RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 5

Tutorial 5 : Complex Numbers

Section A (Basic Questions)

1 It is given that $z=1+2\mathrm{i}$.

(i) Without using a calculator, find the values of z^2 and $\dfrac{1}{z^3}$ in Cartesian form $x+\mathrm{i}y$, showing your working. [4]

(ii) The real numbers p and q are such that $pz^2+\dfrac{q}{z^3}$ is real. Find, in terms of p, the value of q and the value of $pz^2+\dfrac{q}{z^3}$. [3]

[**(i)** $z^2=-3+4\mathrm{i}$, $\dfrac{1}{z^3}=-\dfrac{11}{125}+\dfrac{2}{125}\mathrm{i}$ **(ii)** $q=-250p$, $19p$]

Solution:	
(i)	$z=1+2\mathrm{i}$ $z^2=(1+2\mathrm{i})^2=1+4\mathrm{i}-4=-3+4\mathrm{i}$ $z^3=(-3+4\mathrm{i})(1+2\mathrm{i})=-3+4\mathrm{i}-6\mathrm{i}-8=-11-2\mathrm{i}$ $\dfrac{1}{z^3}=\dfrac{1}{-11-2\mathrm{i}}=\dfrac{1}{-11-2\mathrm{i}}\times\dfrac{-11+2\mathrm{i}}{-11+2\mathrm{i}}=\dfrac{1}{125}(-11+2\mathrm{i})=-\dfrac{11}{125}+\dfrac{2}{125}\mathrm{i}$
(ii)	$pz^2+\dfrac{q}{z^3}=p(-3+4\mathrm{i})+q\left(-\dfrac{11}{125}+\dfrac{2}{125}\mathrm{i}\right)$ is real $\therefore \mathrm{Im}\left(pz^2+\dfrac{q}{z^3}\right)=0$ $4p+\dfrac{2}{125}q=0\Rightarrow q=-250p$ $pz^2+\dfrac{q}{z^3}=p(-3+4\mathrm{i})-250p\left(-\dfrac{11}{125}+\dfrac{2}{125}\mathrm{i}\right)$ $=-3p+4p\mathrm{i}+22p-4p\mathrm{i}$ $=19p$

2 Solve the following simultaneous equations:

(a) $w+z=6+2\mathrm{i}$, $w-3z=\dfrac{20}{2-\mathrm{i}}$; **(b)** $z=w+3\mathrm{i}+2$, $z^2-\mathrm{i}w+5-2\mathrm{i}=0$.

[**(a)** $z=-\frac{1}{2}-\frac{1}{2}\mathrm{i}$, $w=\frac{13}{2}+\frac{5}{2}\mathrm{i}$ **(b)** $z=2\mathrm{i}$, $w=-2-\mathrm{i}$ or $z=-\mathrm{i}$, $w=-2-4\mathrm{i}$]

Solution:	
(a)	$w+z=6+2\mathrm{i}$ ---------(1) $w-3z=\dfrac{20}{2-\mathrm{i}}$ ---------(2) (1)–(2), $4z=6+2\mathrm{i}-\dfrac{20}{2-\mathrm{i}}\times\dfrac{2+\mathrm{i}}{2+\mathrm{i}}$ $=6+2\mathrm{i}-\dfrac{1}{5}(40+20\mathrm{i})=-2-2\mathrm{i}$ $z=-\dfrac{1}{2}-\dfrac{1}{2}\mathrm{i}$ $\therefore w=6+2\mathrm{i}+\dfrac{1}{2}+\dfrac{1}{2}\mathrm{i}$ $=\dfrac{13}{2}+\dfrac{5}{2}\mathrm{i}$
(b)	$z=w+3\mathrm{i}+2$ --------------(1) $z^2-\mathrm{i}w+5-2\mathrm{i}=0$ --------(2) From (1), $w=z-3\mathrm{i}-2$ --------(3) Substitute (3) into (2), $z^2-\mathrm{i}(z-3\mathrm{i}-2)+5-2\mathrm{i}=0$ $z^2-\mathrm{i}z-3+2\mathrm{i}+5-2\mathrm{i}=0$ $z^2-\mathrm{i}z+2=0$ $z=\dfrac{\mathrm{i}\pm\sqrt{(-\mathrm{i})^2-4(1)(2)}}{2(1)}$ $=\dfrac{\mathrm{i}\pm\sqrt{-9}}{2}$ $=\dfrac{\mathrm{i}\pm 3\mathrm{i}}{2}=2\mathrm{i}$ or $-\mathrm{i}$ $w=-2-\mathrm{i}$ or $w=-2-4\mathrm{i}$

