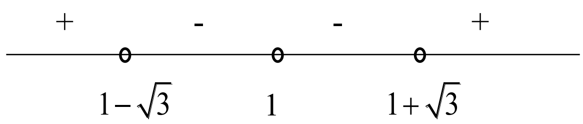
HCI 2025 C2 H2 Mathematics Preliminary Examination Paper 1 Suggested Solutions

1 A curve has equation $y = x + 1 + \dfrac{3}{x-1}$. Using differentiation, find the set of values of x where the curve is strictly decreasing. Give your answers in exact form. [4]

Suggested Solutions
$y = x+1+\dfrac{3}{x-1}$ $\quad = x+1+3\left(x-1\right)^{-1}$ Hence $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1+0+3(-1)(x-1)^{-2}$ $\quad = 1-\dfrac{3}{\left(x-1\right)^2}$ For strictly decreasing, $\dfrac{\mathrm{d}y}{\mathrm{d}x} < 0$ $\Rightarrow 1-\dfrac{3}{\left(x-1\right)^2} < 0$ **<u>Method 1</u>** $\dfrac{x^2-2x+1-3}{\left(x-1\right)^2} < 0$ $\dfrac{x^2-2x-2}{\left(x-1\right)^2} < 0$ Critical values: $x = 1$ and $x^2-2x-2=0$ $\Rightarrow x = \dfrac{2\pm\sqrt{4-4\left(-2\right)}}{2} = 1\pm\sqrt{3}$  $\therefore \left\{x\in\mathbb{R} \mid 1-\sqrt{3} < x < 1 \text{ or } 1 < x < 1+\sqrt{3}\right\}$

Method 2

$$1 < \frac{3}{(x-1)^2}$$

Since $(x-1)^2 \geq 0$,

$(x-1)^2 < 3$ for $x \neq 1$ (IMPT!!)

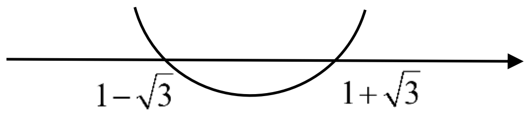
$\Rightarrow x^2 - 2x + 1 - 3 < 0$

$x^2 - 2x - 2 < 0$

Consider

$x^2 - 2x - 2 = 0$

$$\Rightarrow x = \frac{2 \pm \sqrt{4 - 4(-2)}}{2} = 1 \pm \sqrt{3}$$



$$\therefore \left\{ x \in \mathbb{R} \mid 1 - \sqrt{3} < x < 1 + \sqrt{3},\ \ x \neq 1 \right\}$$

2 A sequence of real numbers, x_n, satisfies the recurrence relation

$$x_{n+1} = a(-3)^n + bn^2 + cx_n \text{, for } n,a,b,c \in \mathbb{Z} \text{ and } n \geq 1.$$

Given that $x_1 = 1$, $x_2 = -8$, $x_3 = 70$, and $x_4 = -377$, find the values of a, b and c. [3]

Hence, find the values of x_{10} and x_{11}. [2]

Suggested Solutions

$x_2 = a(-3) + b + c$

$-8 = -3a + b + c$ ---------- (1)

$x_3 = a(-3)^2 + b(2)^2 + c(-8)$

$70 = 9a + 4b - 8c$ ---------- (2)

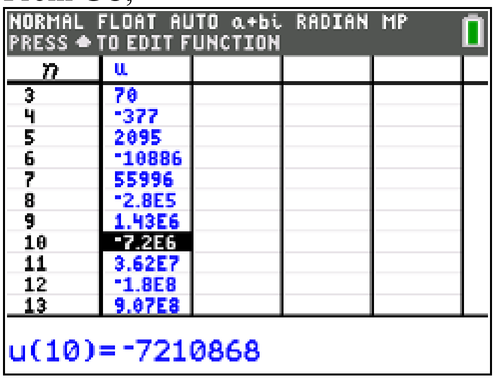
$x_4 = a(-3)^3 + b(3)^2 + c(70)$

$-377 = -27a + 9b + 70c$ --------- (3)

