

NANYANG JUNIOR COLLEGE

JC2 Preliminary Examination

Higher 2

FURTHER MATHEMATICS

9649/01

Paper 1

16 September 2026

3 hours

Additional Materials: Printed Answer Booklet
List of Formulae and Results (MF27)

READ THESE INSTRUCTIONS FIRST

Answer **all** the questions.

Write your answers on the Printed Answer Booklet. Follow the instructions on the front cover of the answer booklet.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are **not** allowed in a question, you must present the mathematical steps using mathematical notations and not calculator commands.

You must show all necessary working clearly.

The number of marks is given in brackets [] at the end of each question or part question.

This document consists of 7 printed pages and 1 blank page.



NANYANG JUNIOR COLLEGE
Internal Examinations

- 1 A sequence $\{u_n\}$ is given by

$$u_1 = M \quad \text{and} \quad u_{n+1} = \frac{u_n^3 + 6}{7} \quad \text{for } n \geq 1,$$

where M is a constant.

It is given that the sequence converges to the limit L .

- (a) Find the possible values of L . [1]

- (b) Show that if $1 < u_n < 2$, then

$$1 < u_{n+1} < 2 \quad \text{and} \quad u_{n+1} < u_n. \quad [3]$$

- (c) Given that $M = 1.5$, deduce the value of L . [1]

- 2 The complex variables w and z , represented by the points W and Z respectively in an Argand diagram, are related by the equation

$$w = \frac{z + 4i}{z - 2}, \quad z \neq 2.$$

- (a) Given that the locus of W is the imaginary axis, find the cartesian equation of the locus of Z and give a geometrical description of this locus. [4]

- (b) Given instead that the locus of W is the real axis, show that the locus of Z is a straight line, and write the equation of the line in the form

$$pz + qz^* = r$$

where z^* denote the complex conjugate of z and p, q and r are complex numbers to be determined. [3]

- 3 (a) It is given that for non-negative integer n , $I_n = \int_0^{\frac{\pi}{6}} \sin^n \theta \, d\theta$.

Show that

$$nI_n = -\frac{\sqrt{3}}{2^n} + (n-1)I_{n-2}, \quad \text{for } n \geq 2. \quad [3]$$

- (b) The region R is bounded by the curve $y = x^2 \sqrt{1-x^2}$, the x -axis and the line $x = \frac{1}{2}$ where $x \in \left[0, \frac{1}{2}\right]$.

Using the shell method and the substitution $x = \sin \theta$, show that the volume of solid generated when R is rotated through 2π radians about the y -axis is given by

$$\frac{\pi}{k} (m + n\sqrt{3}),$$

where k, m and n are integers to be determined. [5]

- 4 Ethan uses the expression

$$E = af(0) + bf\left(\frac{1}{2}\right) + cf(1)$$

to estimate the integral $I = \int_0^1 (1+x)f(x) dx$, where a , b and c are constants. It is given that Ethan's method gives the exact value of I for $f(x) = 1$, x and x^2 .

- (a) Find the values of a , b and c . [4]
- (b) The Simpson's rule with two strips is applied directly to estimate I . Show that Simpson's approximation is identical to Ethan's method. [2]
- (c) Tang applies the trapezium rule with two strips to estimate I . Show that his approximation can be written in the form

$$T = \alpha f(0) + \beta f\left(\frac{1}{2}\right) + \gamma f(1)$$

where α , β and γ are values to be determined. [2]

- (d) If $f(x)$ is a quadratic function, determine whose estimate is more accurate. [2]

- 5 The variables x , y and z are related by the equation $z = f(x, y)$ where f is a function of the independent variables x and y .

We define the independent *differentials* dx and dy to be the small increments of x and y respectively. The *total differential* dz , which is the small increment of z corresponding to small increments of x and y , is given by

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy.$$

A right circular cone has a circular base radius of 9 cm and a height of 20 cm. Denote

- the radius of the circular base of the cone by r ,
- the height of the cone by h ,
- the volume of the cone by V .

- (a) Find expressions in terms of r and h where appropriate, for $\frac{\partial V}{\partial r}$ and $\frac{\partial V}{\partial h}$. [2]

In one instance, in measuring r and h , a man made a maximum error of 0.01 cm in each of them.

- (b) Use the above result to estimate the maximum possible error in calculating V , leaving your answer in the form $k\pi$ where k is to be determined to two decimal places. [3]

In another instance, a man used the same faulty instrument to measure r and h and gave an overestimation of each by an amount ε cm but maintained that the error in calculating V is no more than π cm³.

- (c) Determine in fractional form, the largest possible value of ε . [3]

- 6 A water authority is considering a few potential service reservoirs to supply water to a prospective new town for short-term usage.

For a particular service reservoir, let w_n denote the volumes of water (in million litres) in this reservoir at the start of month n . The volume of water in the reservoir is modelled by

$$w_n = 0.8w_{n-1} + 0.32w_{n-2},$$

where $n \geq 3$, $w_1 = 100$ and $w_2 = 80$.

- (a) Find an exact expression for w_n in terms of n and $\sqrt{3}$. [6]

Another service reservoir is identified to be a nearer water source. Let x_n denote the volumes of water (in million litres) in this reservoir at the **end** of month n . The volume of water in the reservoir is modelled by

$$x_{n+2} = 1.4x_{n+1} - 0.45x_n + (F - Gn),$$

where $n \geq 0$, $x_0 = 100$ and $x_1 = 200$.

- (b) State the significance of the positive constants F and G . [1]

To remain sustainable, the reservoir must not be depleted. To determine whether the reservoir can sustain the new town over a 30-month horizon ($0 \leq n \leq 30$), the value of F is initially fixed at 100 while G is varied, where $4 \leq G \leq 9$.

- (c) For the case when $G = 4$, use your graphing calculator to investigate the behaviour of x_n over the 30-month horizon and comment on its sustainability. [1]
- (d) By considering the behaviour of x_n over the 30-month horizon when $G = 6$ and $G = 9$, comment on the proposed model and how the value of F should be adjusted, so that the reservoir remains sustainable over the 30-month horizon for all values of G satisfying $4 \leq G \leq 9$. [2]

7 Consider the system of equations
$$\begin{cases} x + (\sin \theta)y & = b_1 \\ y + (\cos \theta)z & = b_2, \\ x + (1 + \sin \theta)y + (\alpha + \cos \theta)z & = b_3 \end{cases}$$

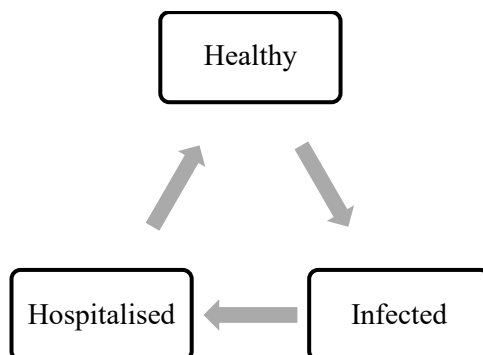
where α , b_1 , b_2 , b_3 are real constants and $\theta \in [0, 2\pi)$.

