RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 5

Tutorial 5 : Complex Numbers

Section A (Basic Questions)

1 It is given that $z=1+2\text{i}$.

(i) Without using a calculator, find the values of z^2 and $\frac{1}{z^3}$ in Cartesian form $x+\text{i}y$, showing your working. [4]

(ii) The real numbers p and q are such that $pz^2+\frac{q}{z^3}$ is real. Find, in terms of p, the value of q and the value of $pz^2+\frac{q}{z^3}$. [3]

[**(i)** $z^2=-3+4\text{i}$, $\frac{1}{z^3}=-\frac{11}{125}+\frac{2}{125}\text{i}$ **(ii)** $q=-250p$, $19p$]

2 Solve the following simultaneous equations:

(a) $w+z=6+2\text{i}$, $w-3z=\frac{20}{2-\text{i}}$; **(b)** $z=w+3\text{i}+2$, $z^2-\text{i}w+5-2\text{i}=0$.

[**(a)** $z=-\frac{1}{2}-\frac{1}{2}\text{i}$, $w=\frac{13}{2}+\frac{5}{2}\text{i}$ **(b)** $z=2\text{i}$, $w=-2-\text{i}$ or $z=-\text{i}$, $w=-2-4\text{i}$]

3 The complex number w is such that $ww^*+2w=3+4\text{i}$, where w^* is the complex conjugate of w. Find w in the form $a+b\text{i}$, where a and b are real. [4]

[$w=-1+2\text{i}$]

4 For each of the following complex number, represent it on an Argand diagram and find its modulus and argument.

(a) $\sqrt{3}+\text{i}$ **(b)** $-1+\text{i}\sqrt{3}$ **(c)** $\text{i}^2(1+\text{i})$ **(d)** $-\text{i}(1+\text{i})$

5 The complex number z is such that $|z|=1$ and $\arg z=\theta$, where $0<\theta<\frac{\pi}{4}$.

(i) Mark a possible point A representing z on an Argand diagram. Given that $\arg\left(z^2\right)=2\arg z$, mark the points B and C representing z^2 and $z+z^2$ respectively on the same Argand diagram corresponding to point A. [2]

(ii) State the geometrical shape of $OACB$. [1]

6 **(i)** Find the roots of the equation $\mathrm{i}z^2-(5+\mathrm{i})z+2-6\mathrm{i}=0$, giving your answers in cartesian form $a+b\mathrm{i}$, where $a,b\in\mathbb{R}$. [2]

(ii) Hence find the roots of the equation $-\mathrm{i}w^2-(1-5\mathrm{i})w+2-6\mathrm{i}=0$, giving your answers in cartesian form $a+b\mathrm{i}$, where $a,b\in\mathbb{R}$. [2]

(iii) Given that the roots found in part **(i)** are also roots of the equation $P(z)=0$, where $P(z)$ is a polynomial of degree 4 with real coefficients, find $P(z)$. [3]

[**(i)** $1-3\mathrm{i}$ or $-2\mathrm{i}$ **(ii)** $w=3+\mathrm{i}$ or 2 **(iii)** $z^4-2z^3+14z^2-8z+40$]

7 The cubic equation $az^3-31z^2+212z+b=0$, where a and b are real numbers, has a complex root $z=1-3\mathrm{i}$.

(i) Explain why the equation must have a real root. [2]

(ii) Find the values of a and b and the real root, showing your working clearly. [5]

[**(ii)** $a=25, b=190$, $-\frac{19}{25}$]

Section B (Discussion Questions)

1 Solve for $z=a+\mathrm{i}b$, $a,\ b\in\mathbb{R}$ if $(z+\mathrm{i})^*=2\mathrm{i}z+\mathrm{i}$.

[$z=-\frac{4}{3}+\frac{2}{3}\mathrm{i}$]

2 **Do not use a calculator in answering this question.**

(i) Given that $x+y\mathrm{i}=\sqrt{8-6\mathrm{i}}$, determine the possible values of x and y, where x and y are real numbers. [2]

(ii) Hence solve $w^2+2\mathrm{i}w-9+6\mathrm{i}=0$, giving your answer in the form $a+b\mathrm{i}$, where a and b are real numbers. [2]

[**(i)** $x=3, y=-1$ or $x=-3, y=1$ **(ii)** $w=3-2\mathrm{i}$ or $w=-3$]

3 It is given that $w=\dfrac{z-1}{z^*+1}$, where $z=a+\mathrm{i}b$, $a,\ b\in\mathbb{R}$ and $b\neq 0$.
By expressing w in the form $u+\mathrm{i}v$, $u,\ v\in\mathbb{R}$, find the conditions under which
(a) w is real, **(b)** w is purely imaginary.