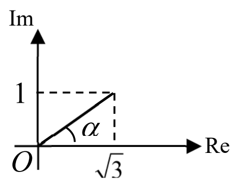
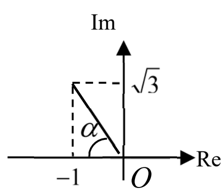
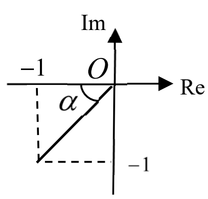
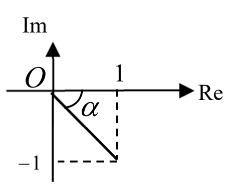
3 The complex number w is such that $ww^*+2w=3+4\text{i}$, where w^* is the complex conjugate of w. Find w in the form $a+b\text{i}$, where a and b are real. [4]

[$w=-1+2\text{i}$]

Solution:
$ww^*+2w=3+4\text{i}$ $(a+\text{i}b)(a-\text{i}b)+2(a+\text{i}b)=3+4\text{i}$ $a^2-(\text{i}b)^2+2a+2b\text{i}=3+4\text{i}$ $a^2+b^2+2a+2b\text{i}=3+4\text{i}$ Comparing real and imaginary parts, $a^2+b^2+2a=3$ and $2b=4 \Rightarrow b=2$ $a^2+4+2a=3$ $a^2+2a+1=0$ $(a+1)^2=0$ $a=-1$ $\therefore w=-1+2\text{i}$

4 For each of the following complex number, represent it on an Argand diagram and find its modulus and argument.

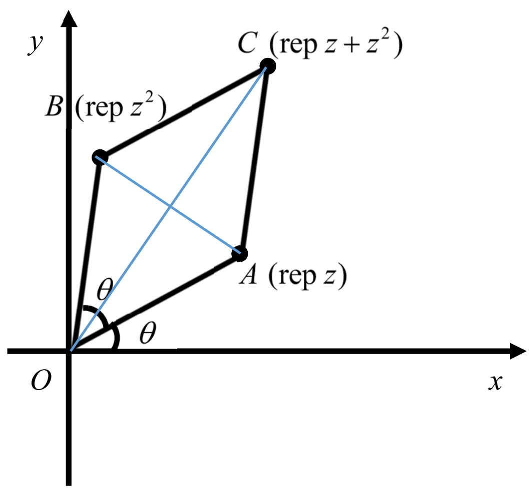
(a) $\sqrt{3}+\mathrm{i}$ **(b)** $-1+\mathrm{i}\sqrt{3}$ **(c)** $\mathrm{i}^2(1+\mathrm{i})$ **(d)** $-\mathrm{i}(1+\mathrm{i})$

Solution:		
(a)		**(a)** $\sqrt{3}+\mathrm{i}$ $\left\|\sqrt{3}+\mathrm{i}\right\| = \sqrt{3+1} = 2$ $\tan\alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6} \Rightarrow \arg(\sqrt{3}+\mathrm{i}) = \alpha = \frac{\pi}{6}$
(b)		**(b)** $-1+\mathrm{i}\sqrt{3}$ $\left\|-1+\mathrm{i}\sqrt{3}\right\| = \sqrt{1+3} = 2$ $\tan\alpha = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3} \Rightarrow \arg(-1+\mathrm{i}\sqrt{3}) = \pi-\alpha = \frac{2\pi}{3}$
(c)		**(c)** $\mathrm{i}^2(1+\mathrm{i}) = -1-\mathrm{i}$ $\left\|-1-\mathrm{i}\right\| = \sqrt{1+1} = \sqrt{2}$ $\tan\alpha = 1 \Rightarrow \alpha = \frac{\pi}{4} \Rightarrow \arg(-1-\mathrm{i}) = -(\pi-\alpha) = -\frac{3\pi}{4}$
(d)		**(d)** $-\mathrm{i}(1+\mathrm{i}) = 1-\mathrm{i}$ $\left\|1-\mathrm{i}\right\| = \sqrt{1+1} = \sqrt{2}$ $\tan\alpha = 1 \Rightarrow \alpha = \frac{\pi}{4} \Rightarrow \arg(1-\mathrm{i}) = -\alpha = -\frac{\pi}{4}$

5 The complex number z is such that $|z|=1$ and $\arg z=\theta$, where $0<\theta<\dfrac{\pi}{4}$.

(i) Mark a possible point A representing z on an Argand diagram. Given that $\arg\left(z^2\right)=2\arg z$, mark the points B and C representing z^2 and $z+z^2$ respectively on the same Argand diagram corresponding to point A. [2]

(ii) State the geometrical shape of $OACB$. [1]

Solution:	
(i)	
(ii)	Since $OACB$ is a parallelogram with 4 equal sides, it is a rhombus.

