

Solutions

1	Solution [11] Inequalities
(a)	By AM-GM inequality, $2a + b + 3c \geq 3\sqrt[3]{2a \cdot b \cdot 3c}$ $= 3\sqrt[3]{6abc}$ $= 3(6)$ $= 18$
(b)	By Cauchy Schwarz inequality $\left[\frac{4a^2}{b+3c} + \frac{b^2}{3c+2a} + \frac{9c^2}{2a+b} \right] [(b+3c) + (3c+2a) + (2a+b)] \geq (2a+b+3c)^2$ $\left[\frac{4a^2}{b+3c} + \frac{b^2}{3c+2a} + \frac{9c^2}{2a+b} \right] [4a+2b+6c] \geq (2a+b+3c)^2$ $\frac{4a^2}{b+3c} + \frac{b^2}{3c+2a} + \frac{9c^2}{2a+b} \geq \frac{2a+b+3c}{2} \text{ since } 2a+b+3c > 0$ Hence $\frac{4a^2}{b+3c} + \frac{b^2}{3c+2a} + \frac{9c^2}{2a+b} \geq 9$ Equality holds for AM-GM inequality when $2a = b = 3c$. Substituting into $abc = 36$ yields $a(2a)\left(\frac{2}{3}a\right) = 36$ $a^3 = 27$ $a = 3$ $\therefore a = 3, b = 6, c = 2$ Check against original expression $\frac{4a^2}{b+3c} + \frac{b^2}{3c+2a} + \frac{9c^2}{2a+b} = \frac{4(3)^2}{6+3(2)} + \frac{(6)^2}{3(2)+2(3)} + \frac{9(2)^2}{2(3)+6} = 3+3+3=9$

(c)

Consider the substitution $a = \frac{1}{p}, b = \frac{1}{q}, c = \frac{1}{r}$

$$\text{Then } abc = \frac{1}{p} \cdot \frac{1}{q} \cdot \frac{1}{r} = \frac{1}{pqr} = 36$$

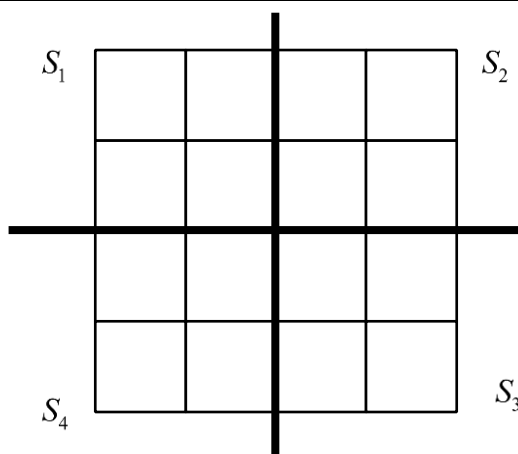
$$\begin{aligned} & \frac{1}{9p^3(r+3q)} + \frac{1}{36q^3(3p+2r)} + \frac{1}{4r^3(2q+p)} \\ &= \left[\frac{1}{9p^3(r+3q)} + \frac{1}{36q^3(3p+2r)} + \frac{1}{4r^3(2q+p)} \right] \times 36pqr \\ &= \frac{4qr}{p^2r+3p^2q} + \frac{pr}{3q^2p+2q^2r} + \frac{9pq}{2r^2q+r^2p} \\ &= \frac{4qr}{p^2r+3p^2q} \times \frac{\frac{1}{p^2qr}}{\frac{1}{p^2qr}} + \frac{pr}{3q^2p+2q^2r} \times \frac{\frac{1}{pq^2r}}{\frac{1}{pq^2r}} + \frac{9pq}{2r^2q+r^2p} \times \frac{\frac{1}{pqr^2}}{\frac{1}{pqr^2}} \\ &= \frac{4\left(\frac{1}{p}\right)^2}{\left(\frac{1}{q}\right)+3\left(\frac{1}{r}\right)} + \frac{\left(\frac{1}{q}\right)^2}{3\left(\frac{1}{r}\right)+2\left(\frac{1}{p}\right)} + \frac{9\left(\frac{1}{r}\right)^2}{2\left(\frac{1}{p}\right)+\left(\frac{1}{q}\right)} \\ &= \frac{4a^2}{b+3c} + \frac{b^2}{3c+2a} + \frac{9c^2}{2a+b} \end{aligned}$$

$$\geq 9$$

by the result of the earlier part.

