JURONG PIONEER JUNIOR COLLEGE
9478 H2 PHYSICS

Projectile Motion

Content

- **Free fall**
- **Gravitational potential energy in a uniform field**
- **Effects of air resistance**

Learning Outcomes

Candidates should be able to:

(a) describe and use the concept of weight as the force experienced by a mass in a gravitational field

(b) describe and explain motion due to a uniform velocity in one direction and a uniform acceleration in a perpendicular direction

(c) derive, from the definition of work done by a force, the equation $\Delta E\text{p} = mg\Delta h$ for gravitational potential energy changes in a uniform gravitational field (e.g. near the Earth's surface)

(d) recall and use the equation $\Delta E\text{p} = mg\Delta h$ to solve problems

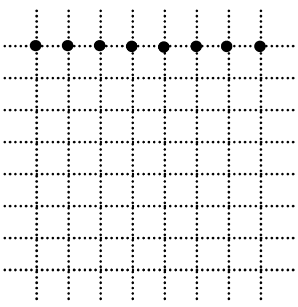
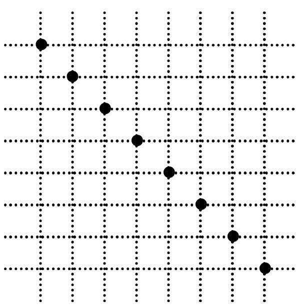
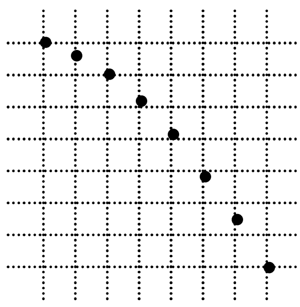
(e) describe qualitatively, with reference to forces and energy, the motion of bodies falling in a uniform gravitational field with air resistance, including the phenomenon of terminal velocity.

1 Two Dimensional Motion

(b) Describe and explain motion due to a uniform velocity in one direction and a uniform acceleration in a perpendicular direction.

1.1 Comparison between 1-D motion and 2-D motion

Below shows the snapshots of three different types of motion:

Path (snapshots of an object against a grid as background, at regular time intervals)			
Description	Uniform velocity in x-direction No velocity and acceleration in y-direction.	Uniform velocity in x-direction. Uniform velocity in y-direction.	Uniform velocity in x-direction Uniform *acceleration* in y-direction
Variables	v_x = constant $a_x = 0$	v_x = constant, v_y = constant $a_x = 0$ $a_y = 0$	v_x = constant, $v_y \neq$ constant $a_x = 0$ $a_y \neq 0$

An example of the third type of motion shown above is **projectile motion**, where an object moving in air (with no air resistance) has acceleration only in the vertical direction.

The key idea in solving projectile motion problems is that the horizontal and vertical components of the motion can be treated independently. Kinematic quantities like displacement, velocity and acceleration along one direction is independent of kinematic quantities in the perpendicular direction.

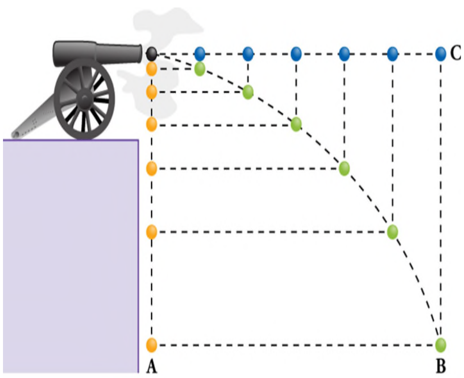
1.2 Analysing Projectile Motion

(a) Horizontal projection

A cannon is shot out horizontally at a velocity u. There is no horizontal acceleration.

If there was no gravitational force, it would be moving to point C in a horizontal line, where the cannon's position is equally spaced at equal time interval. ($s_x = ut$, constant u)

If there was no horizontal velocity, it would be dropping vertically to point A. The position/velocity of the cannon can be calculated using kinematic equations. ($s_y = \frac{1}{2}gt^2, v_y = gt$, initial u_y = 0).



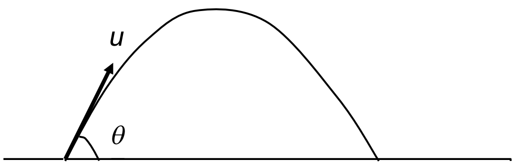
The actual path of a horizontally projected cannon ball is shown in the curved path to point B. The exact horizontal and vertical position at any time t is obtained by combining the positions along C and along A.

