MOE Advanced Level Higher 3 (H3) Math

Mathematical Proofs and Reasoning

Lecture 5

Telegram Chat

Content

1. Cardinality
2. Principle of Inclusion and Exclusion
3. Mathematical Investigation and Reading Mathematical Texts

Injective, Surjective, Bijective

Definition

Let $f : A \to B$ be a function.

- f is injective if whenever $f(a) = f(b)$ for $a, b \in A$, then $a = b$.
- f is surjective if for every $y \in B$, there is a $a \in A$ such that $f(a) = y$.
- f is bijective if it is both injective and surjective.

Remark. Alternatively, by contrapositive, f is injective if $f(a) \neq f(b)$ whenever $a \neq b$.

Definition

Let $f : A \to B$ be a bijection. Then for any $b \in B$, there is a unique $a \in A$ such that $f(a) = b$. The inverse of f, denoted as f^{-1}, is the function $f^{-1}(b) = a$.

This is equivalent to $f(f^{-1}(b)) = b$ and $f^{-1}(f(a)) = a$ for all $a \in A$ and $b \in B$.

Cardinality

Definition

- Sets A and B are said to have the same cardinality, denoted as $|A| = |B|$ if there is a bijection $f : A \to B$.
- A set A is said to be finite if its has the same cardinality as the set $\{1, 2, ..., n\}$ for some positive integer n. In this case, we will denote it as $|A| = n$.
- A set A is said to be countably infinite if its cardinality is equal to the set of natural numbers, $|A| = \mathbb{N}$.
- A set A is said to be uncountable if its neither finite nor countably infinite.

Example

Theorem

The set of integers is countably infinite.

Proof.

Define $f : \mathbb{Z} \to \mathbb{N}$, $f(k) = \begin{cases} 2k & k \geq 0, \\ 2(-k) - 1 & k < 0 \end{cases}$. Then f is a bijection from $\mathbb{Z}$ to $\mathbb{N}$.

k	0	-1	1	-2	2	-3	3	$\cdots$
$f(k)$	0	1	2	3	4	5	6	$\cdots$

$\square$

Example

- The previous theorem gives us the impression that 2 copies of countably infinite set is countably infinite.
- It is in fact true that a finite union of countably infinite set is countably infinite.
- Further, it is true that a union of countably infinite set is countably infinite. https://www.youtube.com/watch?v=OxGsU8oIWjY
- As a consequence, it can be shown that the set of rational numbers are countably finite (https://www.youtube.com/watch?v=WQWkG9cQ8NQ&t=188s)! This is counter-intuitive since the set of rational numbers is dense in $\mathbb{R}$.

As a fun fact, we will show that the set $(0,1) = \{x \mid 0 < x < 1\}$ is uncountable.

Uncountable Set

Theorem

The set of real numbers between 0 and 1 is uncountable.

Proof.

Suppose to the contradiction that there is a bijection $f : \mathbb{N} \to (0, 1)$. Enumerate the numbers in $(0, 1)$, that is, write $f(k) = x_k$, and write them in their decimal expansion form

$$\begin{aligned} x_1 &= 0.a_{11}a_{12}a_{13}a_{14}\dots \\ x_2 &= 0.a_{21}a_{22}a_{23}a_{24}\dots \\ x_3 &= 0.a_{31}a_{32}a_{33}a_{34}\dots \\ &\vdots \end{aligned}$$

where $a_{ij} \in \mathbb{N}$ denotes the j-th decimal digit of x_i.

Now consider number $x = 0.b_1b_2b_3b_4...$, where

$$b_i = \begin{cases} a_{ii} + 1 & \text{if } a_{ii} \leq 5 \\ a_{ii} - 1 & \text{if } a_{ii} > 5 \end{cases}.$$

By construction, it is clear that $x \neq x_k$ for all $k \in \mathbb{N}$, since the k-th decimal place defers. This contradicts the assumption that $f : \mathbb{N} \to (0, 1)$ is a bijection. Hence, $(0, 1)$ is uncountable. $\square$

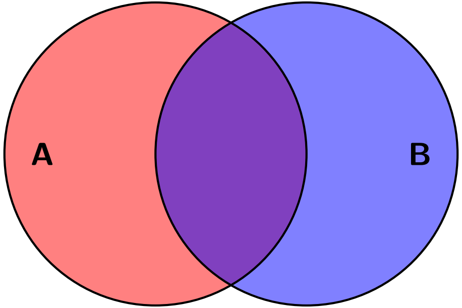
In fact, the function $f : (0, 1) \to \mathbb{R}$, $f(x) = \tan\left(\pi\left(x - \frac{1}{2}\right)\right)$ is a bijection, thus proving that $(0, 1)$ has the same cardinality as $\mathbb{R}$.

Continuum Hypothesis

- The Continuum Hypothesis (CH) is a fundamental conjecture in set theory proposed by Georg Cantor. It concerns the possible sizes of infinite sets and states:
- There is no set whose cardinality is strictly between that of the integers and the real numbers. That is, there is no set S such that
$$|\mathbb{N}| < |S| < |\mathbb{R}|.$$
- It turns out that this set is independent of the Zermelo-Fraenkel set theory, even with the axiom of choice (AFC).
- This means that given the axioms that most mathematician would agree with, the ZFC, one could neither prove nor disprove the continuum hypothesis. https://www.youtube.com/watch?v=HeQX2HjkcNo.