2 (a) (b)	Solutions [11] Differential Equations, Conics Differentiating the substitution with respect to x, $\frac{dy}{dx} = u + x \frac{du}{dx}$ Substituting $\frac{dy}{dx} = u + x \frac{du}{dx}$ and $y = ux$ into the differential equation, we have $u + x \frac{du}{dx} = \frac{2x^2u}{x^2 - 2x^2u^2}$ $x \frac{du}{dx} = \frac{2u}{1 - 2u^2} - u$ $x \frac{du}{dx} = \frac{2u - u + 2u^3}{1 - 2u^2}$ $x \frac{du}{dx} = \frac{u(2u^2 + 1)}{1 - 2u^2}$ Hence, $\int \frac{1 - 2u^2}{u(2u^2 + 1)} du = \int \frac{1}{x} dx$ $\int \frac{1}{u} - \frac{4u}{2u^2 + 1} du = \int \frac{1}{x} dx$ $\ln \left \frac{u}{2u^2 + 1} \right = \ln Ax $ $\frac{u}{2u^2 + 1} = Bx, \text{ where } B = \pm A$ $\frac{y}{x} = 2B \frac{y^2}{x} + Bx$ $y = 2By^2 + Bx^2$ Case 1: $B \neq 0$. $y = 2By^2 + Bx^2$ $\frac{x^2}{2} + \left(y - \frac{1}{4B} \right)^2 = \frac{1}{16B^2}$ $\frac{(x - 0)^2}{\left(\frac{1}{\sqrt{8B}} \right)^2} + \frac{\left(y - \frac{1}{4B} \right)^2}{\left(\frac{1}{4B} \right)^2} = 1$ The general solution represents an ellipse centre at $\left(0, \frac{1}{4B} \right)$ with horizontal radius $\frac{1}{\sqrt{8B}}$ and vertical radius $\frac{1}{4B}$ with $x \neq \sqrt{2}y$ for all $x, y \in \mathbb{R}$. Case 2: $B = 0$ The particular solution is the line $y = 0$.
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3	Solutions [12] PIE, Pigeonhole Principle, Distribution Qns																
(a)(i)	Total number of grids = $2^{16} = 65536$																
(a)(ii)	Label the squares as shown below <table><tr><td>1</td><td>2</td><td></td><td></td></tr><tr><td>3</td><td>4</td><td></td><td></td></tr><tr><td></td><td></td><td></td><td></td></tr><tr><td></td><td></td><td></td><td></td></tr></table> Let A_i be the number of grids with a 3-by-3 shaded square that's top-left square is labelled i. $A_i = 2^7$, $A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_4 = A_3 \cap A_4 = 2^4$, $A_1 \cap A_4 = A_2 \cap A_3 = 2^2$, $A_i \cap A_j \cap A_k = 2$ and $A_1 \cap A_2 \cap A_3 \cap A_4 = 1$. By PIE, $A_1 \cup A_2 \cup A_3 \cup A_4$ $= \sum_{i=1}^4 A_i - \sum_{1 \leq i < j \leq 4} A_i \cap A_j$ $\quad + \sum_{1 \leq i < j < k \leq 4} A_i \cap A_j \cap A_k - A_1 \cap A_2 \cap A_3 \cap A_4$ $= 4(2^7) - (4 \times 2^4 + 2 \times 2^2) + (4 \times 2) - 1$ $= 447$.	1	2			3	4										
1	2																
3	4																
(b)	We require more than 12 from the following construct: <table><tr><td></td><td></td><td></td><td></td></tr><tr><td></td><td></td><td></td><td></td></tr><tr><td></td><td></td><td></td><td></td></tr><tr><td></td><td></td><td></td><td></td></tr></table> Now we show that every grid with 13 shaded unit square will satisfy the required condition. Consider the partitioning for the grid by the four 2-by-2 squares as shown below:																



We have 13 shaded squares (pigeons) and four 2-by-2 squares (pigeonholes). By pigeonhole principle, there will be one 2-by-2 square with at least $\left\lceil \frac{13}{4} \right\rceil = 4$ shaded squares. This would be we have at least one shaded 2-by-2 square if we have 13 shaded square in the grid.

Therefore, at minimum, we need 13 shaded squares.