- (a) Show that if $\alpha \neq 0$, the system is always consistent for any values of b_1 , b_2 , b_3 and the solution set is a point that does not pass through the origin when $b_1 \neq 0$, $b_2 \neq 0$, $b_3 \neq 0$. [3]
- (b) Explain the geometrical relationship of the solution set when $\alpha = 0$, $\theta = \frac{\pi}{2}$.
Discuss how this relationship depends on the values of b_1 , b_2 , b_3 . [2]
- (c) Write down the coefficient matrix of the above system of linear equations and find its determinant in terms of α and θ . Explain how the determinant confirms the condition in part (a). [2]
- (d) Given $\alpha = 1$ and $\theta = \pi$, determine if the coefficient matrix is diagonalisable over $\mathbb{R}$. [4]

- 8 A public health researcher studies the spread of an infectious disease in a closed population. Individuals are classified into three groups:

- **Healthy:** individuals who are not infected but remain susceptible to the disease.
- **Infected:** individuals who carry the disease but do not require medical treatment.
- **Hospitalised:** individuals whose symptoms are severe enough to require hospital care.

Individuals who recover from hospital treatment become healthy again but may later become infected. The movement between the three groups is illustrated below:



The model assumes that each week:

- 10% of the healthy individuals become infected.
- 20% of the infected individuals become hospitalised.
- 60% of the hospitalised individuals recover and become healthy.

Let x , y and z (in millions) be the number of healthy, infected and hospitalised individuals at time t weeks respectively. Assume that the total number of individuals in the population is 1 million throughout the period of research. Initially, $x = 0.7$ and $y = 0.3$. Assume that x , y and z can be treated as continuous variables.

- (a) It is given that $\frac{dx}{dt} = -0.1x + 0.6z$.

Write down, in a similar form in terms of x , y and z where appropriate, $\frac{dy}{dt}$ and $\frac{dz}{dt}$. [2]

- (b) Show that

$$\frac{d^2x}{dt^2} + 0.9\frac{dx}{dt} + 0.2x = 0.12. \quad [3]$$

- (c) Hence, or otherwise, find x in terms of t . [5]

- (d) Determine what happens to the number of healthy, infected and hospitalised individuals in the long term. [3]

- 9 A horizontal plane can be modelled by the cartesian xy -plane with origin O at the point $(0,0)$. A transparent structure S is built with its base on the horizontal plane such that the height, h , of a point on S vertically above the point (x,y) on its horizontal base is given by

$$h = 36 - 3x^2 - y^2$$

where x and y are independent variables satisfying $3x^2 + y^2 \leq 36$.

Points A and B on the horizontal base of S have coordinates $(1,0)$ and $(2,2)$ respectively. Three tiny light bulbs are placed at the points O , A and B .

A tiny AI-enabled robot equipped with a light sensor starts at a point on S and moves continuously on its surface in the direction of increasing x and y in such a way that the vertical projection of its path onto the horizontal base of S is the line L with equation of the form $y = ax + b$ such that the quantity

$$\sum_{i=1}^3 (ax_i + b - y_i)^2$$

where (x_i, y_i) , $i = 1, 2, 3$ represent the points O , A and B , is **minimised**.

- (a) Use partial differentiation to show that the equation of L is

$$y = x - \frac{1}{3},$$

justifying your conclusion.

[6]

At the instant when the robot is at the point on S vertically above the point P on the horizontal base of S with coordinates $\left(\frac{4}{3}, 1\right)$, it shoots out a beam of laser in the direction of its velocity and strikes the horizontal xy -plane at the point Q .

- (b) By calculating an appropriate directional derivative, find a vector equation of the line represented by the beam of laser. [6]
- (c) Find the length of PQ . [2]

- 10 (a)** Given that $0 < \theta < \frac{1}{2}\pi$, express

$$\frac{1 + i \tan \theta}{1 - i \tan \theta}$$

in the form $re^{i\phi}$ where $r > 0$ and ϕ is a function of θ which is to be stated. [2]

- (b)** Hence find the three complex roots of the equation

$$z^3 = \frac{1 + i\sqrt{3}}{1 - i\sqrt{3}},$$

giving your answers in the form $re^{i\theta}$ where $r > 0$ and $-\pi < \theta \leq \pi$. [4]

- (c)** Deduce the roots of the equation

$$z^3 = \frac{1 - i\sqrt{3}}{1 + i\sqrt{3}},$$

giving your answers in exponential form. [2]

The three complex roots in part **(b)** are denoted by a , b and c and are such that $\arg a < \arg b < \arg c$.

- (d)** Mark the complex numbers a , b and c as points on an Argand diagram, indicating clearly their modulus and arguments. [3]

- (e)** With the aid of your diagram in part **(d)**, or otherwise, evaluate

$$|(a-1)(b-1)(c-1)|,$$

giving your answer in the form

$$k \sin \frac{p\pi}{9} \sin \frac{q\pi}{9} \sin \frac{r\pi}{9}$$

where k , p , q and r are positive integers to be determined. [3]

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2026 Preliminary Examination
Further Mathematics Paper 1
Solutions and Marking Scheme

1 A sequence $\{u_n\}$ is given by

$$u_1 = M \quad \text{and} \quad u_{n+1} = \frac{u_n^3 + 6}{7} \quad \text{for } n \geq 1,$$

where M is a constant.

It is given that the sequence converges to the limit L .

(a) Find the possible values of L . [1]

(b) Show that if $1 < u_n < 2$, then [3]

$$1 < u_{n+1} < 2 \quad \text{and} \quad u_{n+1} < u_n.$$

(c) Given that $M = 1.5$, deduce the value of L . [1]

1(a)	As $n \rightarrow \infty$, $u_n \rightarrow L$, $u_{n+1} \rightarrow L$. $L = \frac{L^3 + 6}{7} \Rightarrow L^3 - 7L + 6 = 0$ Using G.C., $L = -3, 2$ or 1	
(b)	$1 < u_n < 2 \Rightarrow \frac{1^3 + 6}{7} < \frac{u_n^3 + 6}{7} < \frac{2^3 + 6}{7}$ $\Rightarrow 1 < u_{n+1} < 2$ $u_{n+1} - u_n = \frac{u_n^3 + 6}{7} - u_n = \frac{u_n^3 - 7u_n + 6}{7} = \frac{(u_n - 1)(u_n - 2)(u_n + 3)}{7}$ $1 < u_n < 2 \Rightarrow u_n - 1 > 0, u_n - 2 < 0 \quad \text{and} \quad u_n + 3 > 0$ Thus $u_{n+1} - u_n < 0 \Rightarrow u_{n+1} < u_n$.	
(c)	By (b), since $1 < u_1 < 2 \Rightarrow 1 < u_2, u_3, \dots < 2$ and since the sequence is decreasing, $L = 1$.	