Principle of Inclusion and Exclusion

Given sets A and B,

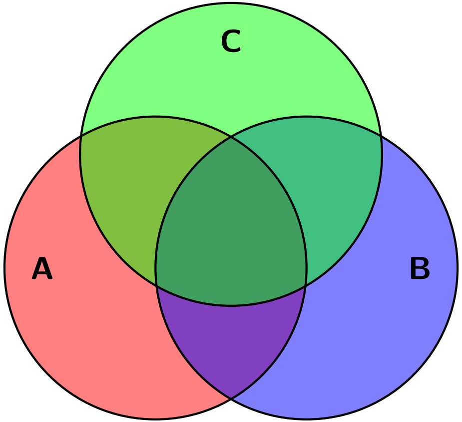


the cardinality of the union satisfies the equation

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Principle of Inclusion and Exclusion

Given sets A, B, and C



the cardinality of the union satisfies the equation

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Discussion

What is the cardinality of the union of the sets A_1, A_2, A_3, A_4, A_5?

Example

How many integers from 1 to 100 are divisible by 3, 5, or 7?

- Let A, B, and C be the set of numbers from 1 to 100 divisible by 3, 5, and 7, respectively.
- Then $|A| = \left\lfloor \frac{100}{3} \right\rfloor = 33$, $|B| = \left\lfloor \frac{100}{5} \right\rfloor = 20$, $|C| = \left\lfloor \frac{100}{7} \right\rfloor = 14$.
- $|A \cap B| = \left\lfloor \frac{100}{15} \right\rfloor = 6$, $|A \cap C| = \left\lfloor \frac{100}{21} \right\rfloor = 4$, $|B \cap C| = \left\lfloor \frac{100}{35} \right\rfloor = 2$.
- $|A \cap B \cap C| = \left\lfloor \frac{100}{105} \right\rfloor = 0$.

Hence, $|A \cup B \cup C| = 33 + 20 + 14 - 6 - 2 - 4 + 0 = 55$.

Example

A class consists of m boys and n girls. On Valentine's day, each boy independently and randomly gives a box of chocolates to one of the girls in class. What is the total number of combinations such that every girl receives at least one box of chocolates?

Label the boys $B = \{b_1, b_2, ..., b_m\}$ and the girls $G = \{g_1, g_2, ..., g_n\}$. Define a function $f : B \to G$ such that $f(b_i) = g_j$ if b_i gives his box of chocolates to g_j. Then every girl receives at least one box of chocolates if and only if f is surjective. There are a total of n^m functions. We will now count the number of surjective functions. Let A_j denote the set of functions such that g_j is not in its range,

$$A_j = \{f : B \to G \mid \forall b_i \in B, f(b_i) \neq g_j\}.$$

Then $|A_j| = (n-1)^m$, since each boy now has only $n-1$ choices of girls. Similarly, for each $k < n$,

$$|A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_k}| = (n-k)^m.$$

By symmetry,

$$\sum_{1\le j_1<j_2<\cdots<j_k\le n} |A_{j_1}\cap A_{j_2}\cap\cdots\cap A_{j_k}| = \binom{n}{k}(n-k)^m,$$

since there are a total of $\binom{n}{k}$ choices of $\{j_1, j_2, ..., j_k\} \subseteq \{1, 2, ..., n\}$. Then by principle of inclusion and exclusion, the total number of surjective functions is

$$\begin{aligned} n^m - \sum_{k=1}^{n-1}(-1)^{k+1}\binom{n}{k}(n-k)^m &= n^m + \sum_{k=1}^{n-1}(-1)^{k}\binom{n}{k}(n-k)^m \\ &= \sum_{k=0}^{n-1}(-1)^{k}\binom{n}{k}(n-k)^m. \end{aligned}$$

Probability Interpretation of the Sum of Reciprocals of Factorial

The sum of reciprocals of factorial is the series

$$\sum_{n=0}^{\infty} \frac{1}{n!}.$$

Since $\frac{1}{n!} \leq \frac{1}{2^n}$ for all $n \geq 4$, the series converges.

The sum of the reciprocals of factorials naturally appears in the Taylor series expansion of e^x,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

by setting $x = 1$. Using the series, we obtain the approximation $e \approx 2.718$.

This series has an interesting probabilistic meaning. A derangement of n objects is a permutation in which no element appears in its original position. The number of derangements of n objects, denoted as $!n$ is given by

$$!n = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}.$$

Then the probability that a random permutation of n objects is a derangement is

$$P(\text{derangement of } n \text{ objects}) = \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Taking the limit, and using the Taylor series expansion of e^x, we conclude that for large n, the probability that a random permutation of n objects is a derangement is

$$P(\text{derangement of } n \text{ objects}) \approx \frac{1}{e} \approx 0.368.$$

(a) Prove the identity $\dfrac{1}{n!} \leq \dfrac{1}{2^n}$ for all $n \geq 4$.

(b) Deduce that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$ is convergent.

(c) (i) Show that $\sum_{k=7}^{\infty} \frac{1}{k!} \leq \frac{1}{7!} \sum_{k=0}^{\infty} \frac{1}{7^k}$.

(ii) Hence, up to 4 significant figures, $e \approx 2.718$.

(d) Use principle of inclusion and exclusion to prove that $!n = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}$.

(e) Deduce that the probability that a random permutation of n objects is a derangement for $n \geq 7$ is approximately 0.368.