(c)(i)

Method 1:

This is equivalent to

$$c_1 + c_2 + c_3 + c_4 = 4, \text{ where } c_i \in \mathbb{Z}_0^+.$$

$$\text{Hence, } \binom{4+4-1}{4-1} = 35.$$

Method 2:

By listing cases,

$$\begin{aligned} & \underbrace{4}_{\text{1 column with 4}} + \underbrace{{}^4C_3 \times 3}_{\text{2 columns with 1, 1 with 2}} \\ & + \underbrace{{}^4P_2}_{\text{1 column with 3, 1 with 1}} + \underbrace{{}^4C_2}_{\text{2 columns with 2}} \\ & + \underbrace{1}_{\text{4 columns with 1}} = 35 \end{aligned}$$

(c)(ii)

As the number of white squares is 16 – the number of shaded squares.

Counting this is equivalent to counting if the grids have at most 4 white squares.

By symmetry this is the same as counting the grids that have at most 4 shaded squares.

Method 1:

So, we require,

$$\begin{aligned} & \binom{4+4-1}{4-1} + \binom{3+4-1}{4-1} + \binom{2+4-1}{4-1} + \binom{1+4-1}{4-1} + \binom{0+4-1}{4-1} \\ & = 70. \end{aligned}$$

Method 2:

This is equivalent to

$$c_1 + c_2 + c_3 + c_4 + c_5 = 4, \text{ where } c_i \in \mathbb{Z}_0^+.$$

We treat c_5 as the number of shaded squares not in the grid if there are less than 4 shaded squares in the grid.

$$\binom{4+5-1}{5-1} = 70.$$

Method 3:

By listing cases:

$$\begin{aligned} & (4 + {}^4C_3 \times 3 + {}^4P_2 + {}^4C_2 + 1) + (4 + {}^4P_2 + 4) \\ & \quad + (4 + {}^4C_2) + (4) + (1) = 70 \end{aligned}$$

4	Solutions [12] Probability
(a)	In constructing a subset of S, we have 2 choices for each of the n integers from 1 to n: include or exclude. By multiplication principle, there are 2^n subsets of S.
(b)	Then, (i) $ \begin{aligned} P(i \notin A_1 \cap A_2) &= 1 - P(i \in A_1 \cap A_2) \\ &= 1 - P(i \in A_1)P(i \in A_2) \quad (\text{by independence}) \\ &= 1 - \left(\frac{1}{2}\right)^2 \\ &= \frac{3}{4} \end{aligned} $ Alternatively, Consider the partition: $(A_1' \cap A_2') \cup (A_1' \cap A_2) \cup (A_1 \cap A_2') \cup (A_1 \cap A_2)$ Then i has equal probability to be in one of these four sets. Therefore, $P(i \notin A_1 \cap A_2) = \frac{3}{4}$.
(b) (ii)	$ \begin{aligned} P(A_1 \cap A_2 = \emptyset) &= \prod_{i=1}^n P(i \notin A_1 \cap A_2) \\ &= \left(\frac{3}{4}\right)^n \end{aligned} $
(b) (iii)	Let the k elements of A_1 be $a_1, \dots, a_k$. $ \begin{aligned} P(A_1 \cap A_2 = \emptyset \mid A_1 = k) &= \prod_{i=1}^k P(a_i \notin A_2) \\ &= \frac{1}{2^k} \end{aligned} $ Alternatively,

	$P(A_1 \cap A_2 = \emptyset A_1 = k) = \frac{P(A_1 \cap A_2 = \emptyset \text{ and } A_1 = k)}{P(A_1 = k)}$ $= \frac{\left[\frac{\binom{n}{k} 2^{n-k}}{4^n} \right]}{\left[\frac{\binom{n}{k}}{2^n} \right]}$ $= \frac{1}{2^k}$
(b) (iv)	$P(A_1 \cap A_2 = \emptyset) = \sum_{k=0}^n P(A_1 = k) P(A_1 \cap A_2 = \emptyset A_1 = k)$ $= \sum_{k=0}^n \left[\binom{n}{k} \frac{1}{2^n} \times \frac{1}{2^k} \right] \text{ (there are } \binom{n}{k} \text{ subsets with } k \text{ elements; part (bii))}$ $= \sum_{k=0}^n \binom{n}{k} \frac{1}{2^{n+k}}$ From (bi), we have $\sum_{k=0}^n \binom{n}{k} \frac{1}{2^{n+k}} = \left(\frac{3}{4}\right)^n$.
(c) (i)	WLOG, let the k elements of their intersection be $1, \dots, k$. Then, we must have $i \in A_1 \cap A_2$ for $1 \leq i \leq k$ and $j \notin A_1 \cap A_2$ for $k+1 \leq j \leq n$. By (bi), $P(i \in A_1 \cap A_2) = \frac{1}{4}$ and $P(i \notin A_1 \cap A_2) = \frac{3}{4}$. Moreover, there are $\binom{n}{k}$ ways to pick the k elements. So $P(A_1 \cap A_2 = k) = \binom{n}{k} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k}$
(c) (ii)	By (ci), $A_1 \cap A_2 \sim B(n, \frac{1}{4})$. So $E(A_1 \cap A_2) = n \times \frac{1}{4} = \frac{n}{4}$.