- 2 The complex variables w and z , represented by the points W and Z respectively in an Argand diagram, are related by the equation

$$w = \frac{z + 4i}{z - 2}, \quad z \neq 2.$$

- (a) Given that the locus of W is the imaginary axis, find the cartesian equation of the locus of Z and give a geometrical description of this locus. [4]
- (b) Given instead that the locus of W is the real axis, show that the locus of Z is a straight line, and write the equation of the line in the form

$$pz + qz^* = r \quad [3]$$

where z^* denote the complex conjugate of z and p , q and r are complex numbers to be determined.

2(a)	Since W lies on the imaginary axis, $\operatorname{Re}(w) = 0$. Let $z = x + yi$ where $(x, y) \neq (2, 0)$. $w = \frac{z + 4i}{z - 2} = \frac{x + yi + 4i}{x + yi - 2}$ $= \frac{[x + (y + 4)i][\overline{(x - 2 + yi)}]}{[(x - 2) + yi][\overline{(x - 2 + yi)}]}$ $= \frac{x(x - 2) + y(y + 4) + [(x - 2)(y + 4) - xy]i}{(x - 2)^2 + y^2}$ $\operatorname{Re}(w) = 0 \Rightarrow x(x - 2) + y(y + 4) = 0$ $\Rightarrow (x - 1)^2 + (y + 2)^2 = 5, \quad (x, y) \neq (2, 0)$ which is a circle of radius $\sqrt{5}$ and centre at $(1, -2)$ with the point $(2, 0)$ removed.	
(b)	Since W lies on the real axis, $\operatorname{Im}(w) = 0$. $\operatorname{Im}(w) = 0 \Rightarrow (x - 2)(y + 4) - xy = 0$ from (a) $\Rightarrow 2x - y = 4, \quad (x, y) \neq (2, 0)$ which is the cartesian equation of a straight line with the point $(2, 0)$ removed. $z + z^* = 2\operatorname{Re}(z) = 2x \Rightarrow x = \frac{z + z^*}{2}$ and $z - z^* = 2i\operatorname{Im}(z) = 2iy \Rightarrow y = \frac{z - z^*}{2i}$ Substitute above into $2x - y = 4$ gives $2\left(\frac{z + z^*}{2}\right) - \left(\frac{z - z^*}{2i}\right) = 4 \Rightarrow \left(1 + \frac{1}{2}i\right)z + \left(1 - \frac{1}{2}i\right)z^* = 4.$	

- 3 (a) It is given that for non-negative integer n , $I_n = \int_0^{\frac{\pi}{6}} \sin^n \theta \, d\theta$. Show that

$$nI_n = -\frac{\sqrt{3}}{2^n} + (n-1)I_{n-2}, \text{ for } n \geq 2. \quad [3]$$

- (b) The region R is bounded by the curve $y = x^2 \sqrt{1-x^2}$, the x -axis and the line $x = \frac{1}{2}$ where $x \in \left[0, \frac{1}{2}\right]$.

Using the shell method and the substitution $x = \sin \theta$, show that the volume of solid generated when R is rotated through 2π radians about the y -axis is given by

$$\frac{\pi}{k}(m + n\sqrt{3}),$$

where k , m and n are integers to be determined.

[5]

(a)	$ \begin{aligned} I_n &= \int_0^{\frac{\pi}{6}} \sin \theta \cdot \sin^{n-1} \theta \, d\theta \\ &= \left[-\cos \theta \cdot \sin^{n-1} \theta \right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} -\cos \theta \cdot (n-1) \sin^{n-2} \theta \cos \theta \, d\theta \\ &= -\frac{\sqrt{3}}{2} \left(\frac{1}{2} \right)^{n-1} + (n-1) \int_0^{\frac{\pi}{6}} \cos^2 \theta \cdot \sin^{n-2} \theta \, d\theta \\ &= -\frac{\sqrt{3}}{2^n} + (n-1) \int (1 - \sin^2 \theta) \cdot \sin^{n-2} \theta \, d\theta \\ &= -\frac{\sqrt{3}}{2^n} + (n-1) \left[\int \sin^{n-2} \theta \, d\theta - \int \sin^n \theta \, d\theta \right] \\ &= -\frac{\sqrt{3}}{2^n} + (n-1) [I_{n-2} - I_n] \\ &= -\frac{\sqrt{3}}{2^n} + (n-1) I_{n-2} - (n-1) I_n \\ nI_n &= -\frac{\sqrt{3}}{2^n} + (n-1) I_{n-2} \end{aligned} $	
(b)	Volume of solid generated $ \begin{aligned} &2\pi \int_0^{\frac{1}{2}} x (x^2 \sqrt{1-x^2}) \, dx \\ &= 2\pi \int_0^{\frac{1}{2}} x^3 \sqrt{1-x^2} \, dx \\ &= 2\pi \int_0^{\frac{\pi}{6}} \sin^3 \theta \sqrt{1-\sin^2 \theta} \cos \theta \, d\theta \\ &= 2\pi \int_0^{\frac{\pi}{6}} \sin^3 \theta \cos^2 \theta \, d\theta \\ &= 2\pi \int_0^{\frac{\pi}{6}} \sin^3 \theta (1 - \sin^2 \theta) \, d\theta \\ &= 2\pi \left[\int_0^{\frac{\pi}{6}} \sin^3 \theta \, d\theta - \int_0^{\frac{\pi}{6}} \sin^5 \theta \, d\theta \right] \\ &= 2\pi (I_3 - I_5) \end{aligned} $ From (a), $I_n = -\frac{\sqrt{3}}{n(2^n)} + \frac{n-1}{n} I_{n-2}$.	

$I_1 = \int_0^{\frac{\pi}{6}} \sin \theta \, d\theta = -[\cos \theta]_0^{\frac{\pi}{6}} = 1 - \frac{\sqrt{3}}{2}$ $I_3 = -\frac{\sqrt{3}}{3(2^3)} + \frac{2}{3}I_1 = -\frac{\sqrt{3}}{24} + \frac{2}{3}\left(1 - \frac{\sqrt{3}}{2}\right) = \frac{2}{3} - \frac{3}{8}\sqrt{3}$ $I_5 = -\frac{\sqrt{3}}{5(2^5)} + \frac{4}{5}I_3 = -\frac{\sqrt{3}}{160} + \frac{4}{5}I_3$ $2\pi(I_3 - I_5) = 2\pi\left[I_3 - \left(-\frac{\sqrt{3}}{160} + \frac{4}{5}I_3\right)\right]$ $= 2\pi\left[\frac{\sqrt{3}}{160} + \frac{1}{5}I_3\right]$ $= 2\pi\left[\frac{\sqrt{3}}{160} + \frac{1}{5}\left(\frac{2}{3} - \frac{3}{8}\sqrt{3}\right)\right]$ $= 2\pi\left(\frac{2}{15} - \frac{11}{160}\sqrt{3}\right)$ $= \frac{\pi}{240}(64 - 33\sqrt{3})$ $k = 240, m = 64, n = -33$	
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4 Ethan uses the expression

$$E = af(0) + bf\left(\frac{1}{2}\right) + cf(1)$$

to estimate the integral $I = \int_0^1 (1+x)f(x) \, dx$, where a, b and c are constants. It is given that Ethan's method gives the exact value of I for $f(x) = 1, x$ and x^2 .

