Mathematics

Lecture 1: Introduction and the Usual Suspects

Overview

- Essential Questions
- Preliminaries/ Definitions
- The Mathematical Method vs the Scientific Method
- A Priori vs A Posteriori?
- The Usual Suspects: Empiricism vs Platonism

Essential Questions

- How do we construct Mathematical Truths? Reason or Experience? Inductive or Deductive reasoning?
- Are Mathematical Truths Created or Discovered?
- Do such Truths exist in the Physical Realm or a Mental Realm?
- How certain are Mathematical Truths? Absolutely or relatively?
- Are Mathematical Truths universal, holding true for all times and place? Or only relative? i.e. Objective vs Subjective?
- Is Mathematics useful?
- How is Mathematics different from other areas of knowledge?
  - Is it more **certain**? (e.g. Social Science)
  - Is it more **fundamental**? (e.g. Science)
  - Is it more **objective/universal**? (e.g. Ethics)
  - Is it more **justified**? (e.g. Aesthetics)
  - Is it more **useful** (E.g. History)

Preliminaries/ Definitions

- Axioms: **assumed** statements in a theory; **no proof** is given
- Theorem: statements in a theory that are implied by the axioms
- Theory: a set of sentences expressed in a formal language
- Formal language: a language two of whose features are formally specified: the linguistic symbols of the language and rules for joining together or concatenating these symbols into well-formed formulae or words which can be assigned precise meanings. (Oxford Dictionary of Philosophy)
  - Essentially, a formal language is the **symbolic version** of the normal, everyday language that we use.
  - E.g. The sentence “For all X that belongs to the set of natural no.” can be translated to something like this

$$\forall x \in \mathbb{N}$$

E.g.: Euclid's axioms

- 1. It shall be possible to draw a straight line joining any two points.
- 2. A finite straight line may be extended without limit in either direction.
- 3. It shall be possible to draw a circle with a given centre and through a given point.
- 4. All right angles are equal to one another.
- 5. There is just one straight line through a given point which is parallel to a given line.

E.g.: Euclid’s theorems

- 1. Lines perpendicular to the same line are parallel.
- 2. Two straight lines do not enclose an area.
- 3. The sum of the angles of a triangle is 180 degrees.
- 4. The angles on a straight line sum to 180 degrees.

HDM vs ATM

Axiom-Theorem nature of Math

Method	Example
1. Start with **Question** / problem	Do two odd numbers added together always give an even number?
2. **Axiom**: Start with assumptions and definitions; check these definitions.	Even number = 2n Odd number = 2n + 1, where n is whole number Laws of arithmetic apply
3. **Theorem**: Through deductive reasoning, find a proof to solve the problem.	Let the odd numbers be a and b Then a = 2n+1 b = 2m+1 for whole numbers n and m a+b = 2n+1+2m+1 = 2m+2n+2 = 2(m+n+1) = 2p where p is whole no. But this is of the form 2n and hence even (QED)

Hypothetico-Deductive Method in Science

Method	Example
1. Science begins with an interesting observation	Diet Coke (DC) floats!
2. Observation leads to a question	‘Why does DC float while regular coke sinks?’
3. Question leads to hypotheses (via inductive reasoning) which allows us to make a prediction (using deductive reasoning).	‘DC floats because aspartame is less dense than sugar’ ‘Aspartame + water will also be lower density than sugar + water’
4. Experiments are conducted to test the hypotheses: If the hypothesis is correct, then prediction will be verified by the experiment. If the hypothesis is wrong, then it’s eliminated and a new hypothesis composed.	All other variables constant, it is the presence of aspartame in place of sugar that causes the diet coke to float. ‘DC floats because the DC cans are not filled to the brim’
5. After repeated tests, a hypothesis is corroborated and becomes scientific fact / theory.	Aspartame is of lower density than sugar.

Inductive or Deductive?

- Might seem a little duh to ask this question since Mathematics seems to be the one area of knowledge which is **totally** deductive in nature.
- For example, 2+2=4 seems a deductive entreprise. We don't go out there, take 2 items, and another 2 items, and then count that there is 4 and then from there generalize it to all items. We *could* but we don't have to.
- Why?

Inductive or Deductive?

- Suppose a child is starting to learn about even and odd numbers. She is presented first with a list of odd numbers (1, 3, 5, 7, ….) and then a list of even numbers (2, 4, 6, 8….)
- By the by, she realises that there seems to be a few patterns:
  - Adding 1 to an odd numbers gives an even number
  - Adding 1 to an even number gives an odd number
  - Adding 2 odd numbers gives an even number
  - Multiplying 2 even numbers gives an even number
- Can she now be said to know that these rules for **all** even and odd numbers?
- After all, hasn’t she done what most of us did when we first started learning math, i.e. learn by trial and error?

