $x = 1$, $B = 1$

$\therefore \dfrac{x^2 - 4x + 2}{x^2 - 3x + 2} = \underline{\underline{1 + \dfrac{1}{x-1} - \dfrac{2}{x-2}}}$

4 It is given that $3^{2x-2}(2^{x+3}) = 9^{2x-1}$.

(i) Find the exact value of $\left(\dfrac{2}{9}\right)^x$. [3]

$3^{2x-2}(2^{x+3}) = 9^{2x-1}$

$\Rightarrow \dfrac{3^{2x}}{3^2} \times 2^x \times 2^3 = \dfrac{9^{2x}}{9}$

$\Rightarrow \dfrac{3^{2x} \times 2^x}{9^{2x}} = \dfrac{3^2}{2^3 \times 9}$

$\Rightarrow \left(\dfrac{3^2 \times 2}{9^2}\right)^x = \dfrac{1}{8}$

$\therefore \underline{\underline{\left(\dfrac{2}{9}\right)^x = \dfrac{1}{8}}}$ (shown)

(ii) Hence find the value of x. [2]

$x = \log_{\frac{2}{9}} \dfrac{1}{8}$

$= \dfrac{\lg \frac{1}{8}}{\lg \frac{2}{9}}$

$= \underline{1.38}$ (3 s.f.)

5 **(i)** Given that $\sin\theta - \cos\theta = \dfrac{3}{4}$, show that $\sin\theta\cos\theta = \dfrac{7}{32}$. [3]

$\sin\theta - \cos\theta = \dfrac{3}{4}$

$(\sin\theta - \cos\theta)^2 = \dfrac{9}{16}$

$\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta = \dfrac{9}{16}$

$1 - 2\sin\theta\cos\theta = \dfrac{9}{16}$

$2\sin\theta\cos\theta = \dfrac{7}{16}$

$\underline{\underline{\sin\theta\cos\theta = \dfrac{7}{32}}}$ (shown)

(iii) Hence find the value of $3(\cot\theta + \tan\theta)$. [2]

$3(\cot\theta + \tan\theta) = 3\left(\dfrac{\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{\cos\theta}\right)$

$= 3\left(\dfrac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}\right)$

$= 3\left(\dfrac{1}{\frac{7}{32}}\right)$

$= \underline{\underline{\dfrac{96}{7}}}$

6 The polynomial P(x) is of degree three and has a constant term 20. The roots of the equation P(x) = 0 are 1, –2 and k. When P(x) is divided by $(x + 1)$, the remainder is 24.

(i) Show that the value of k is 5. [4]

$P(x) = a(x - 1)(x + 2)(x - k)$

$P(0) = 20 \Rightarrow 2ak = 20$

$\Rightarrow ak = 10$ . . . (1)

$P(-1) = 24 \Rightarrow a(-2)(-1 - k) = 24$

$\Rightarrow a + ak = 12$ . . . (2)

Sub (1) into (2), $a + 10 = 12$

$a = 2$

$\underline{k = 5}$ (shown)

(ii) Find the remainder when P(x) is divided by $(x - 2)$. [1]

$P(x) = 2(x - 1)(x + 2)(x - 5)$

Remainder $= P(2)$

$= 2(4)(-3)$

$= \underline{\underline{-24}}$

7 (a) Express each of $2x^2-6x+9$ and $-x^2-6x-8$ in the form $a(x+b)^2+c$, where a, b and c are constants. [3]

$$2x^2-6x+9 = 2(x^2-3x)+9$$
$$=2\left[\left(x-\frac{3}{2}\right)^2-\frac{9}{4}\right]+9$$
$$=2\left(x-\frac{3}{2}\right)^2-\frac{9}{2}+9$$
$$=\underline{\underline{2\left(x-\frac{3}{2}\right)^2+\frac{9}{2}}}$$

$$-x^2-6x-8 = -(x^2+6x)-8$$
$$=-\left[(x+3)^2-9\right]-8$$
$$=-(x+3)^2+9-8$$
$$=\underline{\underline{-(x+3)^2+1}}$$

(b) Use your answers from part (a) to explain why the curve with equations $y=2x^2-6x+9$ and $y=-x^2-6x-8$ will not intersect. [2]

From **(a)**, we know that the graph of $y=2x^2-6x+9$ has a minimum point at $\left(\frac{3}{2}, \frac{9}{2}\right)$ while the graph of $y=-x^2-6x-8$ has a maximum point at $(-3, 1)$. Thus the two graphs will not meet.

