1

Consider

$$U_{n-2}+U_{n-1}=U_n$$

$$1+\frac{U_{n-1}}{U_{n-2}}=\frac{U_n}{U_{n-2}}$$

$$1+\frac{U_{n-1}}{U_{n-2}}=\frac{U_n}{U_{n-1}}\left(\frac{U_{n-1}}{U_{n-2}}\right)$$

$$1+R_{n-1}=R_n\left(R_{n-1}\right)$$

If sequence converges, $R_{n-1}\to L$, $R_n\to L$ when $n\to\infty$ where L is the limit of the sequence

$$\therefore 1+L=L^2$$

$$L^2-L-1=0$$

$$L=\frac{1\pm\sqrt{1-4(1)(-1)}}{2}=\frac{1\pm\sqrt{5}}{2}$$

$=\frac{1+\sqrt{5}}{2}$ since all the Fibonacci numbers are positive, where a = 1 and b = 5

2

(a) $\dfrac{\mathrm{f}(x_n)^2\mathrm{f}''(x_n)}{2\left[\mathrm{f}'(x_n)\right]^3}=0 \quad\Rightarrow\quad \mathrm{f}(x_n)=0 \text{ or } \mathrm{f}''(x_n)=0$

This implies that the function f is a linear function, i.e. it is of the form $\mathrm{f}(x)=ax+b$.

(b) $\mathrm{f}(x)=x\ln x,\ \mathrm{f}'(x)=\ln x+1,\ \mathrm{f}''(x)=\dfrac{1}{x}$

n	x_n (Newton-Raphson)	x_n (Chebyshev)
1	5	5
2	1.916121	1.551663
3	1.161072	1.027829
4	1.010204	1.000013
5	1.000051	1.000000

At each iteration, Chebyshev's results are markedly closer to the actual root, demonstrating a better convergence rate compared to Newton-Raphson method.

(c) Using Chebyshev's method, $x_2=-2.5277868<0$, which is outside the domain of f. We cannot continue to compute x_3 as $\ln x_2$ is undefined.

3

(a) $\mathrm{f}_x=4x\left(x^2+y-11\right)+2\left(x+y^2-7\right)$

$\mathrm{f}_y=2\left(x^2+y-11\right)+4y\left(x+y^2-7\right)$

$$\nabla\mathrm{f}\left(1,2\right)=\begin{pmatrix}4\left(1^2+2-11\right)+2\left(1+2^2-7\right)\\ 2\left(1^2+2-11\right)+4\left(2\right)\left(1+2^2-7\right)\end{pmatrix}=\begin{pmatrix}-36\\-32\end{pmatrix}=-4\begin{pmatrix}9\\8\end{pmatrix}$$

For steepest descent, the unit vector is in the opposite direction of the gradient vector, hence unit vector is $\frac{1}{\sqrt{145}}\begin{pmatrix}9\\8\end{pmatrix}$.

$D_u \mathrm{f}(1,2) = -4\sqrt{145}$

(b) $\mathrm{f}_{xx} = 4\left(x^2+y-11\right)+8x^2+2$

$\mathrm{f}_{yy} = 2+8y^2+4\left(x+y^2-7\right)$

$\mathrm{f}_{xy} = 4x+4y$

At $(3,0,\mathrm{f}(3,0))$, $\mathrm{f}(3,0)=20$, $\mathrm{f}_x=-32$, $\mathrm{f}_y=-4$, $\mathrm{f}_{xx}=66$, $\mathrm{f}_{yy}=-14$, $\mathrm{f}_{xy}=12$

$$Q(x,y)\approx \mathrm{f}(3,0)+(x-3)\mathrm{f}_x(3,0)+y\,\mathrm{f}_y(3,0)+\frac{1}{2}(x-3)^2\,\mathrm{f}_{xx}(3,0)+y(x-3)\mathrm{f}_{xy}(3,0)$$

$$+\frac{1}{2}y^2\,\mathrm{f}_{yy}(3,0)$$

$$Q(x,y)\approx 20-32(x-3)-4y+33(x-3)^2+12y(x-3)-7y^2$$

For saddle point,

$Q_x(x,y)=0 \Rightarrow -32+66(x-3)+12y=0 \Rightarrow 33x+6y=115-----(1)$

$Q_y(x,y)=0 \Rightarrow -4+12(x-3)-14y=0 \Rightarrow 6x-7y=20-----(2)$

Solving gives $x=\frac{925}{267}\approx 3.46, y=\frac{10}{89}\approx 0.112$

$Q_{xx}=66>0, Q_{yy}=-14, Q_{xy}=12 \Rightarrow Q_{xx}Q_{yy}-Q_{xy}{}^2=-1068<0 \Rightarrow$ saddle point, shown

Coordinates are $(3.46,0,112,13.7)$.

