Integration Techniques

1. **AJC/2009/I/8**

Find

(i) $\int \mathrm{e}^{2x} \tan^{-1}\left(\mathrm{e}^{-2x}\right) \, \mathrm{d}x$, [3]

(ii) $\int \frac{x}{\sqrt{1-4x-2x^2}} \, \mathrm{d}x$. [3]

2. **CJC/I/3**

Find

(i) $\int x \sin x \, \mathrm{d}x$, [2]

(ii) $\int \frac{1}{x\sqrt{1-x}} \, \mathrm{d}x$, where $0 < x < 1$, using the substitution $x = \cos^2 u$. [4]

3. **PJC/I/10**

(a) Use the fact that $5\sin x - 3\cos x = (\cos x + \sin x) - 4(\cos x - \sin x)$ to find the exact value of $\int_{\frac{\pi}{6}}^{0} \frac{5\sin x - 3\cos x}{\cos x - \sin x} \, dx$. [4]

(b) Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(x\sqrt{1-x^2}\right)$, expressing your result in the form $\frac{a+bx^2}{\sqrt{1-x^2}}$, where a and b are constants to be determined. [2]

Hence, or otherwise, obtain the exact value of $\int_0^{\frac{1}{2}} \frac{3-2x^2}{\sqrt{1-x^2}} \, \mathrm{d}x$. [3]

4. **MJC/I/3**

Using the substitution $x = 2\mathrm{e}^t$, show that

$$\int \frac{\ln x - \ln 2}{x\sqrt{\ln x - \ln 2 - 2}} \, \mathrm{d}x = \int \left[\sqrt{t-2} + \frac{2}{\sqrt{t-2}}\right] \mathrm{d}t.$$

Hence evaluate $\int_{2\mathrm{e}^2}^{2\mathrm{e}^4} \frac{\ln x - \ln 2}{x\sqrt{\ln x - \ln 2 - 2}} \, \mathrm{d}x$, giving your answer in the form $\frac{a\sqrt{2}}{b}$ where $a, b \in \mathbb{Z}$.

[7]

5. **MJC/II/1**

(i) Differentiate $e^{\cos x}$ with respect to x. [1]

(ii) Find $\int e^{\cos x} \sin 2x \, dx$. [3]

6. **SAJC/I/8**

(a) Show that $\dfrac{2x^2-5x+13}{x^2-2x+5}$ can be written in the form $A+\dfrac{B(2x-2)+C}{x^2-2x+5}$ where A, B and C are constants to be determined.

Hence find the integral $\int \frac{2x^2-5x+13}{x^2-2x+5} \, dx$. [5]

(b) Find, in terms of n and e, $\int_1^{2e} x^{n-1} \ln x \, dx$. [4]

7. **ACJC/2010/II/1**

Find the exact value of $\int_{-1}^{1} \left| e^{2x} - \frac{1}{e^{2(x-1)}} \right| dx$. [4]

8. **AJC/2011/I/1**

Find $\int x \cos^{-1} x^2 \, dx$. [3]

9. **IJC/2011/I/3**

Using the substitution $x=\frac{1}{2}e^u$, find $\displaystyle\int \frac{\left[\ln(2x)\right]^2}{x\left\{25-2\left[\ln(2x)\right]^2\right\}} \, dx$.

10. **NYJC/2011/I/2**

(a) Sketch the graph of $y=\left|x-\frac{a}{2}\right|$ for $-a \le x \le a$ and $a>0$. [1]

Hence, find k such that $\displaystyle\int_{-a}^{0} \left|x-\frac{a}{2}\right| dx = k\int_0^a \left|x-\frac{a}{2}\right| dx$. [4]

(b) Find

(i) $\dfrac{d}{dx}(x^2 \sin 2x)$,

(ii) $\int x^2 \cos 2x \, dx$. [3]

11. **ACJC/2011/I/3**

(a) Find

(i) $\int x\tan\left(x^2\right)\mathrm{d}x$, [2]

(ii) $\int \frac{x}{x^2+x+3}\,\mathrm{d}x$. [4]

(b) Evaluate, exactly,

(i) $\int_0^{\frac{1}{\sqrt{2}}} x\sin^{-1}\left(x^2\right)\mathrm{d}x$, [4]

(ii) $\int_0^1 x|x-b|\,\mathrm{d}x$ where $0<b<1$. [3]

12. **DHS/2012/I/2**

Obtain a formula for $\int_{\frac{1}{2}}^{n} \frac{\left(\tan^{-1}2x\right)^2}{1+4x^2}\,\mathrm{d}x$ in terms of n, where $n>0$.