Q5	Solutions [14] Polynomial, Numbers, Induction
(a)(i)	$f(x) - f(\alpha)$ is a polynomial. When $x = \alpha$, $f(\alpha) - f(\alpha) = 0$. By factor theorem, $x - \alpha$ is a factor of $f(x) - f(\alpha)$. i.e. $f(x) - f(\alpha) = (x - \alpha)f_0(x)$ for some integer polynomial $f_0(x)$. When $x = \beta$, $f(\beta) - f(\alpha) = (\beta - \alpha)f_0(\beta) \equiv 0 \pmod{\beta - \alpha}$. <u>Alternatively</u> By remainder theorem, $f(x) = (x - \alpha)g(x) + f(\alpha)$, where g is a polynomial with integer coefficients. Sub $x = \beta$: $f(\beta) \equiv f(\alpha) \pmod{\beta - \alpha}$.
(a)(ii)	Suppose there exists such a polynomial. $f(3) - f(1) \equiv 0 \pmod{2}$ from (a)(i). But $f(3) - f(1) = 5 - 2 = 3 \equiv 1 \pmod{2}$ a contradiction.
(a)(iii)	$f(2025) - f(1) \equiv 0 \pmod{2024}$ $f(2025) \equiv 0 \pmod{2024}$ Since $f(2025)$ is a multiple of 2024, it cannot be strictly between $0 \times 2024 = 0$ and $1 \times 2024 = 2024$.
(b)(i)	$2 \not\equiv 4 \pmod{7}$ $2^2 + 2 + 1 = 7 \equiv 0 \pmod{7}$ And $4^2 + 4 + 1 = 21 \equiv 0 \pmod{7}$ So, 2 and 4 are incongruent roots modulo 7 of $x^2 + x + 1$.
(b)(ii)	Consider $h(x) - h(\gamma)$. From (a)(i) we have that $h(x) - h(\gamma) = (x - \gamma)h_0(x)$ for some integer polynomial $h_0(x)$ with degree $n - 1$. We note that x^{n-1} coefficient of $h_0(x)$ must be 1 when we compare the x^n coefficient in the equation $h(x) - h(\gamma) = (x - \gamma)h_0(x)$. $h(x) - h(\gamma) = (x - \gamma)h_0(x)$ $h(x) - 0 \equiv (x - \gamma)h_0(x) \pmod{p}$ $h(x) \equiv (x - \gamma)h_0(x) \pmod{p}$ <u>Alternatively</u> $h(x) = (x - \gamma)h_0(x) + h(\gamma)$ by Remainder Theorem, where h is a degree $n - 1$ polynomial with integer coefficients. h is monic by comparing coefficients of x^n. $h(x) = (x - \gamma)h_0(x) + h(\gamma)$ $\equiv (x - \gamma)h_0(x) \pmod{p}$

	since γ is a root modulo p .
(b)(iii)	Let $P(n)$ be the statement that any monic degree n polynomial $h(x)$ has at most n incongruent roots modulo p for $n \geq 0$. Base case: When $n = 0$, $h(x) = 1$ for all x and has no solution for that $h(x) \equiv 0$ (modulo p). Hence it has at most 0 roots. Inductive step: Assume $P(k)$ is true any monic degree k polynomial $h(x)$ has at most k incongruent roots modulo p for some $k \geq 0$. Now suppose $h(x)$ is a degree $k + 1$ polynomial. If it has no roots we are done. If it has a root, say q (modulo p), then $h(q) \equiv 0$ (modulo p). From (b)(i) we have that there exist monic degree $n - 1$ integer polynomial $h_0(x)$ such that $h(x) \equiv (x - q)h_0(x) \pmod{p}.$ This implies that h will have at most $k + 1$ roots modulo p that are incongruent modulo p (at most k from $h_0(x)$ and additionally q). Hence, $P(k + 1)$ is true. Conclusion: Since $P(0)$ is true and $P(k)$ is true $\Rightarrow P(k + 1)$ is true, by Principle of Mathematical Induction, $P(n)$ is true for all $n \geq 0$.

