SCHOOL OF THE ARTS SINGAPORE

Year 6 Mathematics: analysis and approaches
Higher Level
Paper 3
Preliminary Examinations

Tuesday 30 August 2022

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all questions on the answer sheets provided. Write your name and class on each answer sheet, and attach them to this examination paper.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is [**55 marks**].

Question	Marks
1	
2	

Name:______________________________

Class: ______________________________

Index: ______________________________

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Answer **all** questions on the answer sheets provided. Please start each question on a new page.

Full marks are not necessary awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 28]

This question asks you to explore the angle between two vectors using the scalar product and the vector product.

The angle between the vector $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and the vector $\mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + m\mathbf{k}$, $m \in \mathbb{R}$ is given by p in **degrees**.

You will use the scalar product to find the angle between the two vectors.

(a) Show that $p = \arccos\left(\dfrac{7+3m}{\sqrt{14}\sqrt{13+m^2}}\right)$. [4]

(b) (i) Write down the sign of $\cos p$ when $m < -3$.

(ii) Show that when $p = 150$, $3m^2 - 84m + 175 = 0$.

(iii) Hence, deduce that there are no valid values of m for which $p = 150$. [6]

(c) By considering the graph of p, find the smallest angle between the two vectors and state the corresponding value of m for which this occurs. [2]

(d) (i) Let $g(m) = \dfrac{7+3m}{\sqrt{14}\sqrt{13+m^2}}$. Find $\lim\limits_{m\to\infty} g(m)$, expressing your answer as $\dfrac{s}{\sqrt{t}}$ where $s, t \in \mathbb{Z}^+$.

(ii) Hence, find $\lim\limits_{m\to\infty} p$, giving your answer to one decimal place.

(iii) Deduce $\lim\limits_{m\to-\infty} p$. [5]

(e) Using part (c) and part (d), deduce the range of values of p. [2]

(This question continues on the following page)

(Question 1 continued)

You will use the vector product to find the angle between the two vectors.

(f) By considering $|\mathbf{a}\times\mathbf{b}|$, show that $p=\arcsin\left(\sqrt{\dfrac{5m^2-42m+133}{14\left(13+m^2\right)}}\right)$. [5]

(g) By considering the graph of p in part (f), find the range of values of p. [2]

(h) Explain the difference between the answers you obtained in part (e) and part (g). [2]

2. [Maximum mark: 27]

This question investigates different proof techniques to lead to the proof for infinite primes.

The kth Fermat number, F_k, is defined by

$$F_k = 2^{\left(2^k\right)} + 1,\ k = 0, 1, 2, 3, \ldots$$

(a) Find F_0, F_1, F_2 and F_3. [3]

Consider the relation given by $F_0F_1F_2 \ldots F_{n-1} = F_n - 2$. ---------------------------- ($*$)

[$F_0F_1F_2 \ldots F_{n-1} = F_n - 2$ means $F_0 \times F_1 \times F_2 \times \ldots \times F_{n-1} = F_n - 2$]

(b) When $n = 1$, the relation ($*$) becomes $F_0 = F_1 - 2$. Verify that this result is true. [1]

(c) Show that the relation ($*$) is true for $n = 2$ and $n = 3$. [4]

(d) Prove by induction that $F_0F_1F_2 \ldots F_{n-1} = F_n - 2$ where $F_n = 2^{\left(2^n\right)} + 1,\ n \in \mathbb{Z}^+$. [7]

You will now deduce that no two Fermat numbers have a common factor greater than 1.

Suppose $p > 1$ is a common factor of two Fermat numbers F_j and F_m, where $j < m$.

Then $F_j = ap$ and $F_m = bp$, where $a, b \in \mathbb{Z}$.

(e) (i) Explain why p is also a factor of $F_0F_1 \ldots F_j \ldots F_{m-1}$.

(ii) Hence, deduce that p is a factor of $F_m - 2$. [2]

(f) Use part (e) to write down the relationship between p, F_m and $F_m - 2$. [1]

(g) (i) Justify that the relationship in part (f) is impossible.

(ii) Hence, using proof by contradiction, write down the conclusion. [3]

(This question continues on the following page)

(Question 2 continued)

You will now use the result above that “no two Fermat numbers have a common factor greater than 1” to prove that there are infinite primes.

(h) By considering F_1 and F_2, list down the number of distinct positive factors of these Fermat numbers. [2]

(i) Using the result “no two Fermat numbers have a common factor greater than 1”, write down the number of common factors between two Fermat numbers. [1]

It is given that any positive integer can be written as a product of two distinct prime numbers.

(j) Using part (i) and the above given result, deduce the conclusion about the *occurrence* of a prime number in a Fermat number. [2]

(k) Hence, deduce that there are infinitely many primes, justifying your answer. [1]

Please **do not** write on this page.

Answers written on this page
will not be marked.