4

(a) $\det \mathbf{E} = 1(7-12)-2(0-4)+3(0-1)=0$

Since $\det \mathbf{E}=0$, the word vectors are linearly dependent.

(b) Set containing any 2 columns in the matrix **E**.

(c)

$$\left(\begin{array}{ccc|c}1&2&3&6\\0&1&4&9\\1&3&7&18\end{array}\right)\xrightarrow{-R_1+R_3}\left(\begin{array}{ccc|c}1&2&3&6\\0&1&4&9\\0&1&4&12\end{array}\right)\xrightarrow{-R_1+R_3}\left(\begin{array}{ccc|c}1&2&3&6\\0&1&4&9\\0&0&0&3\end{array}\right)$$

The system is inconsistent (last row is not a whole row of zeros). The vector **v** cannot be represented in the model's current embedding space. The model lacks the capacity to understand or process this particular word/concept, as it lies outside the span of the stored word vectors.

(d) $\mathbf{AE}=\begin{pmatrix}1&0&-1\\0&2&1\end{pmatrix}\begin{pmatrix}1&2&3\\0&1&4\\1&3&7\end{pmatrix}=\begin{pmatrix}0&-1&-4\\1&5&15\end{pmatrix}$.

Note that the two rows are linearly independent. So rank(**AE**) = 2.

Using rank-nullity theorem, $\text{nullity}(\mathbf{AE})=3-2=1$.

Solving $\begin{pmatrix} 0 & -1 & -4 \\ 1 & 5 & 15 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$:

$$\begin{pmatrix} 0 & -1 & -4 \\ 1 & 5 & 15 \end{pmatrix} \xrightarrow{5R_1+R_2} \begin{pmatrix} 0 & -1 & -4 \\ 1 & 0 & -5 \end{pmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{-R_1} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \end{pmatrix}$$

$x - 5z = 0 \Rightarrow x = 5z$

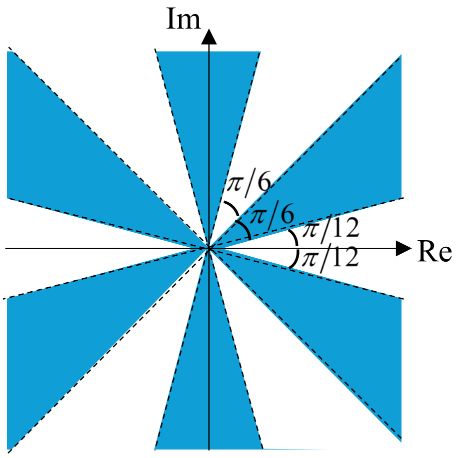
$y + 4z = 0 \Rightarrow y = -4z$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z\begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}$, $z \in \mathbb{R}$. A basis of the null space is $\left\{\begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}\right\}$.

5

(a) Let $u = re^{i\theta}$. Then $u^6 = r^6 e^{i6\theta}$. So $\mathrm{Re}(z^6) = r^6 \cos 6\theta < 0$.

Solving $\cos 6\theta < 0$, we get $\theta \in \left(\dfrac{(1+4k)\pi}{12}, \dfrac{(3+4k)\pi}{12}\right)$ for $k = -3, -2, -1, 0, 1, 2$.



(b) $\arg\left(\dfrac{v^3-1}{v^3+1}\right) = -\dfrac{\pi}{2}$

$\dfrac{v^3-1}{v^3+1} = k\mathrm{i}, \quad$ where $k < 0$

$v^3 - 1 = k\mathrm{i}(v^3+1)$

$v^3(1 - k\mathrm{i}) = 1 + k\mathrm{i}$

$v^3 = \dfrac{1+k\mathrm{i}}{1-k\mathrm{i}}$

$|v|^3 = \dfrac{|1+k\mathrm{i}|}{|1-k\mathrm{i}|} = \dfrac{\sqrt{1+k^2}}{\sqrt{1+k^2}} = 1 \Rightarrow |v| = 1$