6 **(i)** Find the roots of the equation $\mathrm{i}z^2-(5+\mathrm{i})z+2-6\mathrm{i}=0$, giving your answers in cartesian form $a+b\mathrm{i}$, where $a,b\in\mathbb{R}$. [2]

(ii) Hence find the roots of the equation $-\mathrm{i}w^2-(1-5\mathrm{i})w+2-6\mathrm{i}=0$, giving your answers in cartesian form $a+b\mathrm{i}$, where $a,b\in\mathbb{R}$. [2]

(iii) Given that the roots found in part **(i)** are also roots of the equation $P(z)=0$, where $P(z)$ is a polynomial of degree 4 with real coefficients, find $P(z)$. [3]

[**(i)** $1-3\mathrm{i}$ or $-2\mathrm{i}$ **(ii)** $w=3+\mathrm{i}$ or 2 **(iii)** $z^4-2z^3+14z^2-8z+40$]

Solution:	
(i)	$\mathrm{i}z^2-(5+\mathrm{i})z+2-6\mathrm{i}=0$ $$\begin{aligned} z&=\frac{5+\mathrm{i}\pm\sqrt{[-(5+\mathrm{i})]^2-4(\mathrm{i})(2-6\mathrm{i})}}{2\mathrm{i}}\\ &=\frac{5+\mathrm{i}\pm\sqrt{2\mathrm{i}}}{2\mathrm{i}}\\ &=\frac{5+\mathrm{i}\pm(1+\mathrm{i})}{2\mathrm{i}}\\ &=\frac{6+2\mathrm{i}}{2\mathrm{i}} \text{ or } \frac{4}{2\mathrm{i}}\\ &=1-3\mathrm{i} \text{ or } -2\mathrm{i}\end{aligned}$$
(ii)	$-\mathrm{i}w^2-(1-5\mathrm{i})w+2-6\mathrm{i}=0$ Since $w=\mathrm{i}z$, $w=\mathrm{i}(1-3\mathrm{i})$ or $\mathrm{i}(-2\mathrm{i})$ $=3+\mathrm{i}$ or 2
(iii)	Since $P(z)$ is a polynomial of degree 4 with real coefficient, $1+3\mathrm{i}$ and $2\mathrm{i}$ are also the roots. $$\begin{aligned} P(z)&=(z+2\mathrm{i})(z-2\mathrm{i})[z-(1-3\mathrm{i})][z-(1+3\mathrm{i})]\\ &=(z^2-4\mathrm{i}^2)[(z-1)+3\mathrm{i}][(z-1)-3\mathrm{i}]\\ &=(z^2+4)[(z-1)^2+9]\\ &=(z^2+4)(z^2-2z+10)\\ &=z^4-2z^3+14z^2-8z+40\end{aligned}$$

7 The cubic equation $az^3 - 31z^2 + 212z + b = 0$, where a and b are real numbers, has a complex root $z = 1 - 3\text{i}$.

(i) Explain why the equation must have a real root. [2]

(ii) Find the values of a and b and the real root, showing your working clearly. [5]

[**(ii)** $a = 25, b = 190$, $-\dfrac{19}{25}$]

Solution:	
(i)	Since the **<u>coefficients</u>** of $az^3 - 31z^2 + 212z + b = 0$ are **<u>all real</u>**, **<u>complex roots occur in conjugate pair</u>**. Since a **<u>cubic equation has three roots</u>**, the third root must be a real root.
(ii)	$az^3 - 31z^2 + 212z + b = 0$ Given $1-3\text{i}$ is a root, $1+3\text{i}$ is also a root. $\left[z-(1-3\text{i})\right]\left[z-(1+3\text{i})\right]$ $=\left[(z-1)+3\text{i}\right]\left[(z-1)-3\text{i}\right]$ $=(z-1)^2+9$ $=z^2-2z+10$ $az^3 - 31z^2 + 212z + b = (z^2-2z+10)(az+c)$ By comparing coefficients of z^2 : $-31 = c - 2a$ z : $212 = -2c + 10a$ From GC, $c = 19, a = 25$ z^0 : $b = 10c \Rightarrow b = 190$ $\therefore$ The real root of the equation is $-\dfrac{c}{a} = -\dfrac{19}{25}$. <u>Alternative method</u>: Since $1-3\text{i}$ is a root of $az^3 - 31z^2 + 212z + b = 0$, $a(1-3\text{i})^3 - 31(1-3\text{i})^2 + 212(1-3\text{i}) + b = 0$ $a(-26+18\text{i}) - 31(-8-6\text{i}) + 212(1-3\text{i}) + b = 0$ $(-26a+460+b) + (18a-450)\text{i} = 0$ Comparing real and imaginary parts: $-26a + 460 + b = 0$ ----------- (1) $18a - 450 = 0$ -----------------(2)

	From (2), $a=25$, $b=190$ $\left[z-(1-3\mathrm{i})\right]\left[z-(1+3\mathrm{i})\right]$ $=\left[(z-1)+3\mathrm{i}\right]\left[(z-1)-3\mathrm{i}\right]$ $=(z-1)^2+9$ $=z^2-2z+10$ $25z^3-31z^2+212z+190=\left(z^2-2z+10\right)\left(25z+d\right)$ Comparing constant: $190=10d$ $d=19$ The real root is $-\dfrac{19}{25}$.