(a) Find the values of a, b and c . [4]

(b) The Simpson's rule with two strips is applied directly to estimate I . Show that Simpson's approximation is identical to Ethan's method. [2]

(c) Tang applies the trapezium rule with two strips to estimate I . Show that his approximation can be written in the form

$$T = \alpha f(0) + \beta f\left(\frac{1}{2}\right) + \gamma f(1)$$

where α, β and γ are values to be determined. [2]

(d) If $f(x)$ is a quadratic function, determine whose estimate is more accurate. [2]

4(a)	$f(x) = 1: I = \int_0^1 (1+x) \, dx = 1.5$ $a + b + c = 1.5 \text{ ---- (1)}$ $f(x) = x: I = \int_0^1 (1+x)x \, dx = \frac{5}{6}$	
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	$\frac{1}{2}b + c = \frac{5}{6} \text{ ----- (2)}$ $f(x) = x^2 : I = \int_0^1 (1+x) x^2 dx = \frac{7}{12}$ $\frac{1}{4}b + c = \frac{7}{12} \text{ ----- (3)}$ Solving (1), (2) and (3) using GC: $a = \frac{1}{6}, b = 1, c = \frac{1}{3}.$	
(b)	Using Simpson's rule, $I \approx \frac{1}{6}(1-0) \left[(1+0)f(0) + 4\left(1+\frac{1}{2}\right)f\left(\frac{1}{2}\right) + (1+1)f(1) \right]$ $= \frac{1}{6} \left[f(0) + 6f\left(\frac{1}{2}\right) + 2f(1) \right]$ $= \frac{1}{6}f(0) + f\left(\frac{1}{2}\right) + \frac{1}{3}f(1) = E.$	
(c)	Using Trapezium rule, $T = \frac{1}{2} \frac{(1-0)}{2} \left[(1+0)f(0) + 2\left(1+\frac{1}{2}\right)f\left(\frac{1}{2}\right) + (1+1)f(1) \right]$ $= \frac{1}{4} \left[f(0) + 3f\left(\frac{1}{2}\right) + 2f(1) \right]$ $= \frac{1}{4}f(0) + \frac{3}{4}f\left(\frac{1}{2}\right) + \frac{1}{2}f(1)$ $\alpha = \frac{1}{4}, \beta = \frac{3}{4}, \gamma = \frac{1}{2}.$	
(d)	Since $f(x)$ is quadratic, the integrand $(1+x)f(x)$ is a cubic <u>polynomial</u>. <u>Ethan's method</u> which is the same as Simpson's Rule would provide exact value for the integrand for cubic polynomial whereas Tang's method which uses <u>the trapezium rule uses straight, linear segments to approximate the curve results in larger gaps under the curve</u>. Hence, Ethan's estimate is more accurate.	

- 5 The variables x , y and z are related by the equation $z = f(x, y)$ where f is a function of the independent variables x and y .

We define the independent *differentials* dx and dy to be the small increments of x and y respectively. The *total differential* dz , which is the small increment of z corresponding to small increments of x and y , is given by

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy.$$

A right circular cone has a circular base radius of 9 cm and a height of 20 cm. Denote

- the radius of the circular base of the cone by r ,

- the height of the cone by h ,
- the volume of the cone by V .

(a) Find expressions in terms of r and h where appropriate, for $\frac{\partial V}{\partial r}$ and $\frac{\partial V}{\partial h}$. [2]

In one instance, in measuring r and h , a man made a maximum error of 0.01 cm in each of them.

(b) Use the above result to estimate the maximum possible error in calculating V , leaving your answer in the form $k\pi$ where k is to be determined to two decimal places. [3]

In another instance, a man used the same faulty instrument to measure r and h and gave an overestimation of each by an amount ε cm but maintained that the error in calculating V is no more than π cm³.

(c) Determine in fractional form, the largest possible value of ε . [3]

5	(a)	$V = V(r, h) = \frac{1}{3}\pi r^2 h.$ $\frac{\partial V}{\partial r} = \frac{2}{3}\pi r h \text{ and } \frac{\partial V}{\partial h} = \frac{1}{3}\pi r^2.$	
	(b)	$dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh$ $= \frac{2}{3}\pi r h dr + \frac{1}{3}\pi r^2 dh \text{ from (a)}$ $= \frac{2}{3}\pi(9)(20)dr + \frac{1}{3}\pi(9^2)dh$ $= 120\pi dr + 27\pi dh$ Since $dr \leq 0.01$ and $dh \leq 0.01$, $ dV _{\max} = 120\pi(0.01) + 27\pi(0.01) = 1.47\pi \text{ cm}^3.$	
	(c)	We want largest ε ($\varepsilon > 0$) such that $dV \leq \pi$. That is, $ 120\pi\varepsilon + 27\pi\varepsilon \leq \pi \text{ since } dr = dh = \varepsilon$ $147\pi\varepsilon \leq \pi \text{ since } \varepsilon > 0$ $\Rightarrow \varepsilon \leq \frac{1}{147}$ Hence largest possible value of ε is $\frac{1}{147}$.	

6 A water authority is considering a few potential service reservoirs to supply water to a prospective new town for short-term usage.

For a particular service reservoir, let w_n denote the volumes of water (in million litres) in this reservoir at the start of month n . The volume of water in the reservoir is modelled by

$$w_n = 0.8w_{n-1} + 0.32w_{n-2},$$

where $n \geq 3$, $w_1 = 100$ and $w_2 = 80$.

(a) Find an exact expression for w_n in terms of n and $\sqrt{3}$. [6]

Another service reservoir is identified to be a nearer water source. Let x_n denote the volumes of water (in million litres) in this reservoir at the **end** of month n . The volume of water in the reservoir is modelled by

$$x_{n+2} = 1.4x_{n+1} - 0.45x_n + (F - Gn),$$

where $n \geq 0$, $x_0 = 100$ and $x_1 = 200$.

(b) State the significance of the positive constants F and G .

[1]

To remain sustainable, the reservoir must not be depleted. To determine whether the reservoir can sustain the new town over a 30-month horizon ($0 \leq n \leq 30$), the value of F is initially fixed at 100 while G is varied, where $4 \leq G \leq 9$.

(c) For the case when $G = 4$, use your graphing calculator to investigate the behaviour of x_n over the 30-month horizon and comment on its sustainability.

[1]

(d) By considering the behaviour of x_n over the 30-month horizon when $G = 6$ and $G = 9$, comment on the proposed model and how the value of F should be adjusted, so that the reservoir remains sustainable over the 30-month horizon for all values of G satisfying $4 \leq G \leq 9$.