Hence evaluate $\int_{\frac{1}{2}}^{\infty} \frac{\left(\tan^{-1}2x\right)^2}{1+4x^2}\,\mathrm{d}x$ exactly. [4]

13. **MJC/2012/I/3(b)**

(i) Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(2^{2x}\right)$. [1]

(ii) Hence find $\int 2^{2x}\ln 2^x\,\mathrm{d}x$. [3]

14. **SRJC/2012/I/12**

(a) Find the integral $\int (\ln\frac{x}{2})^2\,\mathrm{d}x$. [3]

(b) Solve the inequality $x^3 \geq \frac{a^4}{x}$, leaving your answers in terms of a, where $1<a<3$.

Hence find $\int_1^3 \left|x^3-\frac{a^4}{x}\right|\mathrm{d}x$, in terms of a. [6]

15. **CJC/2013/I/2**

Find the value of k such that

$$\int_{-c}^{0}|x-c|\,\mathrm{d}x = k\int_0^{2c}|x-c|\,\mathrm{d}x,$$

where c is a positive constant.

16. **DHS/2013/II/1**

(a) Find $\int \frac{x^2}{(x-1)(x-2)}\,\mathrm{d}x$. [3]

(b) (i) Differentiate $\sin^{-1}(x^2)$ with respect to x. [1]

(ii) Hence or otherwise, find a positive integral value of n such that $\int_0^n x\sin^{-1}(x^2)\,\mathrm{d}x = \frac{\pi}{4}-\frac{1}{2}$. [3]

17. **DHS/2014/I/8**

(a) Find $\int x\sec^2(x+a)\,\mathrm{d}x$, where a is a constant. [3]

(b) Find $\int \frac{x-1}{x^2-2x+2}\,\mathrm{d}x$. [2]

Hence find

(i) the exact value of $\int_1^2 \frac{x-4}{x^2-2x+2}\,\mathrm{d}x$, [4]

(ii) $\int_{2-p}^{p} \left|\frac{x-1}{x^2-2x+2}\right|\mathrm{d}x$ where p is a constant, $p>1$. Leave your answer in terms of p. [3]

18. **HCI/2014/I/2**

By using the substitution $u=1-x$, show that $\int_0^1 x^n(1-x)^m\,\mathrm{d}x = \int_0^1 (1-x)^n x^m\,\mathrm{d}x$. [2]

Hence, or otherwise, evaluate $\int_0^1 x^2\sqrt{1-x}\,\mathrm{d}x$, express your answer in exact form. [3]

19. **JJC/2014/I/6**

(a) (i) Obtain a formula for $\int_1^n \frac{1}{x^2}\ln x\,\mathrm{d}x$ in terms of n, where $n>1$. [3]

(ii) Hence evaluate $\int_1^\infty \frac{1}{x^2}\ln x\,\mathrm{d}x$. [1]

[You may assume that $\frac{1}{n}\ln n \to 0$ as $n\to\infty$.]

(b) Use the substitution $x=a\sec\theta$ to find the exact value of $\displaystyle\int_a^{2a}\frac{\sqrt{x^2-a^2}}{x}\,dx$ in terms of a and π, where a is a positive constant. [4]

20. **PJC/2014/I/7**

It is given that

$$f(x)=\begin{cases}\dfrac{1}{1+\sqrt{x}} & 0 \le x < a^2, \\ -2 & a^2 \le x < 5,\end{cases}$$

and that $f(x+5)=f(x)$ for all real values of x, where $0<a<\sqrt{5}$.

