RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 5

2022 Year 5 H2 Mathematics Common Test (modified): Solutions with Comments

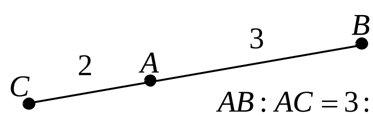
1 It is given that $\mathrm{f}(x)=ax^3-bx^2+bx+c$ and $\mathrm{g}(x)=ax+d$, where a, b, c and d are constants. Given that the curve with equation $y=\mathrm{f}(x)$ has a stationary point at $(1,1)$ and it intersects the curve with equation $y=\mathrm{g}(x)$ at $(3,17)$, determine the values of a, b, c and d. [5]

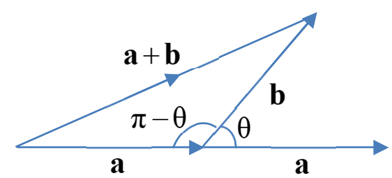
Solutions		Comments
1 **[5]**	Since $y=\mathrm{f}(x)$ passes through $(1,1)$, $a-b+b+c=1$ $a+c=1 \quad \cdots\cdots\cdots (1)$ $\mathrm{f}'(x)=3ax^2-2bx+b$ Since $y=\mathrm{f}(x)$ has a stationary point at $(1,1)$, $\mathrm{f}'(x)=0$ at $x=1$, $3a-2b+b=0$ $3a-b=0 \quad \cdots\cdots\cdots (2)$ Since $y=\mathrm{g}(x)$ passes through $(3,17)$, $3a+d=17 \quad \cdots\cdots\cdots (3)$ For the 2 curves to intersect at $(3,17)$, $ax^3-bx^2+bx+c=ax+d$ $ax^3-bx^2+(b-a)x+c-d=0$ $27a-9b+3(b-a)+c-d=0$ $24a-6b+c-d=0 \quad \cdots\cdots\cdots (4)$ Solving (1), (2), (3) and (4) using GC, $a=2$, $b=6$, $c=-1$ and $d=11$. Or Since $y=\mathrm{f}(x)$ also passes through $(3,17)$, $27a-6b+c=17 \quad \cdots\cdots\cdots (5)$ Solving any 4 equations using GC, $a=2$, $b=6$, $c=-1$ and $d=11$.	Always make full use of the conditions given. The fact that there is a stationary point at (1, 1) gives us 2 conditions • The curve passes through the point • $\mathrm{f}'(x)=0$ at $x=1$. Similarly, that the two curves intersect at (3, 17) gives us 2 conditions as well. Note that at least 4 independent linear equations are needed to solve 4 unknowns completely. It is a lot faster (and less error-prone) to use the GC to solve a system of linear equations.

2 Referred to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Point C lies on BA produced such that $AB : AC = 3 : 2$.

(i) Find $\overrightarrow{OC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ and show that the area of triangle OAC can be written as $k|\mathbf{a}\times\mathbf{b}|$, where k is a constant to be found. [3]

(ii) Given that $\mathbf{a}$ and $\mathbf{b}$ are unit vectors and the magnitude of $\mathbf{a}+\mathbf{b}$ is $\sqrt{3}$, find the cosine of the angle between $\mathbf{a}$ and $\mathbf{b}$. [3]

Solutions		**Comments**
2 (i) **[3]**	 $AB : AC = 3 : 2$ $\dfrac{AB}{AC} = \dfrac{3}{2}$ By ratio theorem, $\overrightarrow{OA} = \dfrac{2\overrightarrow{OB}+3\overrightarrow{OC}}{5}$ $5\overrightarrow{OA} = 2\overrightarrow{OB}+3\overrightarrow{OC}$ $\overrightarrow{OC} = \dfrac{1}{3}\left(5\overrightarrow{OA}-2\overrightarrow{OB}\right) = \dfrac{1}{3}(5\mathbf{a}-2\mathbf{b})$ Area of $\Delta OAC = \dfrac{1}{2}\left\|\overrightarrow{OA}\times\overrightarrow{OC}\right\|$ $=\dfrac{1}{2}\left\|\mathbf{a}\times\dfrac{1}{3}(5\mathbf{a}-2\mathbf{b})\right\|$ $=\dfrac{1}{3}\|\mathbf{a}\times\mathbf{b}\|$ (since $\mathbf{a}\times\mathbf{a}=\mathbf{0}$) where $k=\dfrac{1}{3}$.	If a point is on BA produced, then the order of the points on the line is B, A, C (left to right or right to left). $\dfrac{1}{2}\left\|\mathbf{a}\times\dfrac{1}{3}(5\mathbf{a}-2\mathbf{b})\right\|$ $=\dfrac{1}{2}\left\|\mathbf{a}\times\dfrac{5}{3}\mathbf{a}-\mathbf{a}\times\dfrac{2}{3}\mathbf{b}\right\|$ $=\dfrac{1}{2}\left\|-\dfrac{2}{3}\mathbf{a}\times\mathbf{b}\right\| = \dfrac{1}{2}\left\|-\dfrac{2}{3}\right\|\|\mathbf{a}\times\mathbf{b}\|$ $=\dfrac{1}{2}\left(\dfrac{2}{3}\right)\|\mathbf{a}\times\mathbf{b}\| = \dfrac{1}{3}\|\mathbf{a}\times\mathbf{b}\|$ Note that area cannot be negative.