6	Solution [20] Irrationality, Differentiation, Integration, Functions, Inequality, Limits.
(a)(i)	$\frac{x^n (a - bx)^n}{n!} = \frac{x^n}{n!} \sum_{r=0}^n \binom{n}{r} (-1)^r a^{n-r} b^r x^r$ $= \frac{1}{n!} \sum_{r=0}^n \binom{n}{r} (-1)^r a^{n-r} b^r x^{n+r}$ $= \frac{1}{n!} \sum_{r=n}^{2n} (-1)^{r-n} \binom{n}{r-n} a^{2n-r} b^{r-n} x^r$ $= \frac{1}{n!} \sum_{r=n}^{2n} c_r x^r, \quad c_r = (-1)^{r-n} \binom{n}{r-n} a^{2n-r} b^{r-n}$ We note that $\binom{n}{r-n} \in \mathbb{Z}$ and $a^{2n-r} b^{r-n} \in \mathbb{Z}$ as $n \leq r \leq 2n$. Therefore, $c_r = (-1)^{r-n} \binom{n}{r-n} a^{2n-r} b^{r-n} \in \mathbb{Z}$.
(a)(ii)	For $0 \leq k \leq n-1$, $f^{(k)}(x)$ is a polynomial with no constant term and hence that $f^{(k)}(0) = 0$. For $n \leq k \leq 2n$, $f^{(k)}(x)$ will have a constant term $\frac{k! c_k}{n!}$ which will be an integer as $n \leq k \leq 2n$.
(b)	$f(\pi - x)$ $= \frac{b^n \left[(\pi - x)(\pi - (\pi - x)) \right]^n}{n!}$ $= \frac{b^n \left[(\pi - x)x \right]^n}{n!}$ $= f(x). \quad (\text{Shown})$ $f(\pi - x) = f(x)$ $(-1)f'(\pi - x) = f'(x)$ $(-1)^2 f''(\pi - x) = f''(x)$ $\dots$ $(-1)^k f^{(k)}(\pi - x) = f^{(k)}(x)$ Let $x = 0$: $(-1)^k f^{(k)}(\pi) = f^{(k)}(0) \Rightarrow f^{(k)}(\pi) = (-1)^k f^{(k)}(0) \in \mathbb{Z}.$

(c)(i)	$F(x) = \sum_{r=0}^n \left[(-1)^r f^{(2r)}(x) \right]$ $F'(x) = \sum_{r=0}^n \left[(-1)^r f^{(2r+1)}(x) \right]$ $F''(x) = \sum_{r=0}^n \left[(-1)^r f^{(2r+2)}(x) \right]$ $F''(x) = \sum_{r=0}^{n-1} \left[(-1)^r f^{(2r+2)}(x) \right] + \underbrace{f^{(2n+2)}(x)}_{=0}$ $F''(x) = \sum_{r=1}^n \left[(-1)^{r-1} f^{(2r)}(x) \right] \text{ (replace } r \text{ with } r-1)$ $F''(x) = f(x) - f(x) - \sum_{r=1}^n \left[(-1)^r f^{(2r)}(x) \right]$ $F''(x) = f(x) - \left(\sum_{r=0}^n \left[(-1)^r f^{(2r)}(x) \right] \right)$ $F''(x) = f(x) - F(x) \quad (\text{Shown})$
(c)(ii)	Thus by (c)(i), $\int_0^\pi f(x) \sin(x) \, dx$ $= \int_0^\pi F''(x) \sin(x) + F(x) \sin(x) \, dx$ $= \left[\sin(x) F'(x) \right]_0^\pi - \int_0^\pi F'(x) \cos(x) \, dx$ $- \left[\cos(x) F(x) \right]_0^\pi + \int_0^\pi F'(x) \cos(x) \, dx$ $= (0 - 0) + (-F(\pi) \cos(\pi) + F(0) \cos(0))$ $= F(0) + F(\pi). \text{ (Shown)}$
(d)(i)	<u>Method 1:</u> If $0 < x < \pi$, then $0 < \pi - x < \pi$. And $b^n \pi^n = a^n$ as $\pi = \frac{a}{b}$. Hence, $\frac{(0)(0)}{n!} < \frac{b^n [(\pi - x)x]^n}{n!} < \frac{b^n [(\pi)^n (\pi)^n]}{n!}$ $0 < f(x) = \frac{b^n [(\pi - x)x]^n}{n!} < \frac{\pi^n a^n}{n!} = \frac{\pi^n m^n}{n!} \quad (\text{Shown})$ Where $m = a$. (or $m = b\pi$.) <u>Method 2:</u>