8 In an experiment on the cognitive ability of animals, a scientist recorded the time taken by a mouse to find a piece of cheese in a series of n attempts. The time T, in minutes, the mouse took is modelled by the equation $T=1.6e^{-kn}+3$ where k is a constant. It took the mouse 3.142 minutes to find the cheese on the fourth attempt.

(i) Find the value of k giving your answer correct to 3 significant figures. [2]

When $n=4$, $T=3.142$.

$$\therefore\ 1.6e^{-4k}+3=3.142$$
$$e^{-4k}=\frac{3{,}142-3}{1.6}=0.088\,75$$
$$\therefore -4k=\ln 0.088\,75$$
$$\Rightarrow\ k=\underline{0.605}\ \text{(3s.f.)}$$

(ii) Using your value of k from part **(i)**, find the time taken by the mouse to find the cheese on the sixth attempt. [1]

$$T=1.6e^{-0.605n}+3$$

When $n=6$, $T=1.6e^{-0.605(6)}+3$

$$=\underline{3.04}\text{ minutes}$$

(iii) Will the mouse be able to find the cheese within 1.5 minutes? Explain your answer. [2]

It is not impossible that the mouse can find the cheese within 1.5 minutes since

$T=1.6e^{-0.605n}+3$ and $e^{-0.605n}>0$

$\therefore T>3$ minutes.

9 Given that $y=5xe^{2x}$,

(i) find the value of x when the tangent is parallel to the x-axis. [3]

$$\frac{dy}{dx}=5e^{2x}+5x\left(2e^{2x}\right)=5e^{2x}(1+2x)$$

When the tangent is parallel to the x-axis, then gradient of tangent = 0.

$$\therefore\ 5e^{2x}(1+2x)=0$$

Since $e^{2x}>0$, then $\underline{\underline{x=-\frac{1}{2}}}$.

(ii) find the gradient of the curve when $x=\ln 4$, giving your answer in the form of $80(a\ln 2+b)$. [1]

When $x=\ln 4$, $\frac{dy}{dx}=5e^{2\ln 4}(1+2\ln 4)$

$$=5(4^2)(1+2\ln 2^2)$$
$$=\underline{80(1+4\ln 2)}$$

y is decreasing at a rate of $5(4\ln 2+1)$ units/s when $x=\ln 4$.

(iii) Find the rate of change of x at this instant. [2]

$$\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$$
$$\Rightarrow -5(4\ln 2+1)=80(1+4\ln 2)\times\frac{dx}{dt}$$
$$\therefore\ \frac{dx}{dt}=-\frac{5}{80}=\underline{\underline{-\frac{1}{16}\text{ umits/s}}}$$

10 (a) Solve the equation $\lg(x-4)+2\lg 3=1+\lg\left(\frac{x}{2}\right)$ [3]

$$\lg(x-4)+2\lg 3=1+\lg\left(\frac{x}{2}\right)$$
$$\lg(x-4)+\lg 3^2=\lg 10+\lg\left(\frac{x}{2}\right)$$
$$\lg[(x-4)\times 9]=\lg 10\left(\frac{x}{2}\right)$$
$$\therefore\ 9(x-4)=5x$$
$$4x=36$$
$$\underline{\underline{x=9}}$$

(b) Solve the equation $(\log_{81}x)\left(\frac{1}{\log_x 3}\right)=6\frac{1}{4}$ [4]

$$(\log_{81}x)\left(\frac{1}{\log_x 3}\right)=6\frac{1}{4}$$
$$\left(\frac{\log_3 x}{\log_3 81}\right)\left(\frac{1}{\frac{\log_3 3}{\log_3 x}}\right)=\frac{25}{4}$$

Let $u=\log_3 x$, then the equation becomes

$$\left(\frac{u}{4}\right)\left(\frac{1}{\frac{1}{u}}\right)=\frac{25}{4}$$
$$\therefore\ u^2=25$$
$$u=\pm 5$$
$$\log_3 x=\pm 5$$
$$x=3^5 \text{ or } 3^{-5}$$
$$\underline{\underline{x=243 \text{ or } x=\frac{1}{243}}}$$

11 (a) Solve $2\sqrt{x}=3-5x$. [3]

$2\sqrt{x}=3-5x$ and $3-5x\geq 0$

Squaring both sides, we obtain

$$4x=9-30x+25x^2$$
$$25x^2-34x+9=0$$
$$(25x-9)(x-1)=0$$
$$x=\frac{9}{25}\text{ or } x=1$$

But $x\leq\frac{3}{5}$, $\therefore\ \underline{\underline{x=\frac{9}{25}}}$

(b) A toy train moved at a speed of $(2+\sqrt{3})$ cm per second from point A to point B. Given that the distance covered was $(5\sqrt{75}-10)$ cm, find the time taken to move from point A to B in the form $p\sqrt{3}+q$, where p and q are constants. [3]