(ii) **[3]**	$\lvert\mathbf{a}+\mathbf{b}\rvert=\sqrt{3}$ $\lvert\mathbf{a}+\mathbf{b}\rvert^2=3$ $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})=3$ $\mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}+2\mathbf{a}\cdot\mathbf{b}=3$ $\lvert\mathbf{a}\rvert^2+\lvert\mathbf{b}\rvert^2+2\mathbf{a}\cdot\mathbf{b}=3$ $1+1+2\mathbf{a}\cdot\mathbf{b}=3$ (since $\lvert\mathbf{a}\rvert=\lvert\mathbf{b}\rvert=1$) $\mathbf{a}\cdot\mathbf{b}=0.5$ $\cos\angle AOB=\dfrac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$ $=\dfrac{0.5}{(1)(1)}=\dfrac{1}{2}$ <u>Alternative Method:</u> Let θ be the angle between **a** and **b**. Using cosine rule, $\lvert\mathbf{a}+\mathbf{b}\rvert^2=\lvert\mathbf{a}\rvert^2+\lvert\mathbf{b}\rvert^2-2\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos(\pi-\theta)$ $\left(\sqrt{3}\right)^2=1^2+1^2-2(1)(1)\cos(\pi-\theta)$ $\cos(\pi-\theta)=-\dfrac{1}{2}$ $\cos\theta=-\cos(\pi-\theta)=\dfrac{1}{2}$	Most students are able to arrive at $\cos\angle AOB=\dfrac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}=\dfrac{\mathbf{a}\cdot\mathbf{b}}{(1)(1)}=\mathbf{a}\cdot\mathbf{b}$ To be able to make progress, you must realise that the condition $\lvert\mathbf{a}+\mathbf{b}\rvert=\sqrt{3}$ has to be used. To link it with the scalar product, a common way is to use the fact that for any vector $\mathbf{x}$, $\mathbf{x}\cdot\mathbf{x}=\lvert\mathbf{x}\rvert\lvert\mathbf{x}\rvert\cos 0=\lvert\mathbf{x}\rvert^2$. 

3 The line L_1 has equation $x=3,\ \dfrac{6-y}{3}=z$ and the line L_2 has equation $x-2=y-6,\ z=7$.

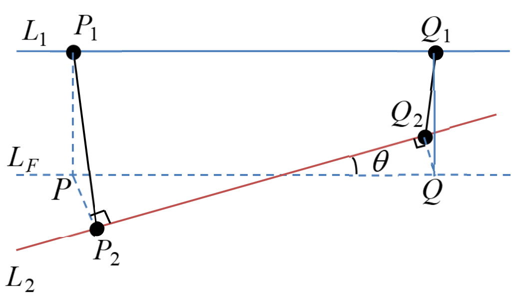
(i) Find the acute angle between L_1 and L_2. [2]

(ii) Show that L_1 and L_2 are skew lines. [2]

(iii) Let P_1 and Q_1 be two points on L_1 such that distance between them is $\sqrt{10}$. Let P_2 and Q_2 be the two respective points on L_2 such that P_1P_2 and Q_1Q_2 are minimum. Find the exact distance between P_2 and Q_2. [2]