	Note that $f(x) = \frac{b^n}{n!} [x(\pi - x)]^n > 0$ for all $x \in (0, \pi)$. Furthermore, the graph of $y = x(\pi - x)$ is concave downwards on the interval $[0, \pi]$ and $f(x)$ is a polynomial in x and so is continuous (and infinitely differentiable) on $\mathbb{R} \supset (0, \pi)$. Hence, $x(\pi - x) \leq \frac{\pi}{2} \text{ for all } x \in (0, \pi). \text{ Then}$ $\begin{aligned} f(x) &= \frac{b^n}{n!} [x(\pi - x)]^n \\ &\leq \frac{b^n}{n!} \cdot \frac{\pi^n}{2^n} \\ &= \frac{\pi^n}{n!} \cdot \frac{b^n}{2^n} = \frac{\pi^n m^n}{n!} \end{aligned}$ Where $m = \frac{b}{2}$.
(d)(ii)	As $0 < f(x) = \frac{b^n [(\pi - x)x]^n}{n!} < \frac{\pi^n m^n}{n!}$ (by (d)(i)), we have: $0 < \int_0^\pi f(x) \sin(x) \, dx < \frac{\pi^n m^n}{n!} \int_0^\pi \sin x \, dx$ $0 < \int_0^\pi f(x) \sin(x) \, dx < \frac{\pi^n m^n}{n!} (-\cos \pi + \cos 0)$ $0 < \int_0^\pi f(x) \sin(x) \, dx < \frac{2\pi^n m^n}{n!}. \quad (\text{Shown})$
(d)(iii)	Method 1: As the Maclaurin series of e^x converges we see that the general term of the series must converge to 0, i.e. $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$. Let $x = \pi m$, we have $\lim_{n \rightarrow \infty} \frac{(\pi m)^n}{n!} = 0$. $\lim_{n \rightarrow \infty} \frac{2\pi^n m^n}{n!} = 2 \lim_{n \rightarrow \infty} \frac{(\pi m)^n}{n!} = 0.$ Method 2: For large enough n, say when $n > m$, $0 < \frac{\pi a}{n} < \frac{1}{2}$. Let $\frac{2\pi^{n_0} m^{n_0}}{n_0!} = M$. $\text{Then } \lim_{n \rightarrow \infty} \frac{2\pi^n a^n}{n!} = \lim_{k \rightarrow \infty} \frac{2\pi^{n_0+k} a^{n_0+k}}{(n_0+k)!} \leq M \lim_{k \rightarrow \infty} \left(\frac{1}{2}\right)^k = 0.$

(e)

By (d)(ii),

$$0 < \int_0^\pi f(x) \sin(x) \, dx < \frac{2\pi^n m^n}{n!}.$$

By (d)(iii), as $\lim_{n \rightarrow \infty} \frac{2\pi^n m^n}{n!} = 0$, by the text, there is sufficiently large n such that

$$\frac{2\pi^n m^n}{n!} < 1.$$

Thus, we select a sufficiently large n to get:

$$0 < \int_0^\pi f(x) \sin(x) \, dx < 1.$$

Moreover, by (a), $f^{(2r)}(0), f^{(2r)}(\pi) \in \mathbb{Z}$ for $0 \leq r \leq n$, thus,

$$\begin{aligned} \int_0^\pi f(x) \sin(x) \, dx &= F(0) + F(\pi) \\ &= \sum_{r=0}^n \left[(-1)^r f^{(2r)}(0) + (-1)^r f^{(2r)}(\pi) \right] \in \mathbb{Z} \end{aligned}$$

However, coupled with $0 < \int_0^\pi f(x) \sin(x) \, dx < 1$, we arrived at a contradiction as there is no integer between 0 to 1.

Therefore, π is irrational.