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}} = \frac{5\sqrt{75}-1}{2+\sqrt{3}}$$
$$= \frac{5\sqrt{75}-1}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}}$$
$$= \frac{10\sqrt{75}-5\sqrt{225}-2+\sqrt{3}}{4-3}$$
$$= 10\times5\sqrt{3}-5\times15-2+\sqrt{3}$$
$$= \underline{\underline{51\sqrt{3}-77}} \text{ cm}$$

12 (i) Write down the first three terms in the expansion, in ascending powers of x, of $\left(3-\frac{x}{5}\right)^n$, where n is a positive integer. [2]

$$\left(3-\frac{x}{5}\right)^n = 3^n + \binom{n}{1}3^{n-1}\left(-\frac{x}{5}\right) + \binom{n}{2}3^{n-2}\left(-\frac{x}{5}\right)^2$$
$$= 3^n + n\times3^{n-1}\left(-\frac{x}{5}\right) + \frac{n(n-1)}{2}3^{n-2}\left(\frac{x^2}{25}\right)$$
$$= 3^n - \frac{n\times3^{n-1}}{5}x + \frac{n(n-1)\times3^{n-2}}{50}x^2$$

The first two terms in the expansion, in ascending powers of x, of $(3+x)\left(3-\frac{x}{5}\right)^n$ are $p + qx^2$, where p and q are constants.

(ii) Show that $n - 5$. [3]

$$(3+x)\left(3-\frac{x}{5}\right)^n$$
$$= (3+x)\left[3^n - \frac{n\times3^{n-1}}{5}x + \frac{n(n-1)\times3^{n-2}}{50}x^2\right]$$

Since coefficient of $x = 0$, then

$$3\times\left(-\frac{n\times3^{n-1}}{5}\right) + 3^n = 0 \Rightarrow 3^n\left(-\frac{n}{5}+1\right) = 0$$

$\therefore n = 5$ (shown)

(iii) Hence find the value of p and q. [3]

So, $p = 3\times3^5 = 3^6 = \underline{\underline{729}}$

q = coefficient of x^2

$$= \left(-\frac{5\times3^4}{5}\right) + 3\times\frac{5\times4\times3^3}{50}$$
$$= -81 + \frac{81\times2}{5}$$
$$= \underline{\underline{-\frac{243}{5}}}$$

13 (a) Find $\int \frac{5}{1+\cos 2x}\,dx$. [3]

$$\int \frac{5}{1+\cos 2x}\,dx = \int \frac{5}{1+2\cos^2 x-1}\,dx$$
$$= \int \frac{5}{2}\sec^2 x\,dx$$
$$= \frac{5}{2}\tan x + c$$

(b) (i) Show that $\frac{d}{dx}\ln\left(\frac{x^2}{\sin x}\right) = \frac{2}{x} - \cot x$ [2]

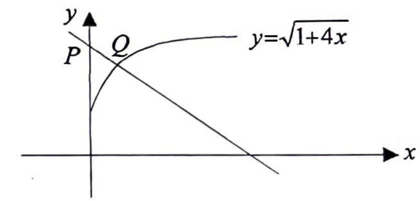
$$\frac{d}{dx}\ln\left(\frac{x^2}{\sin x}\right) = \frac{d}{dx}\left[2\ln x - \ln(\sin x)\right]$$
$$= \frac{2}{x} - \frac{\cos x}{\sin x}$$
$$= \frac{2}{x} - \cot x \text{ (shown)}$$

(ii) Hence evaluate $\int_1^2 \frac{3}{4}\cot x\,dx$. [3]

Integrating both sides of **(i)**, we have

$$\int \frac{2}{x} - \cot x\,dx = \ln\left(\frac{x^2}{\sin x}\right) + c$$
$$\Rightarrow 2\ln x - \int \cot x\,dx = \ln\left(\frac{x^2}{\sin x}\right) + c$$
$$\Rightarrow \int \cot x\,dx = 2\ln x - \ln\left(\frac{x^2}{\sin x}\right) + c$$
$$\Rightarrow \int_1^2 \frac{3}{4}\cot x\,dx = \frac{3}{4}\left[2\ln x - \ln\left(\frac{x^2}{\sin x}\right)\right]_1^2$$
$$\therefore \int_1^2 \frac{3}{4}\cot x\,dx = \frac{3}{4}\left[2\ln 2 - \ln\left(\frac{4}{\sin 2}\right)\right] - \left[0 - \ln\frac{1}{\sin 1}\right]$$
$$= \underline{\underline{0.058\ 1}}$$