Solutions		**Comments**
3 (i) **[2]**	$L_1: \mathbf{r}=\begin{pmatrix}3\\6\\0\end{pmatrix}+\lambda\begin{pmatrix}0\\-3\\1\end{pmatrix}, \lambda\in\mathbb{R},$ $L_2: \mathbf{r}=\begin{pmatrix}2\\6\\7\end{pmatrix}+\mu\begin{pmatrix}1\\1\\0\end{pmatrix}, \mu\in\mathbb{R}$ Let θ be the acute angle between L_1 and L_2. $\cos\theta=\dfrac{\left\lvert\begin{pmatrix}0\\-3\\1\end{pmatrix}\bullet\begin{pmatrix}1\\1\\0\end{pmatrix}\right\rvert}{\sqrt{10}\sqrt{2}}=\dfrac{3}{\sqrt{20}}$ $\Rightarrow\theta=47.9^\circ$ (1 d.p.)	One should be careful when converting cartesian equation of a line to the vector/parametric form. Do it systematically by first introducing the parameter, $x=3,\ \dfrac{6-y}{3}=z=\lambda$ Then express *x*, *y* and *z* in terms of the parameter: $x=3,\ y=6-3\lambda,\ z=\lambda$ before writing it in the vector form as shown. If you have made a mistake here, it will carry over to subsequent parts of the question, which is very costly.
(ii) **[2]**	<u>(1) To show 2 lines are non parallel:</u> Since the angle between L_1 and L_2 is not 0° or 180°, the 2 lines are not parallel. *OR* L_1 is parallel to $\begin{pmatrix}0\\-3\\1\end{pmatrix}$ and L_2 is parallel to $\begin{pmatrix}1\\1\\0\end{pmatrix}$. Since $\begin{pmatrix}0\\-3\\1\end{pmatrix}\neq k\begin{pmatrix}1\\1\\0\end{pmatrix}$ for any real *k*, the two lines are not parallel.	For 2 lines to be skew, they need to be both non parallel and non intersecting. It is not sufficient to just check that there is no intersection between the 2 lines.

	<u>(2) To show 2 lines are non intersecting:</u> Using cartesian equation: If the 2 lines intersect, then $x = 3, z = 7$. The first equation implies that $y = -15$ but the second equation implies that $y = 7$. Hence there are no intersections. Since both lines are not parallel and do not intersect, they are skew lines. <u>Alternative method for checking for intersection between L_1 and L_2 using vector equations:</u> Consider $\begin{pmatrix} 3 \\ 6-3\lambda \\ \lambda \end{pmatrix} = \begin{pmatrix} 2+\mu \\ 6+\mu \\ 7 \end{pmatrix} \Rightarrow \begin{array}{r} \mu = 1 \cdots (1) \\ -3\lambda = \mu \cdots (2) \\ \lambda = 7 \cdots (3) \end{array}$ Substituting (1) and (3), LHS of (2) = –21 and RHS of (2) = 1 This implies equations (1), (2) and (3) have no solutions for λ and μ, so L_1 and L_2 do not intersect. Since both lines are not parallel and do not intersect, they are skew lines.	

(iii) **[2]**	The required distance is given by $P_2Q_2 = P_1Q_1 \cos\theta$, where θ is the acute angle between the 2 lines. Essentially, P_2Q_2 is the length of projection of P_1Q_1 in the direction of line L_2. To see this, we use the following diagram:  Let L_F be the line parallel to L_1 and intersecting L_2. Let P and Q be the respective foot of perpendicular from P_1 and Q_1 to L_F. Then $P_2Q_2 = PQ\cos\theta = P_1Q_1\cos\theta$. We have $\cos\theta = \dfrac{3}{\sqrt{20}}$ and thus the required distance is simply $= \dfrac{3}{\sqrt{20}}\sqrt{10} = \dfrac{3}{\sqrt{2}}$. <u>Alternatively,</u> since P_1 and Q_1 are on L_1 which is parallel to $\begin{pmatrix}0\\-3\\1\end{pmatrix}$ and $\left\|\begin{pmatrix}0\\-3\\1\end{pmatrix}\right\| = \sqrt{10}$, $\overrightarrow{P_1Q_1} = \pm\begin{pmatrix}0\\-3\\1\end{pmatrix}$ $P_2Q_2 =$ length of projection of $\overrightarrow{P_1Q_1}$ on $\begin{pmatrix}1\\1\\0\end{pmatrix}$ $= \dfrac{\left\|\pm\begin{pmatrix}0\\-3\\1\end{pmatrix}.\begin{pmatrix}1\\1\\0\end{pmatrix}\right\|}{\left\|\begin{pmatrix}1\\1\\0\end{pmatrix}\right\|} = \dfrac{3}{\sqrt{2}}$	Students who draw some diagrams usually made the most progress in this question. Although it is possible to solve the problem by taking $\overrightarrow{P_1Q_1} = \pm\begin{pmatrix}0\\-3\\1\end{pmatrix}$, one should not assume the positions of P_1 and Q_1 since they are **any** two points on the line. And the time (and calculation) spent to find the respective foot of perpendiculars is surely not worth the 2 marks, and better spent checking your work in other questions.