(i) Show that $f(100)=1$. [1]

(ii) Sketch the graph of $y=f(x)$ for $-5+a^2 \le x < 5+a^2$. [3]

(iii) By using the substitution $u=1+\sqrt{x}$, find $\int_0^{a^2} f(x)\,dx$, in terms of a. [5]

21. **ACJC Prelim/2020/01/Q1**

(i) Use the substitution $u = 2x - 1$ to find $\displaystyle\int \frac{x}{\sqrt{1-(2x-1)^2}}\,dx$. [4]

(ii) Hence find $\int \sin^{-1}(2x-1)\,dx$. [2]

22. **DHS/2022/I/Q3**

(a) Differentiate $e^{\sin^2 2x}$ with respect to x. [2]

(b) Find $\displaystyle\int \frac{e^{\sin^2 2x}\sin 4x}{\sqrt{1+e^{\sin^2 2x}}}\,dx$. [2]

(c) Find the exact value of $\int_0^{\frac{\pi}{4}} e^{\sin^2 2x}\sin 4x\cos^2 2x\,dx$. [3]

23. **DHS Prelim 9758/2024/01/Q4**

(a) Find $\displaystyle\int \frac{14+3x}{\sqrt{9-8x-x^2}}\,dx$. [4]

(b) Find $\int_0^2 3x^2 e^{kx}\,dx$ in terms of k, where k is a positive constant. Explain whether there exist solutions for k satisfying the equation $\int_0^2 3x^2 e^{kx}\,dx = -\dfrac{6}{k^3}$. [4]

24. **VJC Prelim 9758/2024/01/Q6**

(a) Find the exact value of $\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{2\sin^{-1}x}{\sqrt{1-x^2}}\,\mathrm{d}x$. [3]

(b) Find the exact value of $\int_0^{\frac{\pi}{3}} |\cos 2x|\,\mathrm{d}x$. [3]

(c) Find $\int \frac{1}{-x^2+2kx+3k^2}\,\mathrm{d}x$, where k is a positive constant. [4]

25. **ASRJC Prelim 9758/2025/P2/Q5**

(a) Solve the following integral.

(i) $\int \frac{1-2x}{\sqrt{3+2x-x^2}}\,\mathrm{d}x$. [2]

(ii) $\int x\ln\left(2-x^2\right)\mathrm{d}x$, where $-\sqrt{2}<x<\sqrt{2}$. [3]

(b) A function f is defined by $\mathrm{f}(x)=\mathrm{e}^x-2$.

(i) Sketch the graph of $y=|\mathrm{f}(x)|$, indicating clearly the equation(s) of asymptote(s), if any. [2]

Hence find $\int_0^{\ln 3} |\mathrm{f}(x)|\,\mathrm{d}x$, giving your answer in exact form. [3]

26. **NYJC Prelim 9758/2025/P1/Q7**

(a) Use the substitution $x=\sin^2\theta$, where $0\le\theta\le\frac{\pi}{2}$, to find $\int_0^{\frac{1}{2}} \sqrt{\frac{16x}{1-x}}\,\mathrm{d}x$ exactly. [4]

(b) Find $\int_a^b \frac{|x-1|}{\sqrt{\frac{1}{2}x^2-x+1}}\,\mathrm{d}x$ in terms of a and b, where $a<1<b$. [4]

Answers

1	(i) $\frac{1}{2}e^{2x}\tan^{-1}\left(e^{-2x}\right)+\frac{1}{4}\ln\left(e^{4x}+1\right)+C$ (ii) $-\frac{1}{2}\sqrt{1-4x-2x^2}-\frac{1}{\sqrt{2}}\sin^{-1}\frac{\sqrt{2}\left(x+1\right)}{\sqrt{3}}+C$
2	(i) $\sin x-x\cos x+c$ (ii) $-2\ln\left(\frac{1}{\sqrt{x}}+\sqrt{\frac{1-x}{x}}\right)+c$
3	(a) $\frac{2}{3}\pi+\ln\left(\frac{\sqrt{3}-1}{2}\right)$; (b) $\frac{1-2x^2}{\sqrt{1-x^2}}$, $\frac{\pi}{3}+\frac{\sqrt{3}}{4}$
4	$\frac{16\sqrt{2}}{3}$, $a=16$, $b=3$