Similarly, the velocity at any point is obtained by finding the vector sum of the horizontal and vertical velocities.

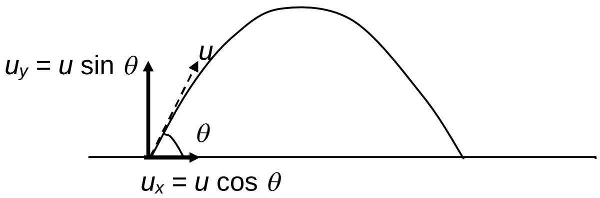
The total time taken is calculated from the vertical components. The horizontal distance is calculated from the horizontal components.

(b) Angled projection

Consider an object projected in air at an angle of θ above the horizontal. The path traced out by the object is parabolic in shape, as shown on the right:



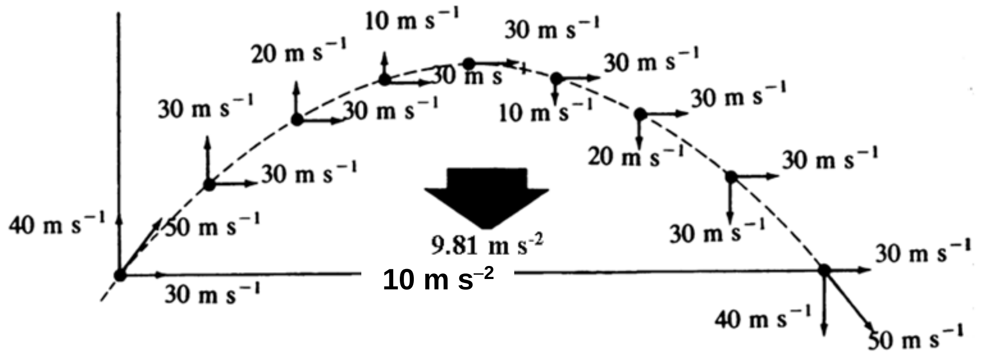
The initial velocity u can be resolved into 2 perpendicular components.
Horizontal component is $u_x = u \cos \theta$ and Vertical component is $u_y = u \sin \theta$.



Assume that air resistance is ignored,

- there is no horizontal acceleration and therefore horizontal velocity u_x is constant;
- vertical acceleration is due to gravity only ($a_y = g$ or $a_y = -g$), therefore vertical motion is uniformly accelerated;
- the equations of motion can be applied to solve for the quantities in the x (horizontal) and y (vertical) directions;
- time t is the same for both x, y components.

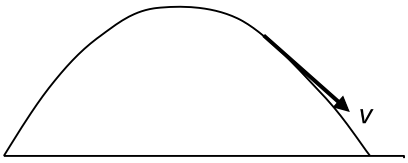
Example of a second-by-second exposure of an object projected at an angle to the horizontal is shown below. (Assume g = 10 m s^{-2}.)

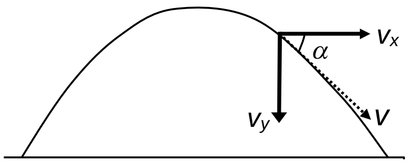


Common calculations:

1. **Time of flight t**: use the vertical component. ($s_y = u_y t - \frac{1}{2}gt^2$, set $s_y = 0$) to find t.

   Can use $v_y = u_y - gt_{up}$ to find t_{up} . Total t = 2t_{up}
2. **Horizontal range**: use horizontal component. ($s_x = u_x t$)
3. **Velocity**: combine both vertical and horizontal velocities to find the magnitude and direction.





$$v^2 = v_x^2 + v_y^2$$

$$\tan\alpha = \frac{v_y}{v_x}$$

The object's velocity is v, at an angle of α below the horizontal.

The direction of the velocity vector v is the tangent to the path at the point.

4. **Maximum height**: use the vertical component when final velocity = 0. The path is symmetrical about the highest point when there is no air resistance.

The same set of kinematic equations in 1 dimension are still applicable. It becomes simpler for horizontal component.