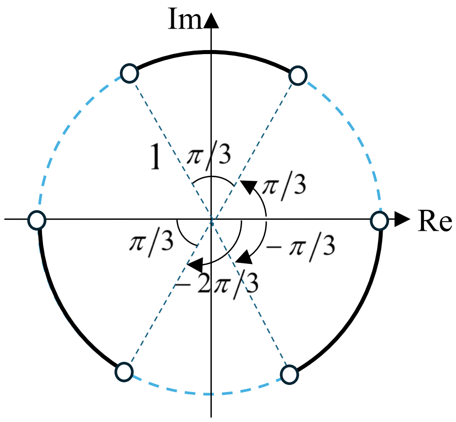
$\arg(v^3) = \arg(1+k\mathrm{i}) - \arg(1-k\mathrm{i}) = 2\arg(1+k\mathrm{i})$.

Since k can be any negative real number, $-\dfrac{\pi}{2} < \arg(1+k\mathrm{i}) < 0$.

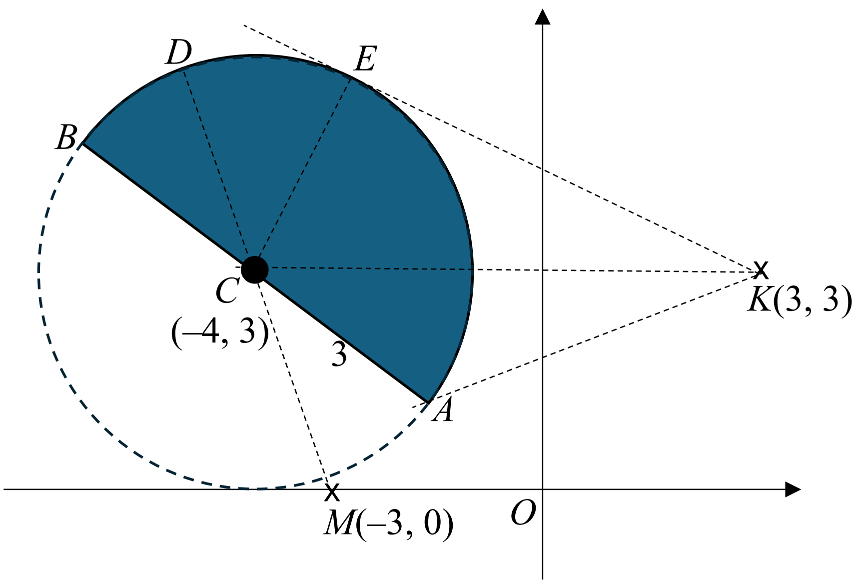
This gives $2k\pi - \pi < 3\arg(v) < 2k\pi \Rightarrow \dfrac{(2k-1)\pi}{3} < \arg(v) < \dfrac{2k\pi}{3}$ for $k = -1, 0, 1$.

The locus consists of three arcs of distance 1 unit from the origin.

Total length of arcs $= 3 \times 1\left(\dfrac{\pi}{3}\right) = \pi$ units.



(c) (i)



(ii) $\max|z+3| = MC + CD = \sqrt{10}+3$

(iii) Point A has coordinates $\left(-2\left(\frac{4}{5}\right),\ 2\left(\frac{3}{5}\right)\right) = \left(-\frac{8}{5},\ \frac{6}{5}\right)$.

$\tan\angle AKC = \dfrac{3-6/5}{3+8/5} = \dfrac{9}{23}$.

Also, $\sin\angle EKC = \dfrac{3}{7}$.

Range of values of $\arg(z-3-3\mathrm{i})$ is $\left(-\pi,\ -\pi+\tan^{-1}\frac{9}{23}\right]\cup\left[\pi-\sin^{-1}\frac{3}{7},\ \pi\right]$.

(iv) Consider the amount of anticlockwise rotation θ about the origin such that the rotated point B would have y-coordinate $= -1$.

$OB = 8$. $\angle MOB = \tan^{-1}\dfrac{3}{4}$.

Required angle $\theta = \angle MOB + \sin^{-1}\dfrac{1}{8} = \tan^{-1}\dfrac{3}{4} + \sin^{-1}\dfrac{1}{8}$ (≈ 0.769 rad)

6

(a) Width of 95% CI $= 2(1.96)\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$

The maximum value that can be attained by $\hat{p}(1-\hat{p})$ occurs when $\hat{p} = \dfrac{1}{2}$.