Inductive or Deductive?

- Answer: No. As all she has done is work with a **limited sample size** and then generalised to a principle, i.e. she has *inductively* arrived at her conclusions.
- But there is an infinite number of numbers and thus, she cannot yet guarantee that these rules hold true for all numbers.
- To do this, she will need to reason **deductively** instead. In other words, she will need to *prove* that these rules hold true for all cases.
- Difference between Proofs and Conjectures:
  - Proof: In a proof, a theorem is shown to follow logically, i.e. *necessarily*, from the relevant axioms.
  - Conjecture: A conjecture is a hypothesis that seems to work, but has not been shown to be *necessarily* true.

E.g.: Goldbach's conjecture

- Conjecture: every even number is the sum of 2 primes
- Let's test it!
- 2 = 1+1
- 4 = 2+2
- 6 = 3+5
- 8 = 5+3
- 10 = 5+5….
- Mathematicians have used computers to test the conjecture for all even numbers up to **100,000,000,000,000**
- But is this yet a proof? No. Because it is still possible that the next even number is not the sum of 2 primes.
- Furthermore, 100,000,000,000,000, while a huge number, is still small compared to infinity!

Theorem: The sum of an odd number and an even number is an odd number

- Let the odd number be o and the even number be e
- Then $o=2n+1$ and $e=2m$ for some whole numbers n and m, by definition
- So $o + e \quad = 2n + 1 + 2m$

  $= 2(n + m) + 1$

  $= 2p + 1$ where p is a whole number

  but this is of the form $2n + 1$ and hence odd.

This is an example of a mathematical **proof** where, proceeding deductively, a theorem is shown to hold true for all cases.

Infinite No. Of Primes (Euclid)

Suppose that there are only finitely many primes*, say *n* of them.

We could then make a list of these *n* primes, say *p*1, *p*2...*pn*.

Then consider the integer *M* = *p*1 x *p*2 x ... *pn* + *1*, where *M* is obtained by multiplying all of the primes, and then adding 1.

This new number *M* is either prime or it is not.

If it is not, then it is divisible by some prime.

But we are supposing that the only primes are *p*1 through *pn* – and none of these primes divides *M*, for when *M* is divided by one of these primes, the remainder is 1.

This **contradicts** the known fact that every positive integer greater than 1 is divisible (evenly) by some prime.

So the conclusion must be that *M* is prime.

But clearly *M* is greater than all of the primes *p*1, *p*2, ..., *pn* , so this **contradicts** the supposition that *p*1, *p*2, ..., *pn* is a finite and complete list of all primes.

Therefore, this must mean that there are **not** finitely many primes; there are infinitely many. That is, the sequence of primes does not end.

Inductive or Deductive?

- Recall: a deductive argument is one where the arguer claims that it is **impossible** for the argument to be false given that the premises are true.
- Same thing happens in Math.
- We begin with axioms that we *assume* to be true.
- And then if we have done our maths right, the conclusion is guaranteed.
- i.e. all maths is **implicit** in the axioms
- In other words, there is **nothing new** in the conclusion that was not already contained within the premises.

"All mathematics exhibits in its conclusions only what is already implicit in its premises. Hence all mathematical derivation can be viewed simply as **change in representation, making evident what was previously true but obscure.**" (H. A. Simon, emphasis added)

- Of course, if the axioms were wrong to begin with….

A Priori or A Posteriori?

- Math is normally seen as an a priori subject and not a posteriori.
- Intuitive! It does seem to be the case that we can simply go into a cave and come to the realisation that 2+2=4 simply by thinking about it.
- For example, once having learnt basic arithmetic, we can compute sums and numbers that we have **never experienced before** and come to a conclusion that all can agree on.
- Also explains why in Math, we are concerned with Proofs (cf. TOK reading, 192-6)
- We are not happy with ‘proofs’ by the ‘trial and error’ method because they are inductive and merely show that for these cases tested **so far**, the conjecture works. But this in no way guarantees that it holds for all possible cases.
- This is the commonly held view of Math.

A case for A Posteriori Math?

- However, some Empiricists actually argue that Math is empirical in nature!
- Proponents: JS Mill, Quine and Putnam
- Rejected the Platonist idea of Math as a priori and existing in a mystical realm
- Putnam: this is **unnecessary** to mathematical practice in any real sense (hence, by Occam's Razor, can be dispensed with)
- Intuitive: seems to be <u>how we learnt math</u> as babies.
- A Realist view: argued that mathematical entities are **mind independent**, to be discovered from the external world.
- Relies on the *Indispensability Argument* (IA)
  - the idea that since mathematics is **indispensable** to all empirical sciences, **if** we want to believe in the reality of the phenomena described by the sciences, then we **ought** also believe in the reality of those entities required for this description