[2]

6(a)	The auxiliary equation is $m^2 - 0.8m - 0.32 = 0$ $m = \frac{0.8 \pm \sqrt{0.64 + 4(0.32)}}{2}$ $= 0.4 \pm \sqrt{0.48}$ $= \frac{2 \pm 2\sqrt{3}}{5}$ General solution: $w_n = A \left(\frac{2+2\sqrt{3}}{5} \right)^n + B \left(\frac{2-2\sqrt{3}}{5} \right)^n$ Initial Conditions $w_1 = 100: 100 = A \left(\frac{2+2\sqrt{3}}{5} \right) + B \left(\frac{2-2\sqrt{3}}{5} \right) \text{ --- (1)}$ $w_2 = 80: 80 = A \left(\frac{2+2\sqrt{3}}{5} \right)^2 + B \left(\frac{2-2\sqrt{3}}{5} \right)^2 \text{ --- (2)}$ From (2), since $m^2 = 0.8m + 0.32$, $80 = 0.8 \left[A \left(\frac{2+2\sqrt{3}}{5} \right) + B \left(\frac{2-2\sqrt{3}}{5} \right) \right] + 0.32(A+B) \text{ --- (3)}$ Sub (1) into (3): $80 = 80 + 0.32(A+B)$ $B = -A \text{ --- (4)}$ Sub (4) into (1): $100 = A \left(\frac{2+2\sqrt{3}}{5} \right) - A \left(\frac{2-2\sqrt{3}}{5} \right)$ $100 = A \frac{4\sqrt{3}}{5} \Rightarrow A = \frac{125\sqrt{3}}{3}$	
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	Hence $B = -\frac{125\sqrt{3}}{3}$ $w_n = \frac{125\sqrt{3}}{3} \left(\frac{2+2\sqrt{3}}{5} \right)^n - \frac{125\sqrt{3}}{3} \left(\frac{2-2\sqrt{3}}{5} \right)^n$	
(b)	F is the initial volume of water added into the reservoir and G is the constant rate of decrease of water from the reservoir for each subsequent month.	
(c)	The reservoir volume increases initially before decreasing gradually. Throughout the first 30 months, the volume remains positive, so the reservoir is not depleted. (The model suggests that the reservoir can sustain the town over the 30-month horizon).	
(d)	As G increases, the depletion will become faster, and the reservoir will be depleted before month 30 which makes it unsustainable for higher values of G . The current value of $F = 100$ is sustainable only for low values of G such as 4. There is a need to increase the value of F as G increases.	

7 Given the system of equations
$$\begin{cases} x + (\sin \theta)y & = b_1 \\ y + (\cos \theta)z & = b_2, \\ x + (1 + \sin \theta)y + (\alpha + \cos \theta)z & = b_3 \end{cases}$$

where α, b_1, b_2, b_3 are real constants and $\theta \in [0, 2\pi)$.

- (a) Show that if $\alpha \neq 0$, the system is always consistent for any values of b_1, b_2, b_3 and the solution set is a point that does not pass through the origin when $b_1 \neq 0, b_2 \neq 0, b_3 \neq 0$. [3]
- (b) Explain the geometrical relationship of the solution set when $\alpha = 0, \theta = \frac{\pi}{2}$. Discuss how this relationship depends on the values of b_1, b_2, b_3 . [2]
- (c) Write down the coefficient matrix of the above system of linear equations and find its determinant in terms of α and θ . Explain how does the determinant confirm the condition in (a). [2]
- (d) Given $\alpha = 1$ and $\theta = \pi$, determine if the coefficient matrix is diagonalisable over $\mathbb{R}$. [4]

$\left(\begin{array}{ccc c} 1 & \sin \theta & 0 & b_1 \\ 0 & 1 & \cos \theta & b_2 \\ 1 & 1 + \sin \theta & \alpha + \cos \theta & b_3 \end{array} \right) \xrightarrow{R_3 \rightarrow R_3 - R_1} \left(\begin{array}{ccc c} 1 & \sin \theta & 0 & b_1 \\ 0 & 1 & \cos \theta & b_2 \\ 0 & 1 & \alpha + \cos \theta & b_3 - b_1 \end{array} \right)$ $\xrightarrow{R_3 \rightarrow R_3 - R_2} \left(\begin{array}{ccc c} 1 & \sin \theta & 0 & b_1 \\ 0 & 1 & \cos \theta & b_2 \\ 0 & 0 & \alpha & b_3 - b_1 - b_2 \end{array} \right)$ Since $\alpha \neq 0, z = \frac{b_3 - b_1 - b_2}{\alpha}$, there is a unique solution and the system is consistent. And since $b_1 \neq 0, b_2 \neq 0, b_3 \neq 0$, origin is not one the solution.	
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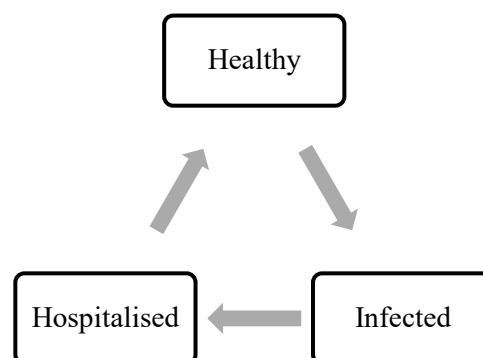
Hence the solution set is a unique point that does not passes through the origin.	
$\alpha = 0, \theta = \frac{\pi}{2}$ $\left(\begin{array}{ccc c} 1 & 1 & 0 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & 0 & 0 & b_3 - b_1 - b_2 \end{array} \right)$ If $b_3 - b_1 - b_2 \neq 0$, The third row creates an impossible equation. The system is inconsistent and has no solution. Geometrically, the three planes represented by the equations do not share any mutual intersection space. the intersection of the first two planes is a line that is parallel to the third plane, so there is no common intersection. If $b_3 - b_1 - b_2 = 0$ The third row vanishes completely. The system is consistent. Geometrically, the three planes intersect in a line. In this specific case, $y = b_2$ and $x = b_1 - b_2$, while z can be any value. This is a vertical line.	
$\mathbf{M} = \begin{pmatrix} 1 & \sin \theta & 0 \\ 0 & 1 & \cos \theta \\ 1 & 1 + \sin \theta & \alpha + \cos \theta \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \sin \theta & 0 \\ 0 & 1 & \cos \theta \\ 0 & 0 & \alpha \end{pmatrix}$ $ \mathbf{M} = \alpha \begin{vmatrix} 1 & \sin \theta \\ 0 & 1 \end{vmatrix} = \alpha$ Since $\mathbf{M} \neq 0$ $\Rightarrow \mathbf{M}$ is invertible. The system of equations can be considered as $\mathbf{M}(\mathbf{x}) = \mathbf{b}$ $\Rightarrow \mathbf{x} = \mathbf{M}^{-1}(\mathbf{b})$ Therefore there is a unique solution. An invertible matrix always yields a unique solution (a point) for any constant vector $\mathbf{b}$, which confirms the condition in part (a).	
$\alpha = 1$ and $\theta = \pi$, $\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$ $ \mathbf{M} - \lambda \mathbf{I} = 0, \text{ i.e. } \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & -1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$	