4 **(a)** Differentiate $\log_3\left(2x^2+1\right)$ with respect to x. [2]

(b) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$, given that $y=\frac{\tan^{-1}x}{1+x^2}$. [2]

(c) Given that $x=1-\cos\theta$, $y=2\sin\theta$, where $0<\theta<\pi$, find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of θ. [2]

Solutions		**Comments**
4(a) **[2]**	$\frac{\mathrm{d}}{\mathrm{d}x}\log_3\left(2x^2+1\right)=\frac{\mathrm{d}}{\mathrm{d}x}\frac{\ln\left(2x^2+1\right)}{\ln 3}$ $=\frac{1}{\ln 3}\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(2x^2+1\right)$ $=\frac{1}{\ln 3}\cdot\frac{1}{\left(2x^2+1\right)}\cdot 4x$ $=\frac{4x}{\left(2x^2+1\right)\ln 3}$	(1) $\log_b a=\frac{\log_c a}{\log_c b}$ for $a,b,c>0$ (2) $\frac{1}{\ln 3}$ is a scalar multiple
(b) **[2]**	$y=\frac{\tan^{-1}x}{1+x^2}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\left(1+x^2\right)\frac{1}{1+x^2}-\left(2x\right)\tan^{-1}x}{\left(1+x^2\right)^2}$ $=\frac{1-\left(2x\right)\tan^{-1}x}{\left(1+x^2\right)^2}$	Note that $\tan^{-1}x$ gives the angle θ such that $\tan\theta=x$. Eg. $\tan^{-1}1=\frac{\pi}{4}$. As such, $\tan^{-1}x\neq\frac{1}{\tan x}$
(c) **[2]**	$x=1-\cos\theta$ $\quad$ $y=2\sin\theta$ $\frac{\mathrm{d}x}{\mathrm{d}\theta}=\sin\theta$ $\quad$ $\frac{\mathrm{d}y}{\mathrm{d}\theta}=2\cos\theta$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}\theta}\div\frac{\mathrm{d}x}{\mathrm{d}\theta}$ $=\frac{2\cos\theta}{\sin\theta}$ $=2\cot\theta$	

5 **(a)** Without using a calculator, solve the inequality

$$\frac{2x^3+x^2-5x+4}{x+1}>x. \qquad [4]$$

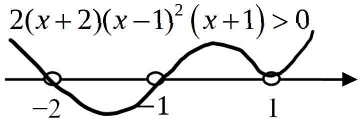
Hence solve the inequalities

(i) $$\frac{2|x|^3+x^2-5|x|+4}{|x|+1}>|x|, \qquad [1]$$

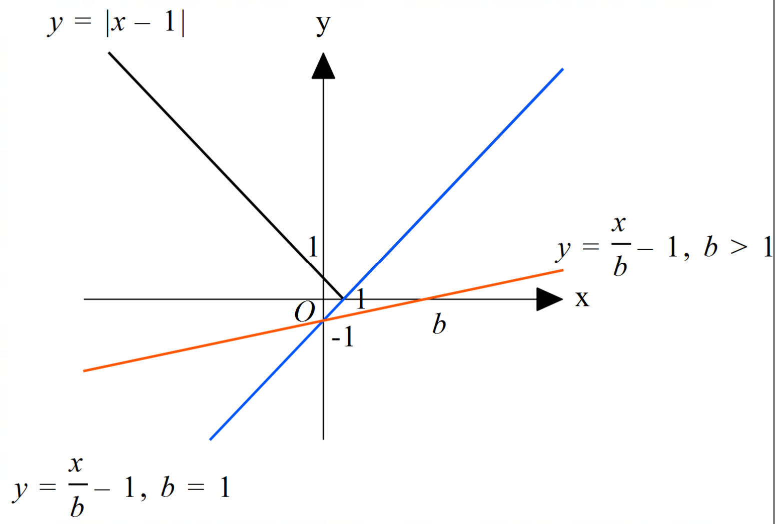
(ii) $$\frac{2\cos^3 x-\cos^2 x-5\cos x-4}{1-\cos x}<\cos x, \text{ for } 0<x<2\pi. \qquad [2]$$