14



The diagram above shows part of the curve $y=\sqrt{1+4x}$ with the normal of the curve at Q. Given that the gradient of the tangent at Q is $\frac{2}{3}$,

(i) find the coordinates of Q. [3]

$$y=\sqrt{1+4x}$$
$$\frac{dy}{dx} = \frac{4}{2\sqrt{1+4x}} = \frac{2}{\sqrt{1+4x}}$$

Gradient of tangent at $Q = \frac{2}{3}$

$$\Rightarrow \frac{2}{\sqrt{1+4x}} = \frac{2}{3}$$
$$\Rightarrow \sqrt{1+4x} = 3$$
$$\Rightarrow 1+4x = 9$$
$$\Rightarrow 4x = 8$$
$$\Rightarrow x = 2$$
$$\therefore y = 3$$

$\therefore \underline{\underline{Q = (2, 3).}}$

(ii) find the coordinates of P, [2]

Gradient of normal at $Q = -\frac{3}{2}$

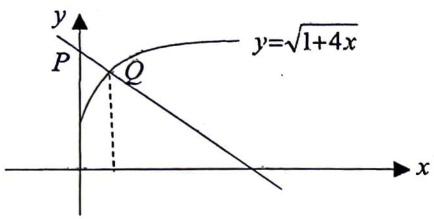
Equation of normal at Q is

$$y - 3 = -\frac{3}{2}(x-2)$$
$$y = -\frac{3}{2}x + 6$$

When $x = 0$, $y = 6$

$\therefore \underline{\underline{P = (0, 6).}}$

(iii) Calculate the area bounded by the y-axis, line PQ and the curve. [4]



Required area = Area of trapezium − Area under the curve

$$= \frac{1}{2}(6+3)\times 2 - \int_0^2 \sqrt{1+4x}\,dx$$

$$= 9 - \left[\frac{(1+4x)^{\frac{3}{2}}}{\frac{3}{2}(4)}\right]_0^2$$

$$= 9 - \left[\frac{1}{6}\sqrt{(1+4x)^3}\right]_0^2$$

$$= 9 - \frac{1}{6}\left[\sqrt{9^3} - 1\right]$$

$$= 9 - \frac{1}{6}(27-1)$$

$$= 9 - \frac{13}{3}$$

$$= \frac{14}{3} \text{ units}^2$$

15 The gradient at any point on a particular curve is given by the expression $x^2 + \frac{16}{x^2}$, where $x > 0$.

Given that the curve passes through the point $P(4, 18)$, find

(i) the equation of the normal to the curve at P, [3]

$$\frac{dy}{dx} = x^2 + \frac{16}{x^2}$$

At P, gradient of tangent $= 4^2 + \frac{16}{4^2}$

$= 17$

Gradient of normal $= -\frac{1}{17}$

$\therefore$ Equation of the normal is

$$y - 18 = -\frac{1}{17}(x-4)$$

$$y = -\frac{1}{17}x + \frac{310}{17}$$ [3]

(ii) the equation of the curve, [3]

$$y = \int x^2 + \frac{16}{x^2}\,dx$$

$$= \frac{x^3}{3} + 16\left(\frac{x^{-1}}{-1}\right) + c$$

$$= \frac{x^3}{3} - \frac{16}{x} + c$$

When $x = 4$, $y = 18$

$$18 = \frac{64}{3} - \frac{16}{4} + c$$

$$\therefore c = \frac{2}{3}$$

Equation of the curve is $y = \frac{x^3}{3} - \frac{16}{x} + \frac{2}{3}$

(iii) the coordinates of the point on the curve where the gradient is a minimum and calculate this minimum value. [4]

gradient function is $\frac{dy}{dx} = x^2 + \frac{16}{x^2}$

$$\frac{d^2y}{dx^2} = 2x - \frac{32}{x^3}$$

For minimum gradient, $\frac{d^2y}{dx^2} = 0$.

$$2x - \frac{32}{x^3} = 0$$

$$2x = \frac{32}{x^3}$$

$$x^4 = 16$$

Since $x > 0$, $x = 2$

$$y = \frac{8}{3} - \frac{16}{2} + \frac{2}{3}$$

$$= -\frac{14}{3}$$

$\therefore$ The point is $\left(2, -\frac{14}{3}\right)$

$$\frac{d^3y}{dx^3} = 2 + \frac{96}{x^4}$$

Since $x = 2$, $\frac{d^3y}{dx^3} > 0$, the gradient is minimum when $x = 2$.

Minimum gradient $= 2^2 + \frac{16}{2^2}$

$= 8$