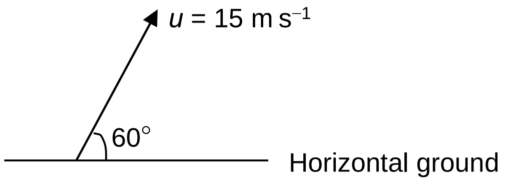
missing variable	general equations of motion	equations for projectile motion:	
		horizontal component $(a_x = 0)$	vertical component $(a_y \neq 0)$
s	$v = u + at$	$v_x = u_x$	$v_y = u_y + a_y t$
v	$s = ut + \frac{1}{2}at^2$	$s_x = u_x t$	$s_y = u_y t + \frac{1}{2}a_y t^2$
a	$s = \left(\frac{u+v}{2}\right)t$	$s_x = u_x t$	$s_y = \left(\frac{u_y + v_y}{2}\right)t$
t	$v^2 = u^2 + 2as$	$v_x = u_x$	$v_y^2 = u_y^2 + 2a_y s_y$

Problem solving strategy for Projectile Motion

1. Sketch the path of the projectile, including initial and final positions.
2. Resolve the initial velocity vector into *x*- and *y*-components.
3. Treat the horizontal motion and the vertical motion independently.
4. Time is the only quantity that is common for both *x*- and *y*- directions.
5. Follow the techniques for solving problems with constant velocity to analyse the horizontal motion of the projectile.
6. Follow the techniques for solving problems with constant acceleration to analyse the vertical motion of the projectile. (Note: vertical acceleration is 9.81 m s^{-2} downwards)

Example 1

A ball is projected from horizontal ground with an initial velocity of 15 m s^{-1} at an angle of 60° to the horizontal, as shown in the figure on the right.



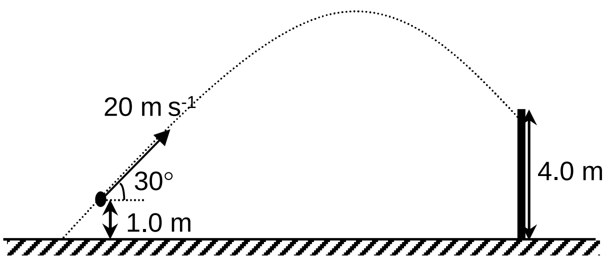
(a) Calculate, for this ball, the initial values of
- (i) the vertical component of the velocity,
- (ii) the horizontal component of the velocity.

(b) Assuming that air resistance can be neglected, use your answers in (a) to determine
- (i) the maximum height *H* to which the ball rises,
- (ii) the time of flight, i.e. the time interval between the instant when the ball is projected and the instant when it returns to the ground level,
- (iii) the range, i.e. the displacement along the same horizontal level (distance between the point from which the ball is projected and the point where it strikes the ground).

Take upward and rightward as positive

Example 2

A tennis ball is thrown from a height of 1.0 m above the ground with a speed of 20 m s^{-1} at an angle of 30° above the horizontal. On its downward motion, it strikes the top of a fence which is 4.0 m above the ground, as shown.



Calculate
(a) the greatest height above the ground reached by the ball,
(b) the time taken by the ball to reach this height,
(c) the time taken by the ball to reach the point of impact with the fence,
(d) the ball's velocity just before it hits the fence.

Take upward and rightward as positive

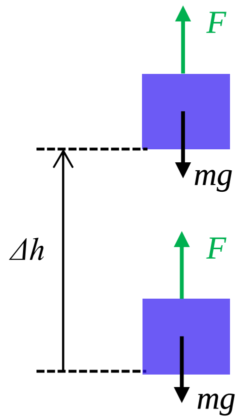
Example 3

A dive bomber, diving at an angle of 60° with the vertical, releases a bomb at an altitude of 600 m. The bomb hits the ground 5.0 s after being released.

(a) Calculate the speed of the bomber.
(b) Determine the horizontal displacement of the bomb.
(c) Determine the velocity of the bomb just before striking the ground.

Take downward as positive.