(b) Show that there are no solutions to the equation

$b|x-1|=x-b$, where $b>1$ is a fixed constant. [4]

Solutions		**Comments**
5(a) **[4]**	$\dfrac{2x^3+x^2-5x+4}{x+1}>x \quad ------(*)$ $\dfrac{2x^3+x^2-5x+4}{x+1}-\dfrac{x^2+x}{x+1}>0$ $\dfrac{2\left(x^3-3x+2\right)}{x+1}>0$ Let $\mathrm{f}(x)=x^3-3x+2$ Since $\mathrm{f}(1)=1^3-3+2=0$, $x-1$ is a factor of f(x). $\therefore \mathrm{f}(x)=(x-1)\left(x^2+ax-2\right)$ By comparing coefficient of $x^2, a-1=0 \Rightarrow a=1$ $\dfrac{2(x-1)\left(x^2+x-2\right)}{x+1}>0$ $2(x+2)(x-1)^2(x+1)>0$  $\therefore\ x<-2 \quad \text{or} \quad x>-1, \quad x\neq 1$ Alternatively, $x<-2 \quad \text{or} \quad -1<x<1 \quad \text{or} \quad x>1$	Note that the question says "w/o use of calculator", so working has to be shown for the factorization of x^3-3x+2 Do not present your answer as $x<-2, -1<x<1, x>1$ "Or" should be used instead of commas, which mean "and".
(i)	$\dfrac{2|x|^3+x^2-5|x|+4}{|x|+1}>|x| \quad -----(\#)$ Replace x by $|x|$ in (*) to obtain (#). So, we do likewise for the solution,	

	$\lvert x\rvert < -2$ or $-1 < \lvert x\rvert < 1$ or $\lvert x\rvert > 1$ No Soln $\quad -1 < x < 1 \quad x < -1$ or $x > 1$ Thus, the solution of (#) is $x < -1$ or $-1 < x < 1$ or $x > 1$	
(ii)	$\dfrac{2\cos^3 x - \cos^2 x - 5\cos x - 4}{1-\cos x} < \cos x \quad -----(@)$ $\times -1,\ -\dfrac{2\cos^3 x - \cos^2 x - 5\cos x - 4}{1-\cos x} > -\cos x$ $\dfrac{-2\cos^3 x + \cos^2 x + 5\cos x + 4}{(-\cos x)+1} > -\cos x$ $\dfrac{2(-\cos x)^2 + (-\cos x)^2 + 5(-\cos x) + 4}{(-\cos x)+1} > -\cos x$ Replace x by $-\cos x$ in (*) to obtain (@). So, we do likewise for the solution, $-\cos x < -2$ or $-1 < -\cos x < 1$ or $-\cos x > 1$ $\cos x > 2$ or $-1 < \cos x < 1$ or $\cos x < -1$ No soln or $0 < x < \pi$ or $\pi < x < 2\pi$ or No soln Thus, the solution of (@) is $0 < x < \pi$ or $\pi < x < 2\pi$.	Have to exclude $x = \pi$ because $\cos\pi = -1$
(b) **[4]**	<u>Method 1:</u> $\lvert x-1\rvert = \begin{cases} x-1, & x \ge 1 \\ 1-x, & x < 1 \end{cases}$ For $x \ge 1,\ bx - b = x - b$ $\quad$ For $x < 1,\ b - bx = x - b$ $(b-1)x = 0$ or $(b+1)x = 2b$ $x = 0$ or $x = \dfrac{2b}{b+1} > \dfrac{2b}{b+b} = 1$ (reject since $x \ge 1$) $\quad$ (reject since $x < 1$) Note: The case for $x < 1$ need not be discussed if the following observation is made. Since $b > 1$ and $\lvert x-1\rvert \ge 0$ for all real x, $b\lvert x-1\rvert = x - b \Rightarrow x - b \ge 0 \Rightarrow x \ge b > 1 \Rightarrow x > 1$. <u>Method 2:</u> $b\lvert x-1\rvert = x - b$ $\lvert x-1\rvert = \dfrac{x-b}{b} = \dfrac{x}{b} - 1$	For the case $x \ge 1$, $bx - b = x - b$ which leads to $bx = x$, many incorrectly concluded that b must be 1, without considering that x can be 0, which needs to be rejected by either considering the domain as shown or substituting it back into the equation.