Math as Empirical – Arguments For

- Intuitive – that's how we learnt math as children
- Math found in nature – Fibonacci sequence, Golden Ratio
- (Seemingly) <u>Different</u> mathematical systems across cultures: different bases 2, 5, 6, 10, 12, 20, 60, different placeholder systems (0 vs space), different symbols for numbers (1, 2, 3 vs i,ii,iii vs Chinese numbering etc) [c.f. Article B)
- Math developed for practical purposes (keeping count of things, how to navigate, how to construct), thereby suggesting that if the world is different and things no longer remain discrete, <u>our math would be different</u>

Counterargument

- But there are also regularities and similarities across mathematical systems (C.f. Article B)
  - The creation of abstract symbols to represent *any* physical objects, i.e. numbers (be it 1, 2, 3 or i,ii,iii etc)
  - Babylonian, Chinese, Mayan, Indian systems all invented the place-value notation (the latter 3 reinventing it)
  - The use of Round Numbers across all cultures (to connote approximation and uncertainty), as well as confidence intervals (like “10 or 15 people”)
- An empiricist view of math **strips** math of its special and distinct nature from the sciences (due to *IA*)
- If math is <u>just as empirical</u> as the sciences, then it seems to follow that it is <u>as fallible</u> as the sciences (just as inductive) and <u>just as contingent</u>
- E.g. in this world, objects are discrete and stay discrete – hence 1+1=2 all the time. But in another world, objects might *not* be discrete, such that 1+1=1.5!
- This seems to point to mathematical truths as **non-necessary** and **non-eternal**.
- Unintuitive!
- We tend to think that 1+1=2 in every possible universe such that mathematics becomes a **universal** language.
- Hence, anyone trying to argue otherwise has <u>a huge burden of proof</u>.

Platonism

- The commonly held view of Math
- Claims: mathematical entities are abstract, have no spatiotemporal or causal properties, are eternal and never changing, and exists in the World of Forms
- Arguments for:
  - **Intuitive**, fits the prevailing view of mathematicians (appeal to authority)
    - E.g. the idea of a point as that which takes up no space cannot be empirically substantiated. Same for a line as that which has no width.
  - Explains why mathematical truths are **necessary, universal** and **eternal**
    - E.g. Leibniz and Newton's independent development of Calculus.
    - Keith Devlin (mathematician): " If Beethoven had not lived, we would never have heard the piece we call his Ninth Symphony. If Shakespeare had not lived, we would never have seen Hamlet. But if, say, Newton had not lived, the world would have gotten calculus sooner or later, and it would have been exactly the same!" (Math and the real world)

Platonism

- Explains why we can arrive at mathematical truths via **reason alone** – we don't need to experience complex numbers (we can't!) to arrive at the truth of complex numbers.
- One can accept such a concept as "set" without worrying about how the mind can construct a set since sets are there for us to discover, and not construct (something that the anti-realists hold; realists thus have the advantage of accepting many more abstract entities in math than anti-realism)
- Explains the A Priori nature of Math – the justification of Math lies not in whether the theorem or math entity can be found in the physical world but whether it enjoys the support (because it is seen as deductively valid), and thus justification, of the math fraternity.
  - Even if 2 lions and 2 zebras in a cage became 2 very fat lions, it doesn't prove that 2+2 is not 4; it just proves a matter of biology.
  - Or think of how we don't disprove that the sum of angles of a triangle is 180 degrees via measuring many physical triangles.
- Explains why people of diverse nationalities, backgrounds and personalities can somehow come together and agree on whether a proof is right or wrong. How is this possible? Because they can all access this same, universal and indubitable set of math rules
  - E.g. the math fraternity agreed that Wiles' 1st attempt at proving Fermat's last theorem was wrong and then agreed again later that his 2nd attempt was right.
- [for the rest, c.f. lecture slides on Rationalism…]

Counterarguments

- If mathematical entities are abstract, non-spatiotemporal and non-causal, then it seems **weird** that we can even come to know of such entities.
- How is it that we, who are embodied flesh, can explore this abstract, mystical realm when these entities don't even have any causal power such that they can cause us to know them, just as physical objects have causal power such that light can bounce off them into our eyes and we can thus see them?
- Also, if there is a form, triangle, then this must be a very **weird** form indeed since it will have to be at once an isosceles triangle, a right-angled one, an equilateral one etc. at the same time.
- If Platonism is right, then there should not be **contradictory** geometry systems since truths do not allow for contradiction (law of non-contradiction)
- But there are contradictory geometry systems (Euclidean vs Riemannian geometry) (TOK 205-6)
- [for the rest, c.f. lecture slides on Rationalism…]

Homework

- TOK Readings (54-58) and (192-202; end just before "Discovered or Invented")
- Article B
- Article C

- Optional: Article A