	Therefore, the maximum width $= 2(1.96)\sqrt{\dfrac{0.5(1-0.5)}{n}} = \dfrac{1.96}{\sqrt{n}} < \dfrac{2}{\sqrt{n}}$. (b) Sample proportion, $\hat{p} = \dfrac{0.27369+0.42631}{2} = 0.35$ $2(1.96)\sqrt{\dfrac{0.35(0.65)}{n}} = 0.42631-0.27369$ $n = 150$ (c) $\overline{x} = 14.125$, $s^2 = 0.50931818$ Degree of freedom = 12 – 1 = 11. $t = 3.105807$ $\overline{x} \pm t\cdot\dfrac{s}{\sqrt{n}} = 14.1525 \pm 3.1058 \times \dfrac{0.7137}{\sqrt{12}}$ $= 14.1525 \pm 0.6399$ $= 13.5126,\ 14.7924$ 99% confidence interval $= (13.51,\ 14.79)$ (d) 99% of intervals calculated in a similar manner would contain the true population mean.
7	(a) H_0: Choice of habitat where Anna's Hummingbird forage is independent of the time of the day. H_1: Choice of habitat where Anna's Hummingbird forage is dependent on the time of the day. Under H_0, Expected frequencies E_i = $\dfrac{(row\ total)(column\ total)}{grand\ total}$

O_i / E_i	Morning	Midday	Evening	Total
Urban Gardens	148/158.6	54/47.58	103/98.82	305
Chaparral (Mediterranean Shrubland)	63/59.8	10/17.94	42/37.26	115
Oak Woodlands	49/41.6	14/12.48	17/25.92	80
Total	260	78	162	500

Test Statistic:

$$\chi^2_{calc} = \sum \frac{(O_i - E_i)^2}{E_i} \sim \chi^2(4)$$

degree of freedom $\nu = (3-1)(3-1) = 4$

	$$\chi^2_{calc} = \sum \frac{(O_i - E_i)^2}{E_i} = 10.611$$ $p - \text{value} = 0.0313$ Since p-value = 0.0313 < 0.05, the null hypothesis is rejected, so there is significant evidence at 5% level of significance that the choice of habitat where the Anna's Hummingbird forage is dependent on the time of the day. (b) Contributions to test statistic

O_i / E_i	Morning	Midday	Evening
Urban Gardens	0.7084	0.8663	0.1768
Chaparral (Mediterranean Shrubland)	0.1712	3.5141	0.6030
Oak Woodlands	1.3163	0.1851	3.0697

The largest contributor to the test statistic is Chaparral at midday and the observation is less than expected. Hence, the conservationist can conclude that the hummingbird prefers to forage in other habitats more than in Chaparral during midday.

8 (a)

$$\int_0^1 \frac{4}{b}x^3 \, dx + \int_1^b \frac{1}{b} \, dx = 1$$

$$y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x}$$

$$\int_{-\infty}^{0} \frac{4}{b}e^{3y}\left(e^{y}\right)dy + \int_{0}^{\ln b} \frac{1}{b}\left(e^{y}\right)dy = 1$$

$$h(y) = \begin{cases} \frac{4}{b}e^{4y}, & y \le 0 \\ \frac{1}{b}e^{y}, & 0 < y < \ln b \\ 0, & \text{otherwise} \end{cases}$$

(b) $$\int_{-\infty}^{0} \frac{4}{b}e^{4y}dy = \frac{1}{b}\left[e^{4y}\right]_{-\infty}^{0} = \frac{1}{b}$$