$(1-\lambda) \begin{vmatrix} 1-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$ $(1-\lambda)[- \lambda(1-\lambda)+1] = 0$ $(1-\lambda)(\lambda^2 - \lambda + 1) = 0$ Discriminant of $(\lambda^2 - \lambda + 1) < 0$ $\Rightarrow$ no real roots Consider $(\lambda^2 - \lambda + 1) = 0$ $\lambda = \frac{1 \pm \sqrt{1-4}}{2}$ $\lambda = \frac{1 \pm i\sqrt{3}}{2}$ A real matrix is diagonalizable over the real numbers if and only if all of its eigenvalues are real and there are enough linearly independent eigenvectors to form a basis. Because two of our eigenvalues are complex conjugate pairs the matrix is not diagonalizable over $\mathbb{R}$.	
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8 A public health researcher studies the spread of an infectious disease in a closed population. Individuals are classified into three groups:

- **Healthy:** individuals who are not infected but remain susceptible to the disease.
- **Infected:** individuals who carry the disease but do not require medical treatment.
- **Hospitalised:** individuals whose symptoms are severe enough to require hospital care.

Individuals who recover from hospital treatment become healthy again but may later become infected. The movement between the three groups is illustrated below:



The model assumes that each week:

- 10% of the healthy individuals become infected.
- 20% of the infected individuals become hospitalised.
- 60% of the hospitalised individuals recover and become healthy.

Let x , y and z (in millions) be the number of healthy, infected and hospitalised individuals at time t weeks respectively. Assume that the total number of individuals in the population is 1 million throughout the

period of research. Initially, $x = 0.7$ and $y = 0.3$. Assume that x , y and z can be treated as continuous variables.

(a) It is given that $\frac{dx}{dt} = -0.1x + 0.6z$.

Write down, in a similar form in terms of x , y and z where appropriate, $\frac{dy}{dt}$ and $\frac{dz}{dt}$. [2]

(b) Show that

$$\frac{d^2x}{dt^2} + 0.9\frac{dx}{dt} + 0.2x = 0.12. \quad [3]$$

(c) Hence, or otherwise, find x in terms of t . [5]

(d) Determine what happens to the number of healthy, infected and hospitalised individuals in the long term. [3]

(a)	$\frac{dx}{dt} = -0.1x + 0.6z \text{ (Given)}$ $\frac{dy}{dt} = -0.2y + 0.1x$ $\frac{dz}{dt} = -0.6z + 0.2y$	
(b)	$\frac{dx}{dt} = -0.1x + 0.6z \text{ ----- (1)}$ $\frac{dy}{dt} = -0.2y + 0.1x \text{ ----- (2)}$ $\frac{dz}{dt} = -0.6z + 0.2y \text{ ----- (3)}$ $x + y + z = 1 \text{ ----- (4)}$ Substitute eqn (4) into eqn (1): $\frac{dx}{dt} = -0.1x + 0.6(1 - x - y)$ $= -0.7x - 0.6y + 0.6 \text{ ----- (5)}$ Differentiate w.r.t t: $\frac{d^2x}{dt^2} = -0.7\frac{dx}{dt} - 0.6\frac{dy}{dt}$ Substitute eqn (2): $\frac{d^2x}{dt^2} = -0.7\frac{dx}{dt} - 0.6(-0.2y + 0.1x)$ $\frac{d^2x}{dt^2} = -0.7\frac{dx}{dt} + 0.12y - 0.06x$ Substitute eqn (5): $\frac{d^2x}{dt^2} = -0.7\frac{dx}{dt} - 0.06x + 0.2\left(-\frac{dx}{dt} - 0.7x + 0.6\right)$ $\frac{d^2x}{dt^2} = -0.9\frac{dx}{dt} - 0.2x + 0.12$	

	$\frac{d^2x}{dt^2} + 0.9\frac{dx}{dt} + 0.2x = 0.12$ Characteristic equation: $\lambda^2 + 0.9\lambda + 0.2 = 0$ $\lambda = -0.4, -0.5$ Complementary solution: $x = Ae^{-0.4t} + Be^{-0.5t}$ Particular integral: $x = \frac{0.12}{0.2} = 0.6$ General solution: $x = Ae^{-0.4t} + Be^{-0.5t} + 0.6$ When $t = 0, x = 0.7, y = 0.3$ and $z = 0$ $0.7 = A + B + 0.6 \Rightarrow A + B = 0.1$ ----- (I) $\frac{dx}{dt} = -0.1(0.7) + 0.6(0) = -0.07$ $\frac{dx}{dt} = -0.4Ae^{-0.4t} - 0.5Be^{-0.5t}$ $-0.4A - 0.5B = -0.07$ ----- (II) Solving (I) and (II): $A = -0.2$ and $B = 0.3$ Hence $x = -0.2e^{-0.4t} + 0.3e^{-0.5t} + 0.6$	
(c)	$x = -0.2e^{-0.4t} + 0.3e^{-0.5t} + 0.6$ $\lim_{t \rightarrow \infty} x = 0 + 0 + 0.6 = 0.6$ $\frac{dx}{dt} = 0.08e^{-0.4t} - 0.15e^{-0.5t}$ $\lim_{t \rightarrow \infty} \frac{dx}{dt} = 0 - 0 = 0$ From eqn (1): $z = \frac{5}{3} \left(\frac{dx}{dt} + 0.1x \right)$ $\lim_{t \rightarrow \infty} z = \frac{5}{3} (0 + 0.1(0.6)) = 0.1$ Since $y = 1 - x - z$, $\lim_{t \rightarrow \infty} y = 1 - 0.6 - 0.1 = 0.3$. Over a long period of time, the number of healthy, infected and hospitalised individuals approached 0.6, 0.3 and 0.1 millions respectively.	

- 9 A horizontal plane can be modelled by the cartesian xy -plane with origin O at the point $(0,0)$. A transparent structure S is built with its base on the horizontal plane such that the height, h , of a point on S vertically above the point (x,y) on its horizontal base is given by

$$h = 36 - 3x^2 - y^2$$

where x and y are independent variables satisfying $3x^2 + y^2 \leq 36$.

Points A and B on the horizontal base of S have coordinates $(1,0)$ and $(2,2)$ respectively. Three tiny light bulbs are placed at the points O, A and B .

A tiny AI-enabled robot equipped with a light sensor starts at a point on S and moves continuously on its surface in the direction of increasing x and y in such a way that the vertical projection of its path onto the horizontal base of S is the line L with equation of the form $y = ax + b$ such that the quantity

$$\sum_{i=1}^3 (ax_i + b - y_i)^2$$

where (x_i, y_i) , $i = 1, 2, 3$ represent the points O , A and B , is **minimised**.