5	(i) $-e^{\cos x}\sin x$ (ii) $-2e^{\cos x}\cos x+2e^{\cos x}+C$
6	(a) $2x-\frac{1}{2}\ln\left\|x^2-2x+5\right\|+2\left(\frac{1}{2}\right)\tan^{-1}\left(\frac{x-1}{2}\right)+c$ (b) $\frac{1}{n^2}\left[n(2e)^n(\ln 2+1)-(2e)^n+1\right]$
7	$\frac{1}{2}\left(e^4+e^2-4e+e^{-2}+1\right)$
8	$\frac{x^2}{2}\cos^{-1}x^2-\frac{1}{2}\sqrt{1-x^4}+C$
9	$\frac{1}{2}\left(\frac{5}{2\sqrt{2}}\ln\left\|\frac{5+\sqrt{2}\ln(2x)}{5-\sqrt{2}\ln(2x)}\right\|-\ln(2x)\right)+c$
10	(b) (i) $2x\sin 2x+2x^2\cos 2x$ (ii) $\frac{1}{2}x\cos 2x-\frac{1}{4}\sin 2x+\frac{1}{2}x^2\sin 2x+C$
11	(a)(i) $-\frac{1}{2}\ln\left\|\cos\left(x^2\right)\right\|+c$ (a)(ii) $\frac{1}{2}\ln\left\|x^2+x+3\right\|-\frac{1}{\sqrt{11}}\tan^{-1}\frac{(2x+1)}{\sqrt{11}}+c$ (b)(i) $\frac{\pi}{24}+\frac{\sqrt{3}}{4}-\frac{1}{2}$ (b)(ii) $\frac{b^3}{3}+\frac{1}{3}-\frac{b}{2}$
12	$\frac{1}{6}\left[\left(\tan^{-1}2n\right)^3-\left(\frac{\pi}{4}\right)^3\right]$, $\frac{7}{384}\pi^3$
13	(i) $2^{2x+1}\ln 2$, (ii) $2^{2x-1}\left(x-\frac{1}{2\ln 2}\right)+C$
14	(a) $x(\ln\frac{x}{2})^2-2x(\ln\frac{x}{2})+2x+c$
15	$k=\frac{3}{2}$
16	(a) $x-\ln\left\|x-1\right\|+4\ln\left\|x-2\right\|+c$ (b)(i) $\frac{2x}{\sqrt{1-x^4}}$ (b)(ii) $n=1$
17	(a) $x\tan(x+a)-\ln\left\|\sec(x+a)\right\|+c$ (b) $\frac{1}{2}\ln(x^2-2x+2)+c$ (b)(i) $\frac{1}{2}\ln 2-\frac{3\pi}{4}$ (ii) $\ln\left(p^2-2p+2\right)$
18	$\frac{16}{105}$
19	(a)(i) $-\frac{\ln n}{n}-\frac{1}{n}+1$ (a)(ii) 1 (b) $a\left(\sqrt{3}-\frac{\pi}{3}\right)$
20	(i) $f(100)=f(0)=1$ (iii) $2\left(a-\ln(1+a)\right)$
21	(i) $\frac{1}{4}\sin^{-1}(2x-1)-\frac{1}{4}\sqrt{1-(2x-1)^2}+C$ (ii) $\left(x-\frac{1}{2}\right)\sin^{-1}(2x-1)+\frac{1}{2}\sqrt{1-(2x-1)^2}+C$

22	(a) $2e^{\sin^2 2x}\sin 4x$ (b) $\sqrt{1+e^{\sin^2 2x}}+c$ (c) $\frac{1}{2}e-1$
23	(a) $2\sin^{-1}\left(\frac{x+4}{5}\right)-3\sqrt{9-8x-x^2}+c$ (b) $\frac{6e^{2k}\left(2k^2-2k+1\right)-6}{k^3}$
24	(a) $\frac{\pi^2}{12}$ (b) $1-\frac{\sqrt{3}}{4}$ (c) $\frac{1}{4k}\ln\left\|\frac{k+x}{3k-x}\right\|+C$
25	(a)(i) $2\sqrt{3+2x-x^2}-\sin^{-1}\left(\frac{x-1}{2}\right)+c$ (ii) $\frac{x^2}{2}\ln\left(2-x^2\right)-\frac{x^2}{2}-\ln\left(2-x^2\right)+c$ (b)(ii) $4\ln 2-2\ln 3$
26	(a) $\pi-2$ (b) $-2\sqrt{2}+2\left(\frac{1}{2}a^2-a+1\right)^{\frac{1}{2}}+2\left(\frac{1}{2}b^2-b+1\right)^{\frac{1}{2}}$