	 Since $b > 1 \Rightarrow \frac{1}{b} < 1$, the line $y = \frac{x}{b} - 1$ is less steep compared with the line $y = x - 1$. Since the line $y = x - 1$ is already above $y = \frac{x}{b} - 1$ at $x = b$, it will continue to be above the line for $x > b$ (in fact the gap increases). When $x \le b$, it is clear there are no intersections since one graph is above the x-axis while the other is below the x-axis and hence there are no intersections.	For a graphical solution, there must be some mention of how the condition $b > 1$ affects the gradient of the lines you have drawn.

6 The function f is defined by

$$\mathrm{f}: x \mapsto (x-3)^2 + 2, \quad x \in \mathbb{R}, \quad x \geq k.$$

(i) Given that the function f^{-1} exists, state the least value of k. [1]

Use the value of k found in part **(i)** for the rest of this question.

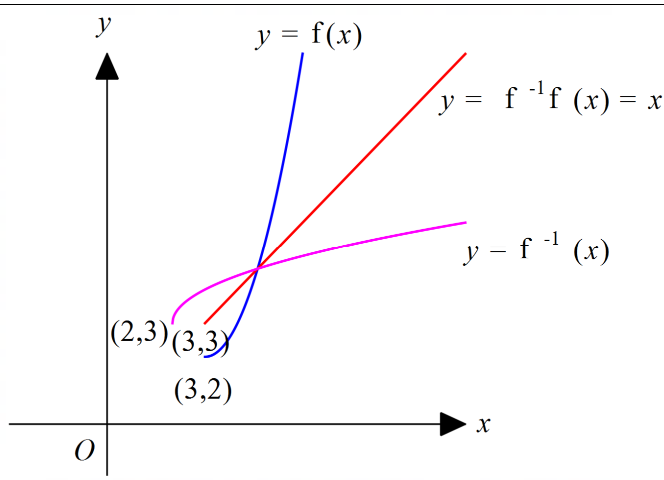
(ii) Without finding f^{-1}, sketch on the same diagram the graphs of $y = \mathrm{f}(x)$, $y = \mathrm{f}^{-1}(x)$ and $y = \mathrm{f}^{-1}\mathrm{f}(x)$. You should label your graphs clearly. [3]

(iii) Hence find the exact solution(s) of the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [3]

The function g is defined by

$$\mathrm{g}: x \mapsto \sqrt{x} + h, \quad x \in \mathbb{R}, \quad x \geq 0.$$

(iv) Find the set of values of h such that composite function fg exists and write down an expression for $\mathrm{fg}(x)$ in terms of h. [3]

Solutions		**Comments**
6(i) **[1]**	Least value of $k = 3$.	
(ii) **[3]**		The graphs of f and f^{-1} are symmetrical about $y = x$, so ensure you have the same scale for both x and y axes. End points should be clearly labelled. Also, note that the domain of $\mathrm{f}^{-1}\mathrm{f}$ is the same as domain of f, so $y = \mathrm{f}^{-1}\mathrm{f}(x) = x$ only for $x \in D_{\mathrm{f}} = [3, \infty)$ and (3, 3) is the end point. We cannot draw the entire line $y = x$

<table>
<tr>
<td>(iii)
[3]</td>
<td>$\mathrm{f}(x)=\mathrm{f}^{-1}(x)=x$
$\mathrm{f}(x)=x$
$(x-3)^2+2=x$
$x^2-7x+11=0$
$x=\dfrac{7\pm\sqrt{7^2-4(1)(11)}}{2}$
$=\dfrac{7+\sqrt{5}}{2}$ or $\dfrac{7-\sqrt{5}}{2}$ (rejected since $x\geq 3$)</td>
<td>The reason for rejecting the other answer should be given as $x\geq 3$, since that is the domain of f. Saying $x\geq 0$ doesn't make sense; in fact, $\dfrac{7-\sqrt{5}}{2}>0$.</td>
</tr>
<tr>
<td>(iv)
[3]</td>
<td colspan="2">For fg to exist, $\mathrm{R_g}\subseteq\mathrm{D_f}$.
$\mathrm{R_g}=[h,\infty)$ and $\mathrm{D_f}=[3,\infty)$
Set of values of h for fg to exist is $=[3,\infty)$.