(a) Use partial differentiation to show that the equation of L is

$$y = x - \frac{1}{3},$$

justifying your conclusion.

[6]

At the instant when the robot is at the point on S vertically above the point P on the horizontal base of S with coordinates $\left(\frac{4}{3}, 1\right)$, it shoots out a beam of laser in the direction of its velocity and strikes the horizontal xy -plane at the point Q .

(b) By calculating an appropriate directional derivative, find a vector equation of the line represented by the beam of laser.

[6]

(c) Find the length of PQ .

[2]

9	(a) We wish to minimise $M = \sum_{i=1}^3 (ax_i + b - y_i)^2$. $M = (a \cdot 0 + b - 0)^2 + (a \cdot 1 + b - 0)^2 + (a \cdot 2 + b - 2)^2$ $= b^2 + (a + b)^2 + (2a + b - 2)^2$ $\frac{\partial M}{\partial a} = 2(a + b) + 2(2a + b - 2)(2) = 10a + 6b - 8$ $\frac{\partial M}{\partial b} = 2b + 2(a + b) + 2(2a + b - 2) = 6a + 6b - 4$ $\frac{\partial M}{\partial a} = 0 \Rightarrow 10a + 6b - 8 = 0 \Rightarrow 5a + 3b = 4$ $\frac{\partial M}{\partial b} = 0 \Rightarrow 6a + 6b - 4 = 0 \Rightarrow 3a + 3b = 2$ Solving gives $a = 1, b = -\frac{1}{3}$. $\frac{\partial^2 M}{\partial a^2} = 10 > 0, \frac{\partial^2 M}{\partial b^2} = 6, \frac{\partial^2 M}{\partial b \partial a} = 6$ $D = \frac{\partial^2 M}{\partial a^2} \cdot \frac{\partial^2 M}{\partial b^2} - \left(\frac{\partial^2 M}{\partial b \partial a} \right)^2 = 10 \cdot 6 - 6^2 = 24 > 0.$ By the second derivative test, $a = 1, b = -\frac{1}{3}$ minimises M and therefore the line L with equation $y = x - \frac{1}{3}$ is the required least-squares line.	
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- (b) Since the vertical projection of the path of the robot is along the line $y = x - \frac{1}{3}$, we calculate the directional derivative of $h = h(x, y) = 36 - 3x^2 - y^2$ in the direction of the unit vector $\mathbf{u} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$ which is parallel to L .

$$\begin{aligned}
 D_{\mathbf{u}}h|_{x=\frac{4}{3}, y=1} &= \nabla h \cdot \mathbf{u}|_{x=\frac{4}{3}, y=1} \\
 &= \left(\frac{\partial h}{\partial x} \mathbf{i} + \frac{\partial h}{\partial y} \mathbf{j} \right) \cdot \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) \Big|_{x=\frac{4}{3}, y=1} \\
 &= (-6x\mathbf{i} - 2y\mathbf{j}) \cdot \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) \Big|_{x=\frac{4}{3}, y=1} \\
 &= \frac{1}{\sqrt{2}} \left[-6\left(\frac{4}{3}\right) - 2(1) \right] \\
 &= -\frac{10}{\sqrt{2}} \\
 &= -5\sqrt{2}
 \end{aligned}$$

Therefore the gradient of T , the tangent to S at P in the direction of the unit vector $\mathbf{u} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$ is $-5\sqrt{2}$.

Along the direction of $\mathbf{u}$, the robot is descending vertically at a rate of $5\sqrt{2}$ as shown in the diagram to the right.

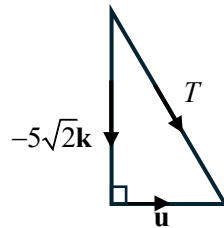
A direction vector of T is

$$\begin{aligned}
 &-5\sqrt{2}\mathbf{k} + \mathbf{u} \\
 &= \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) - 5\sqrt{2}\mathbf{k}
 \end{aligned}$$

$$h\left(\frac{4}{3}, 1\right) = 36 - 3\left(\frac{4}{3}\right)^2 - 1^2 = \frac{89}{3}.$$

Therefore a vector equation of T is

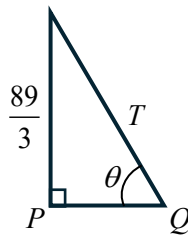
$$\mathbf{r} = \frac{4}{3}\mathbf{i} + \mathbf{j} + \frac{89}{3}\mathbf{k} + \lambda\left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} - 5\sqrt{2}\mathbf{k}\right), \lambda \geq 0.$$



- (c) In the diagram to the right,

$$\frac{89}{3} / PQ = \tan \theta = 5\sqrt{2}$$

$$\Rightarrow PQ = \frac{89}{15\sqrt{2}}$$



Alternative Method:

When T strikes the xy -plane at Q , its z -coordinate is 0.

$$\frac{89}{3} + \lambda(-5\sqrt{2}) = 0 \Rightarrow \lambda = \frac{89}{15\sqrt{2}}.$$

		$\overrightarrow{OQ} = \frac{4}{3}\mathbf{i} + \mathbf{j} + \frac{89}{15\sqrt{2}}\left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}\right).$ $PQ = \overrightarrow{PQ} = \left \frac{89}{15\sqrt{2}}\left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}\right)\right = \frac{89}{15\sqrt{2}} \text{ as before.}$	
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- 10 (a)** Given that $0 < \theta < \frac{1}{2}\pi$, express

$$\frac{1 + i \tan \theta}{1 - i \tan \theta}$$

in the form $re^{i\phi}$ where $r > 0$ and ϕ is a function of θ which is to be stated. [2]

- (b)** Hence find the three complex roots of the equation

$$z^3 = \frac{1 + i\sqrt{3}}{1 - i\sqrt{3}},$$

giving your answers in the form $re^{i\theta}$ where $r > 0$ and $-\pi < \theta \leq \pi$. [4]

- (c)** Deduce the roots of the equation

$$z^3 = \frac{1 - i\sqrt{3}}{1 + i\sqrt{3}},$$

giving your answers in exponential form. [2]

The three complex roots in **(b)** are denoted by a, b and c and are such that $\arg a < \arg b < \arg c$.