2.1 Derivation of $\Delta E_p = mg\Delta h$



object rising at constant velocity

- The gravitational force acting on a mass *m* in a gravitational field is ***mg***, the **weight** of the mass.
- Consider an object of mass *m* lifted by a constant force *F*, vertically upwards at **constant speed** from ground level to a change in height of Δh in a uniform gravitational field.
- At constant speed, the applied force *F* must **balance** the weight of object, $F = mg$
- Since the object's velocity is constant, its kinetic energy is also **constant**.
- The work done by the applied force *F* on the object to change its position in the gravitational field is transferred to an increase in gravitational potential energy ΔE_p (law of conservation of energy).
- Work done by *F* is

$$W = Fs$$
$$W = (mg)\Delta h$$
$$\boxed{\Delta E_p = mg\Delta h}$$

2.2 Recall and use the equation $\Delta Ep = mg\Delta h$

A mass can change its height, and thus its gravitational potential energy, when an external force (excluding gravitational force) does work, either increasing or decreasing its gravitational potential energy.

If there are no external force (excluding gravitational force), a mass can still change its height by changing its kinetic energy ($\frac{1}{2}mv^2$), as long as the sum of kinetic energy and gravitational potential energy remains constant. This is the principle of conservation of energy.

Expressed in equations:

$$KE_i + GPE_i = KE_f + GPE_f = \text{constant}$$
$$\frac{1}{2}mv_i^2 + mgh_i = \frac{1}{2}mv_f^2 + mgh_f$$

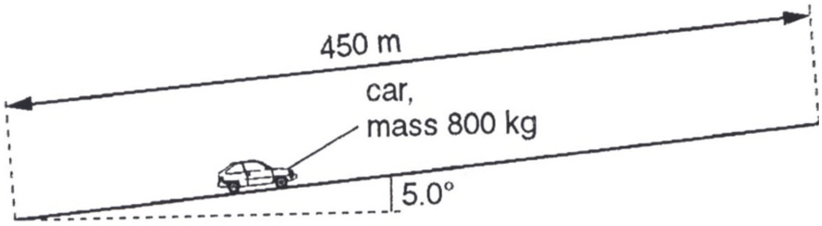
Where i and f refer to the initial and final values.
Rearranging the equation:

$$\frac{1}{2}mv_i^2 - \frac{1}{2}mv_f^2 = mgh_f - mgh_i$$
$$-\Delta KE = \Delta GPE$$
$$0 = \Delta GPE + \Delta KE$$

The sum of change in GPE and change in KE is zero.

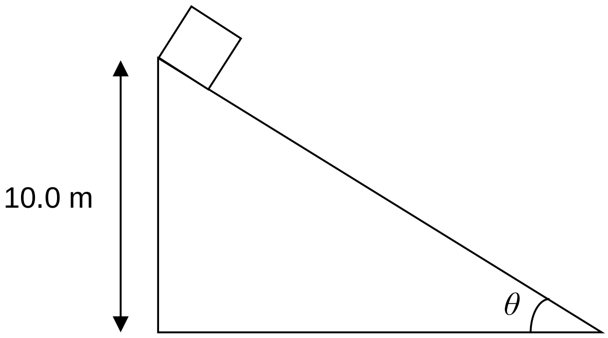
Example 4
A car travels upwards along a straight road inclined at an angle of 5.0º to the horizontal, as shown in the figure below.

The length of the road is 450 m and the car has a mass 800 kg.
What is the gain in gravitational potential energy of the car when it reaches the top of the slope?



Example 5
A block weighing 800 N moves down a frictionless slope inclined at 20º to the horizontal of height 10.0 m as shown below.

Determine the speed of the block at the bottom of the slope, assuming that the block moves off from the top at an initial speed of 5.0 m s^{-1}.



3 Air resistance

(e) Describe qualitatively, with reference to forces and energy, the motion of bodies falling in a uniform gravitational field with air resistance, including the phenomenon of terminal velocity

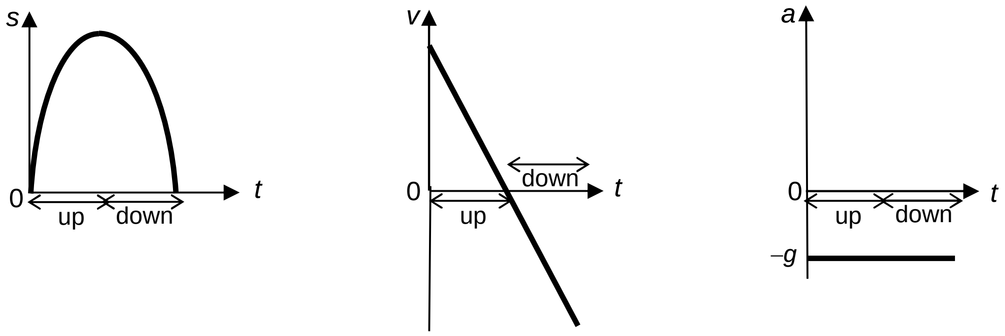
(a) Object thrown vertically up without air resistance (free fall)

If an object is thrown vertically upwards and air resistance can be ignored, the object is said to be in free fall motion.