[**(a)** $a=0$ **(b)** $a^2-b^2=1$]

4 Given that $\left|\dfrac{2\text{i}-z^*}{z}-1\right|^2+z=-\text{i}$, find z in the form $x+\text{i}y$.

State the exact values of $|z|$ and $\arg(z)$. [5]

[$x=4$ and $z=-4-\text{i}$, $|z|=\sqrt{17}$, $\arg z=-\pi+\tan^{-1}\dfrac{1}{4}$]

5 **Do not use a calculator in answering this question.**

It is given that $2-\text{i}$ is a root of the equation $2z^3+az^2-2z+b=0$.

(a) Find the values of the real numbers a and b and the remaining roots of the equation. [4]

(b) Using these values of a and b, deduce the roots of the equation

$$bz^3-2z^2+az+2=0.$$ [2]

[**(a)** $z=2+\text{i}$ or $z=-\dfrac{3}{2}$ **(b)** $z=\dfrac{2}{5}+\dfrac{1}{5}\text{i}$, $\dfrac{2}{5}-\dfrac{1}{5}\text{i}$ or $-\dfrac{2}{3}$]

6 **Do not use a calculator in answering this question.**

Given that one of the roots of the equation $z^4-az^3+10z-25=0$ is $1+2\text{i}$ where a is real, show that $a=2$. Find the other roots of the equation.

Hence find the roots of the equation $(w-1)^4+2(w-1)^3-10(w-1)-25=0$.

[$z=1-2\text{i}$ or $\pm\sqrt{5}$, $w=\pm2\text{i}$ or $1\pm\sqrt{5}$]

7 **Do not use a calculator in answering this question**

(i) Given that $-\text{i}$ is a root of the equation

$$z^3+kz^2+\left(8+2\sqrt{2}\,\text{i}\right)z+8\,\text{i}=0,$$

where k is a constant to be determined, find the other roots, leaving your answers in exact cartesian form $x+y\text{i}$, showing your working. [3]

(ii) Hence solve the equation $\text{i}z^3+kz^2+\left(2\sqrt{2}-8\,\text{i}\right)z-8\,\text{i}=0$, leaving your answers in exact cartesian form. [2]

[**(i)** $k=2\sqrt{2}+\text{i}$, $-\sqrt{2}+\sqrt{6}\text{i}$ and $-\sqrt{2}-\sqrt{6}\text{i}$ **(ii)** $z=-1$, $-\sqrt{6}+\sqrt{2}\text{i}$, or $\sqrt{6}+\sqrt{2}\text{i}$]

8 Given that $z=a+\text{i}b$ and $\arg(z)=\theta$, where $a>0$, $b>0$, find, in terms of θ and π, the values of

(a) $\arg(z^*)$, **(b)** $\arg(-z)$, **(c)** $\arg(-a+\text{i}b)$, **(d)** $\arg(b+\text{i}a)$

[**(a)** $-\theta$, **(b)** $-\pi+\theta$, **(c)** $\pi-\theta$, **(d)** $\dfrac{\pi}{2}-\theta$]

9 In an Argand diagram, the points A, B, C, D represent the complex numbers a, b, c, d respectively. Given that $ABCD$ is a square described in an anticlockwise sense, with $a = 1+\mathrm{i}$ and $c = 7+3\mathrm{i}$, find b and d.

[$b = 5-\mathrm{i}$, $d = 3+5\mathrm{i}$]

10 The complex numbers z and w are given by $z = \sqrt{3}+\mathrm{i}$ and $w = -1+\mathrm{i}\sqrt{3}$.

Sketch an Argand diagram, with origin O, showing the points Z, W and P representing the complex numbers z, w and $z+w$ respectively. Show that $OWPZ$ is a square. [3]

By considering the argument of $z+w$, deduce that $\tan\frac{5\pi}{12} = 2+\sqrt{3}$. [3]

THE END