$\mathrm{fg}(x)=\mathrm{f}\left(\sqrt{x}+h\right)$
$=\left(\sqrt{x}+h-3\right)^2+2$</td>
</tr>
</table>

7 **Do not use a calculator in answering this question.**

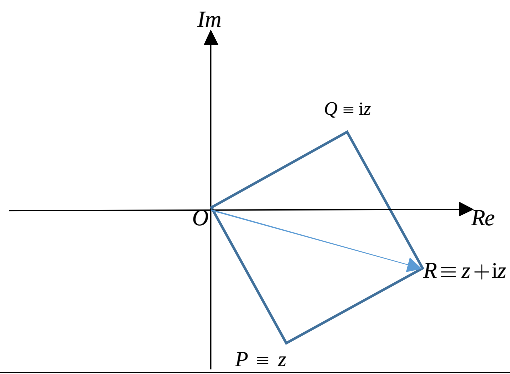
The complex number z is such that $\arg(z)=\theta$ where $-\frac{\pi}{2}<\theta<-\frac{\pi}{4}$.

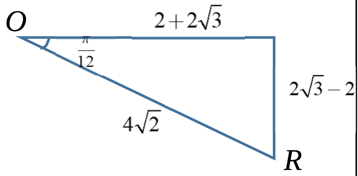
(i) In an Argand diagram, mark and label clearly the points P, Q and R representing the complex numbers z, $\mathrm{i}z$ and $z+\mathrm{i}z$ respectively. [2]

(ii) State the geometrical representation of $OPRQ$ and express $\arg(z+\mathrm{i}z)$ in terms of θ and π. [2]

(iii) It is now given that $z=2-2\sqrt{3}\mathrm{i}$. Show that

$$\sin\left(\frac{\pi}{12}\right)=\frac{\sqrt{3}-1}{2\sqrt{2}}.$$ [5]

<table>
<tr><th colspan="2">Solutions</th><th>Comments</th></tr>
<tr><td>8(i)</td><td></td><td>Note that $\measuredangle QOP=90^\circ$</td></tr>
<tr><td>(ii)</td><td>Since angle $POQ=90^\circ$ and $OP=OQ=|z|$, $OPRQ$ is a square.
$\arg(z+\mathrm{i}z)=\theta+\frac{\pi}{4}$</td><td>Note that $\measuredangle POR=45^\circ$.
So, $\arg(z+\mathrm{i}z)$
$=\arg z+\measuredangle POR$
$=\theta+\frac{\pi}{4}$</td></tr>
<tr><td>(iii)</td><td>$z=2-2\sqrt{3}\,\mathrm{i}$
$\arg(z)=-\frac{\pi}{3}$
$\arg(z+\mathrm{i}z)=-\frac{\pi}{3}+\frac{\pi}{4}=-\frac{\pi}{12}$
$z+\mathrm{i}z=\left(2-2\sqrt{3}\mathrm{i}\right)+\mathrm{i}\left(2-2\sqrt{3}\mathrm{i}\right)$
$=\left(2+2\sqrt{3}\right)+\left(2-2\sqrt{3}\right)\mathrm{i}$
Note that $2-2\sqrt{3}<0$.</td><td></td></tr>
</table>

	Hypotenuse $=\sqrt{\left(2\sqrt{3}-2\right)^2+\left(2+2\sqrt{3}\right)}$ $=\sqrt{12-8\sqrt{3}+4+4+8\sqrt{3}+12}$ $=\sqrt{32}$ $=\sqrt{16\times 2}$ $=4\sqrt{2}$ From the diagram on the right, $\therefore \sin\left(\frac{\pi}{12}\right)=\frac{2\sqrt{3}-2}{4\sqrt{2}}=\frac{\sqrt{3}-1}{2\sqrt{2}}$ (Shown) 	

8 **(a)** A manufacturer can produce q thousand units of a particular machine per week at the unit price $\$p$ according to the equation $q^2 - 3qp + p^2 = 5$, where $p > 3$ and $q > 3$.

Find the rate of change of the supply of the machines when the quantity produced is 4000 units, and the unit price is \$11 which is increasing at a rate of \$0.10 per week. [3]

(b) Isocost lines and isoquant curves are studied in economics. A general class of isoquant curves are the Cobb-Douglas functions of the form $x^{\alpha} y^{\beta} = k$, where α, β and k are positive constants.

(i) Show that $\dfrac{x}{y}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right) = -\dfrac{\alpha}{\beta}$. [2]

The point at which the isocost line is a tangent to the isoquant curve is called an equilibrium point.

(ii) The isocost line $27x + 2y = 6$ is tangential to the isoquant curve $x^{\frac{1}{3}} y^{\frac{2}{3}} = k$, where $x > 0$, $y > 0$ and k is a positive constant.