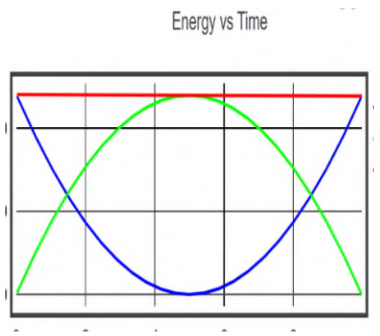
As the object rises, its velocity decreases steadily at a rate of 9.81 m s^{-2}.

When the object has reached a maximum height, it will stop momentarily before falling down.

Taking upward to be positive. Combining both upward and downward motion on the same graph, we get:



The variation of GPE, KE and Total energy with time is shown below



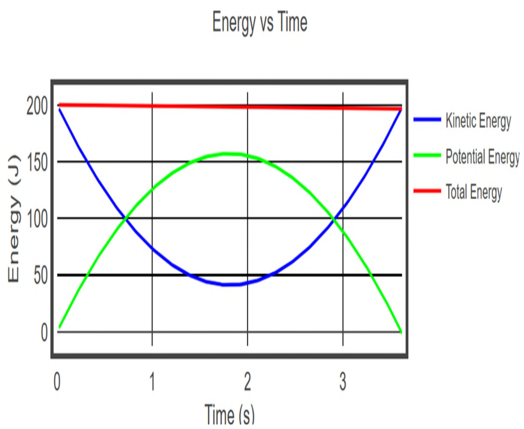
KE decreases from maximum to 0 at the highest point, then increases back to its initial value.

GPE increases from zero to a maximum at the higher point, then decrease back to zero.

The sum of KE and GPE is a constant value.

Example 6

Sketch the variation of gravitational potential energy and kinetic energy with time for a projectile launched at an angle from the horizontal.



1. As the projectile rises and falls, its GPE will change according to its height.
2. The KE will change in the reverse order to the GPE change by the same amount.
3. The KE will reach a non-zero minimum value at the highest point, because the horizontal velocity is not zero even when its vertical velocity is zero.

(b) Object thrown vertically up with air resistance (non-uniform acceleration)

When an object moves through air, it experiences an **opposing force** called air resistance. The following 2 points must be kept in mind when analysing motion involving air resistance:

- Magnitude of air resistance is proportional to the object's speed (at low speed) or square of its speed (at high speed). (Not required to memorise this relationship)
- Direction of air resistance is always opposite to the object's direction of motion or velocity.

If an object is thrown vertically upwards and air resistance is considered, the object is not in free fall motion. At the instant it is thrown, its velocity is the greatest and the opposing air resistance is the largest. Since both Earth's gravitational pull and air resistance are directed downwards, this causes the object's velocity to decrease at a faster rate (larger downward acceleration). The maximum height reached by the object is hence lower than the case without air resistance.

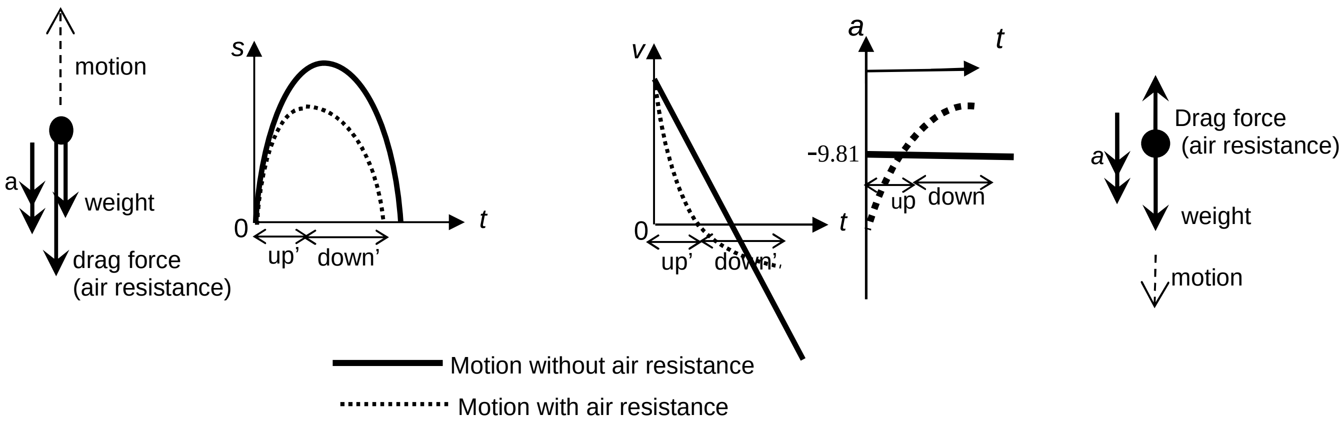
As the object travels up with decreasing velocity, the magnitude of the air resistance decreases, causing the velocity to decrease at a slower rate than its initial motion. Hence, the downward acceleration also decreases (but still larger in magnitude than 9.81 m s^{-2}) until the object reaches maximum height.