Since $b > 2 \Rightarrow 0 < \frac{1}{b} < \frac{1}{2}$

$$\therefore \frac{1}{b} + \int_0^y \frac{1}{b}e^{y}dy = \frac{3}{4}$$

	$\frac{1}{b}+\left[\frac{1}{b}e^{y}\right]_0^y=\frac{3}{4}$ $\frac{1}{b}+\frac{1}{b}e^{y}-\frac{1}{b}=\frac{3}{4}$ $\frac{1}{b}e^{y}=\frac{3}{4}\Rightarrow y=\ln\frac{3b}{4}$. The upper quartile of Y is $\ln\frac{3b}{4}$. (c) $\int_{-\infty}^{y}\frac{4}{b}e^{4y}\mathrm{d}y=\frac{1}{4}$ $\frac{1}{b}\left[e^{4y}\right]_{-\infty}^{y}=\frac{1}{4}$ $\frac{1}{b}e^{4y}=\frac{1}{4}\Rightarrow y=\frac{1}{4}\ln\frac{b}{4}$ Interquartile range: $\ln\frac{3b}{4}-\frac{1}{4}\ln\frac{b}{4}=\ln\left(\frac{375}{128\sqrt{2}}\right)$ $\ln\frac{\frac{3b}{4}}{\left(\frac{b}{4}\right)^{\frac{1}{4}}}=\ln\left(\frac{375}{128\sqrt{2}}\right)$ $\frac{\frac{3b}{4}}{\left(\frac{b}{4}\right)^{\frac{1}{4}}}=\frac{375}{128\sqrt{2}}$ $\frac{3b^{\frac{3}{4}}}{2\sqrt{2}}=\frac{375}{128\sqrt{2}}$ $b^{\frac{3}{4}}=\frac{125}{64}\Rightarrow b=\left(\frac{5}{4}\right)^{4}$ (d) $V\sim \text{Geo}\left(\frac{1}{4}\right)$ $\text{Var}(V)=\frac{1-\frac{1}{4}}{\left(\frac{1}{4}\right)^{2}}=12$
9	(a) The average rate of occurrence of earthquakes with magnitudes 7.0 and above is constant throughout the year. The occurrences of earthquakes with magnitudes 7.0 and above are independent. (b) Let X be the random variable denoting the number of earthquakes with magnitude 7.0 and above in a year.

	$$X \sim \mathrm{Po}(1)$$ $\mathrm{P}(X_1+X_2>2 \mid X_1, X_2 \le 2)$ $= \dfrac{2\mathrm{P}(X_1=1)\mathrm{P}(X_2=2)+\mathrm{P}(X_1=2)\mathrm{P}(X_2=2)}{\mathrm{P}(X_1\le 2)\mathrm{P}(X_2\le 2)}$ $= \dfrac{2(\mathrm{e}^{-1})\left(\frac{1}{2}\mathrm{e}^{-1}\right)+\left(\frac{1}{2}\mathrm{e}^{-1}\right)^2}{\left(\mathrm{e}^{-1}+\mathrm{e}^{-1}+\frac{1}{2}\mathrm{e}^{-1}\right)^2} = \dfrac{1}{5}$ (c) Let T be the random variable denoting the time (in months) between consecutive occurrences of earthquakes with magnitude 7.0 and above. $$T \sim \mathrm{Exp}\left(\frac{1}{12}\right)$$ $\mathrm{P}(T \le 29 \mid T > 12) = \mathrm{P}(T \le 17) = 1-\mathrm{e}^{-\frac{1}{12}(17)} = 0.757$
10	(a) Let X and Y be the random variables denoting the time spent on a yellow flower and on a blue flower respectively and $D = X - Y$. Assume a symmetric distribution for D. Let m_D be the median of D. $\mathrm{H}_0: m_D = 0.5$ $\mathrm{H}_1: m_D > 0.5$ 1-tailed test at 5% significance level, under H_0,

Bee	1	2	3	4	5	6	7	8	9	10
X	4.6	4.7	5.1	4.4	4	5.0	4.3	4.2	5.0	4.3
Y	3.0	3.2	3.0	4.4	4.3	3.0	4.1	3.1	3.3	4.0
$D-0.5 = X-Y-0.5$	1.1	1.0	1.6	-0.5	-0.8	1.5	-0.3	0.6	1.2	-0.2
Rank	7	6	10	3	5	9	2	4	8	1

$P =$ sum of postive ranks $= 44$

$Q =$ sum of negative ranks $= 11$

$T = \min(P, Q) = 11$

From MF27, for $n = 10$, 1-tailed test at 5% significance level, critical region is $T \le 10$.

Since $T = 11$ lies outside the critical region, we do not reject H_0 and conclude that there is insufficient evidence at 5% significance level to support the melittologist's hypothesis.

(b) Let μ_D be the population mean of D.

$\mathrm{H}_0: \mu_D = 0.5$

$\mathrm{H}_1: \mu_D > 0.5$

Assume D follows a normal distribution.

Under H_0,

	Test Statistic: $T = \dfrac{\overline{D} - 0.5}{S_D / \sqrt{10}} \sim t(9)$ $\overline{d} = 1.02,\ s^2 = 0.890443^2$. Using GC, p-value = 0.0489 < 0.05 We reject H_0 and conclude that there is sufficient evidence, at 5% significance level, to support the melittologist's hypothesis. (c) For χ^2-goodness of fit test, p-value = 0.101 > 0.1 or test statistic 11.99 < 12.02 (10 % sig level) There is insufficient evidence that D does not follow a normal distribution. Hence the researcher's suspicion is not supported.