Find the coordinates of the equilibrium point and the value of k. [7]

Solutions		**Comments**
8(a) **[3]**	$q^2 - 3qp + p^2 = 5$ Method 1: Differentiate wrt t $2q\dfrac{\mathrm{d}q}{\mathrm{d}t} - 3\left(q\dfrac{\mathrm{d}p}{\mathrm{d}t} + p\dfrac{\mathrm{d}q}{\mathrm{d}t}\right) + 2p\dfrac{\mathrm{d}p}{\mathrm{d}t} = 0$ When $q = 4$, $p = 11$ and $\dfrac{\mathrm{d}p}{\mathrm{d}t} = 0.1$, $2(4)\dfrac{\mathrm{d}q}{\mathrm{d}t} - 3\left(4(0.1) + (11)\dfrac{\mathrm{d}q}{\mathrm{d}t}\right) + 2(11)(0.1) = 0$ $\dfrac{\mathrm{d}q}{\mathrm{d}t} = \dfrac{1}{25}$ Supply is increasing at a rate of 40 $\left(= \dfrac{1}{25} \times 1000\right)$ units per week. Method 2:	In this question, q represents number of machines produced in **thousands**.

	$q^2-3qp+p^2=5$ Differentiate wrt p: $2q\dfrac{\mathrm{d}q}{\mathrm{d}p}-3\left(q+p\dfrac{\mathrm{d}q}{\mathrm{d}p}\right)+2p=0$ $\dfrac{\mathrm{d}q}{\mathrm{d}p}=\dfrac{3q-2p}{2q-3p}$ $\dfrac{\mathrm{d}q}{\mathrm{d}t}=\dfrac{\mathrm{d}q}{\mathrm{d}p}\times\dfrac{\mathrm{d}p}{\mathrm{d}t}=\dfrac{0.1(3q-2p)}{2q-3p}$ When $q=4$, $p=11$ and $\dfrac{\mathrm{d}p}{\mathrm{d}t}=0.1$, $\dfrac{\mathrm{d}q}{\mathrm{d}t}=\dfrac{0.1(3(4)-2(11))}{2(4)-3(11)}=\dfrac{1}{25}$ Supply is increasing at a rate of 40 units per week.	
(b) (i) [2]	$x^{\alpha}y^{\beta}=k$ Since $\alpha,\ \beta,\ k>0,\ x>0$ and $y>0$. Taking natural logarithm on both sides, $\alpha\ln x+\beta\ln y=\ln k$. Differentiating with respect to x: $\dfrac{\alpha}{x}+\beta\dfrac{1}{y}\dfrac{\mathrm{d}y}{\mathrm{d}x}=0\Rightarrow\dfrac{\beta}{y}\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{\alpha}{x}$ $\Rightarrow\dfrac{x}{y}\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{\alpha}{\beta}$ (shown) <u>Alternative (Implicit differentiation):</u> $x^{\alpha}y^{\beta}=k$ Differentiating implicitly wrt x $\alpha x^{\alpha-1}y^{\beta}+\beta x^{\alpha}y^{\beta-1}\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$ $\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{\alpha x^{\alpha-1}y^{\beta}}{\beta x^{\alpha}y^{\beta-1}}$ (note that $x, y\neq 0$ since $k>0$) $\Rightarrow\dfrac{x}{y}\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{\alpha x^{\alpha}y^{\beta}}{\beta x^{\alpha}y^{\beta}}=-\dfrac{\alpha}{\beta}$ (shown)	

(b) **(ii)** **[7]**	From **(b)(i)** $\frac{x}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{\frac{1}{3}}{\frac{2}{3}}=-\frac{1}{2}$ Tangent line : $27x+2y=6 \Leftrightarrow y=-\frac{27}{2}x+3$ has slope $-\frac{27}{2}$. $\frac{x}{y}\left(-\frac{27}{2}\right)=-\frac{1}{2} \Rightarrow x=\frac{y}{27}$ Using tangent line, $27\left(\frac{y}{27}\right)+2(y)=6 \Rightarrow y=2,\ x=\frac{2}{27}$ Co-ordinates of equilibrium point are $\left(\frac{2}{27},2\right)$ To find k we use equation of curve $x^{\frac{1}{3}}y^{\frac{2}{3}}=k$ and the equilibrium point $\left(\frac{2}{27},2\right)$ $\therefore k=\left(\frac{2}{27}\right)^{\frac{1}{3}}2^{\frac{2}{3}}=\frac{2}{3}$