When the object is at maximum height, its velocity and air resistance at this instant are both zero. The only force acting on the object is the Earth's gravitational pull, so the value of deceleration is 9.81 m s^{-2} .

As the object travels down, its velocity and air resistance both increase. However, the air resistance is now acting upwards, opposite to Earth's gravitational pull. This causes the object's velocity to increase at a slower rate than the case without air resistance (smaller downward acceleration). Hence, as compared to the free fall case, it acquires a lower final velocity.

As compared to its upward motion, the object hence takes a longer time, on its downward motion to return to its starting position, than it took to travel up.

Comparing the *s-t* and *v-t* graphs with and without air resistance, we get the following:



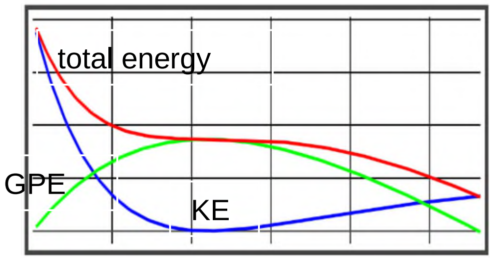
The *s-t* graph shows a lower the maximum height and a shorter time of flight.
The *v-t* graph has a steeper gradient on its way up and a smaller gradient on its way down. The gradient = 9.81 m s^{-2} at the highest point. Since the areas in the + and – part of *v-t* graph are the same, it means that the time taken must be longer on its way down.

The difference in acceleration in the upward and downward journey can also be seen using the free-body diagram and Newton's second law.

In the upward journey (leftmost diagram), the weight and drag force are in the same direction, hence producing a larger acceleration downward. The gradient of the *v*-*t* graph is steeper than *g*.

In the downward journey (rightmost diagram), the drag force is opposite to the weight direction. The net force downward is therefore smaller, giving rise to a smaller downward acceleration. The gradient of the graph is less than *g*.

In terms of energy, the change in gravitational potential energy is not equal to the change in kinetic energy when there is air resistance. The sum of KE and GPE is not constant but decreases continuously.



In the upward journey, the loss in KE = gain in PE + work done against air resistance.

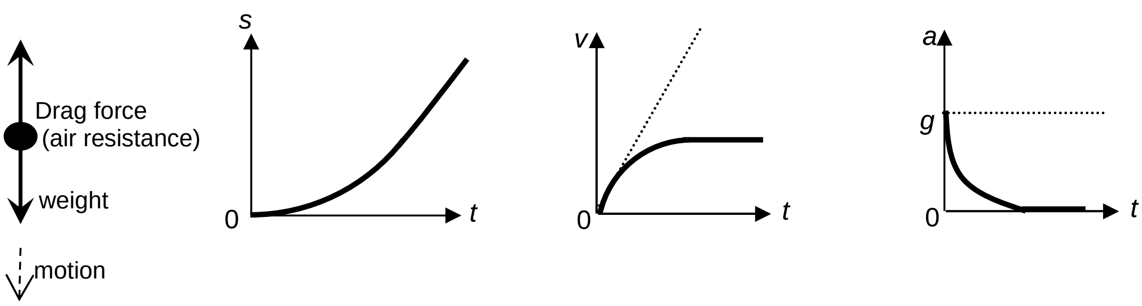
In the downward journey, the loss in GPE = gain in KE + work done against air resistance.