- (d)** Mark the complex numbers a, b and c as points on an Argand diagram, indicating clearly their modulus and arguments. [3]

- (e)** With the aid of your diagram in **(d)**, or otherwise, evaluate

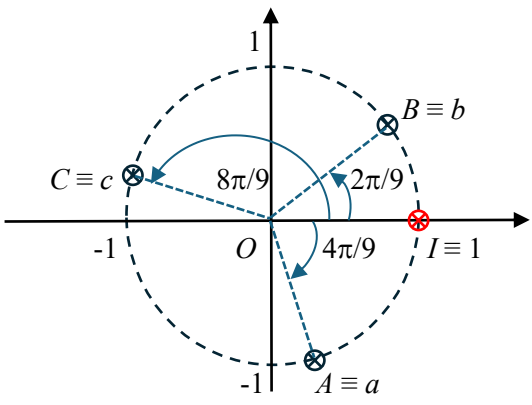
$$|(a-1)(b-1)(c-1)|,$$

giving your answer in the form

$$k \sin \frac{p\pi}{9} \sin \frac{q\pi}{9} \sin \frac{r\pi}{9}$$

where k, p, q and r are positive integers to be determined. [3]

10	(a)	$\frac{1 + i \tan \theta}{1 - i \tan \theta} = \frac{1 + i \left(\frac{\sin \theta}{\cos \theta} \right)}{1 - i \left(\frac{\sin \theta}{\cos \theta} \right)} = \frac{\cos \theta + i \sin \theta}{\cos \theta - i \sin \theta}$ $= \frac{e^{i\theta}}{e^{-i\theta}}$ $= e^{i2\theta}$ That is, $r = 1$ and $\phi = 2\theta$.	
	(b)	$z^3 = \frac{1 + i\sqrt{3}}{1 - i\sqrt{3}} = \frac{1 + i \tan \frac{\pi}{3}}{1 - i \tan \frac{\pi}{3}}$	

		$= e^{i\frac{2\pi}{3}} \text{ by (a).}$ $= e^{i\frac{2\pi}{3} + 2k\pi i}$ $z = e^{\frac{1}{3}(i\frac{2\pi}{3} + 2k\pi i)} = e^{\frac{2\pi}{9}(3k+1)i}, k = -1, 0, 1.$ Therefore the 3 complex roots are $e^{i(-\frac{4\pi}{9})}$, $e^{i\frac{2\pi}{9}}$ and $e^{i\frac{8\pi}{9}}$.	
	(c)	Taking conjugate on both sides of the equation $z^3 = \frac{1-i\sqrt{3}}{1+i\sqrt{3}}$ gives $(z^3)^* = \left(\frac{1-i\sqrt{3}}{1+i\sqrt{3}} \right)^*$ $(z^*)^3 = \frac{1+i\sqrt{3}}{1-i\sqrt{3}}$ $\Rightarrow z^* = e^{i(-\frac{4\pi}{9})}, e^{i\frac{2\pi}{9}} \text{ and } e^{i\frac{8\pi}{9}} \text{ from (b)}$ $\Rightarrow z = e^{i(\frac{4\pi}{9})}, e^{i(-\frac{2\pi}{9})} \text{ and } e^{i(-\frac{8\pi}{9})}$ (c) Alternative Method: Taking reciprocal on both sides of the equation $z^3 = \frac{1-i\sqrt{3}}{1+i\sqrt{3}}$ gives $\frac{1}{z^3} = 1 / \frac{1-i\sqrt{3}}{1+i\sqrt{3}}$ $\left(\frac{1}{z} \right)^3 = \frac{1+i\sqrt{3}}{1-i\sqrt{3}}.$ $\Rightarrow \frac{1}{z} = e^{i(-\frac{4\pi}{9})}, e^{i\frac{2\pi}{9}} \text{ and } e^{i\frac{8\pi}{9}} \text{ from (b)}$ $\Rightarrow z = e^{i(\frac{4\pi}{9})}, e^{i(-\frac{2\pi}{9})} \text{ and } e^{i(-\frac{8\pi}{9})}$	
	(d)	Let the points A, B and C represent a, b and c. 	
	(e)	Let I represent the complex number 1.	

$$\begin{aligned}
|(a-1)(b-1)(c-1)| &= |a-1||b-1||c-1| \\
&= AI \cdot BI \cdot CI \\
&= \left(2 \sin \frac{2\pi}{9}\right) \left(2 \sin \frac{\pi}{9}\right) \left(2 \sin \frac{4\pi}{9}\right) \\
&= 8 \sin \frac{\pi}{9} \sin \frac{2\pi}{9} \sin \frac{4\pi}{9}
\end{aligned}$$

That is, $k = 8, p = 1, q = 2, r = 4$.

Alternative Method 1:

Let I represent the complex number 1. Using cosine rule,

$$\begin{aligned}
&|(a-1)(b-1)(c-1)| \\
&= |a-1||b-1||c-1| \\
&= AI \cdot BI \cdot CI \\
&= \sqrt{\left(1^2 + 1^2 - 2 \cos \frac{4\pi}{9}\right) \left(1^2 + 1^2 - 2 \cos \frac{2\pi}{9}\right) \left(1^2 + 1^2 - 2 \cos \frac{8\pi}{9}\right)} \\
&= \sqrt{8 \left(1 - \cos \frac{2\pi}{9}\right) \left(1 - \cos \frac{4\pi}{9}\right) \left(1 - \cos \frac{8\pi}{9}\right)} \\
&= 2\sqrt{2} \sqrt{\left[1 - \left(1 - 2 \sin^2 \frac{\pi}{9}\right)\right] \left[1 - \left(1 - 2 \sin^2 \frac{2\pi}{9}\right)\right] \left[1 - \left(1 - 2 \sin^2 \frac{4\pi}{9}\right)\right]} \\
&= 8 \sin \frac{\pi}{9} \sin \frac{2\pi}{9} \sin \frac{4\pi}{9}
\end{aligned}$$

Alternative Method 2:

$$\begin{aligned}
&|(a-1)(b-1)(c-1)| \\
&= |a-1||b-1||c-1| \\
&= \left| e^{i\left(-\frac{4\pi}{9}\right)} - 1 \right| \left| e^{i\frac{2\pi}{9}} - 1 \right| \left| e^{i\frac{8\pi}{9}} - 1 \right| \\
&= \left| e^{-i\frac{2\pi}{9}} \left(e^{-i\frac{2\pi}{9}} - e^{i\frac{2\pi}{9}} \right) \right| \left| e^{i\frac{\pi}{9}} \left(e^{i\frac{\pi}{9}} - e^{-i\frac{\pi}{9}} \right) \right| \left| e^{i\frac{4\pi}{9}} \left(e^{i\frac{4\pi}{9}} - e^{-i\frac{4\pi}{9}} \right) \right| \\
&= \left| e^{-i\frac{2\pi}{9}} \left(-2i \sin \frac{2\pi}{9} \right) \right| \left| e^{i\frac{\pi}{9}} \left(2i \sin \frac{\pi}{9} \right) \right| \left| e^{i\frac{4\pi}{9}} \left(2i \sin \frac{4\pi}{9} \right) \right| \\
&= \left(2 \sin \frac{2\pi}{9} \right) \left(2 \sin \frac{\pi}{9} \right) \left(2 \sin \frac{4\pi}{9} \right) \\
&= 8 \sin \frac{\pi}{9} \sin \frac{2\pi}{9} \sin \frac{4\pi}{9}
\end{aligned}$$