The energy used to overcome the force of air resistance is converted into thermal energy within the surrounding air particles, causing a rise in temperature.

(c) Terminal velocity in free fall

If an object falls through a long enough distance, and air resistance is present, the object's velocity will increase until it reaches a maximum velocity known as *terminal velocity*.

As the object falls, its velocity increases. This causes the opposing air resistance to increase. The air resistance increases until it is **equal** to the object's weight. The resultant force on the object is now **zero**. Hence the downward acceleration is zero and the object continues to move down with a constant velocity (terminal velocity).
Taking downward to be positive, the following graphs represent motion leading to terminal velocity.



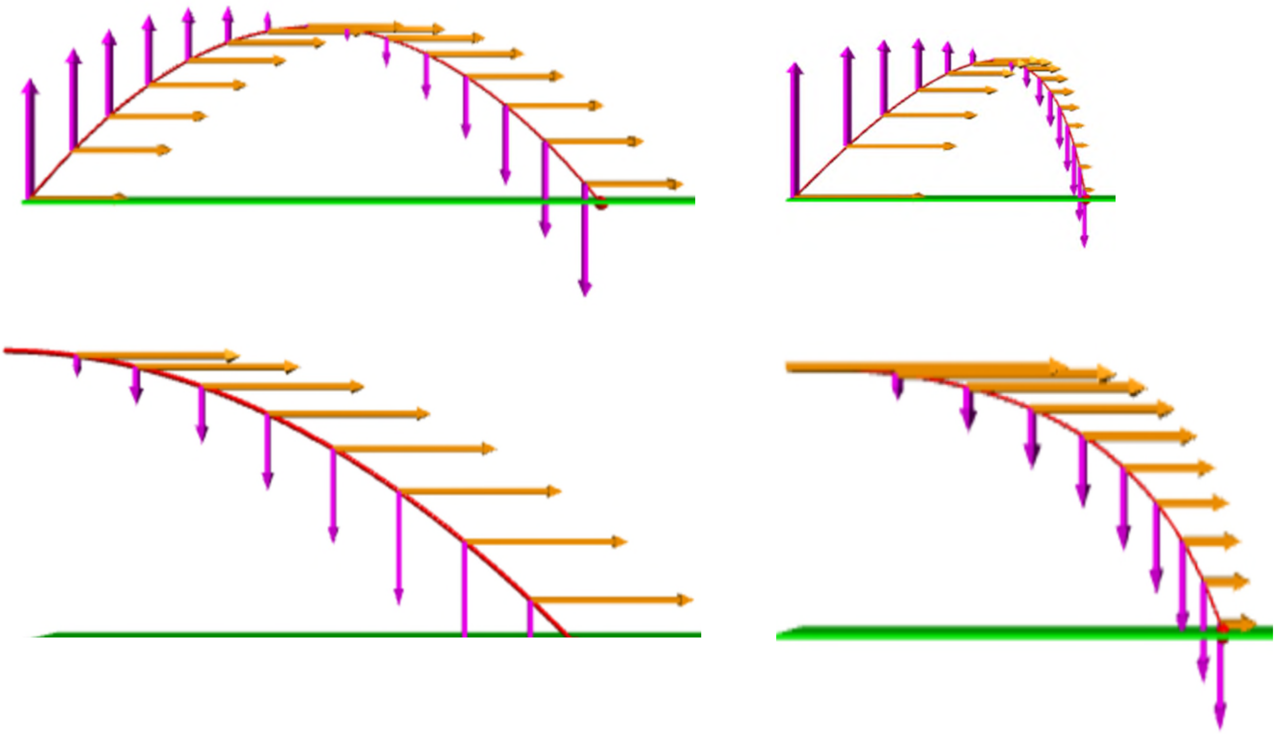
Graphs for terminal velocity (under air resistance)

When an object falls with terminal velocity, there is **no gain in KE**.

The loss in GPE = work done against air resistance.

(d) Projectile motion with air resistance

Air resistance reduces both the horizontal and vertical components of velocity of a projectile. Hence both the maximum height and horizontal distance will be reduced.
The trajectory also becomes asymmetrical about the highest point.



Comparison of motion without air resistance (left) and with air resistance (right)

If the degree of resistance force is increase, the effect will be more pronounced.