By GC,

$a = 2$, $b = 3$, $c = -5$

From GC,



$x_{10} = -7210868$

$x_{11} = 36172738$

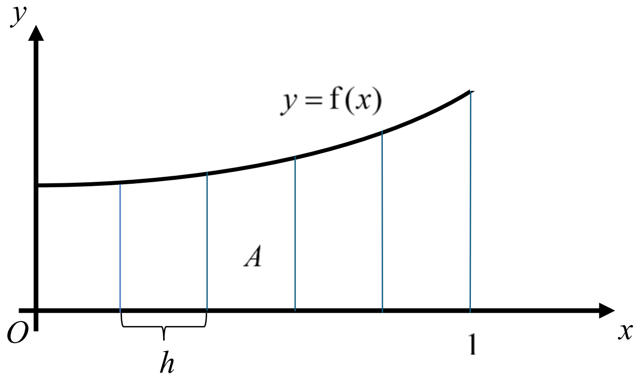
3 It is given that $\left(2xy\dfrac{\mathrm{d}y}{\mathrm{d}x}+y^2\right)\cos x=\dfrac{1}{xy^2}$, where $0<x<\dfrac{\pi}{2}$ and $y\neq 0$.

(a) Using the substitution $u=xy^2$, show that the differential equation can be reduced to $\dfrac{\mathrm{d}u}{\mathrm{d}x}=\dfrac{\sec x}{u}$. [3]

(b) Hence find the general solution to the differential equation. [3]

	Suggested Solutions
(a)	$u=xy^2$ $\dfrac{\mathrm{d}u}{\mathrm{d}x}=y^2+x\left(2y\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)$ $\cos x\left(y^2+2xy\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)=\dfrac{1}{xy^2}$ $\cos x\left(\dfrac{\mathrm{d}u}{\mathrm{d}x}\right)=\dfrac{1}{u}$ $\dfrac{\mathrm{d}u}{\mathrm{d}x}=\dfrac{\sec x}{u}$
(b)	$\int u\ \mathrm{d}u=\int \sec x\ \mathrm{d}x$ $\dfrac{1}{2}u^2=\ln\left(\sec x+\tan x\right)+C$ $\left(\because \sec x,\tan x>0 \text{ for } 0<x<\dfrac{\pi}{2}\right)$ $x^2y^4=2\ln\left(\sec x+\tan x\right)+C$

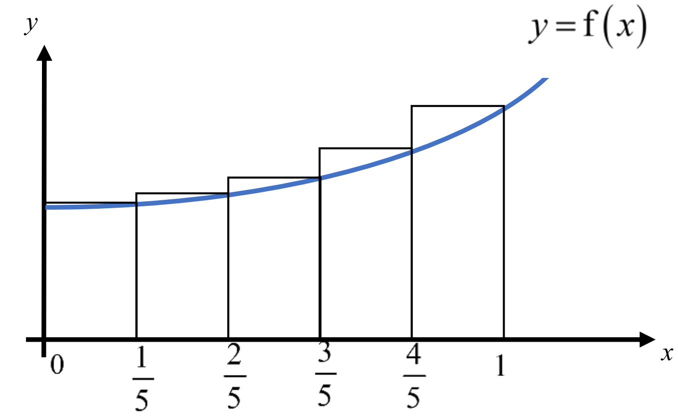
4 The diagram shows a sketch of the function $f(x)=e^{3x}+1$, for $0 \le x \le 1$. The region bounded by the curve and the lines $y=0$, $x=0$ and $x=1$ is A. The region A is split into 5 vertical strips of equal width h, as shown in the diagram.



(a) State the value of h and using a suitable sketch, explain whether $\sum_{k=1}^{5}\left(h\,f(kh)\right)$ is less or more than the area of A. [3]

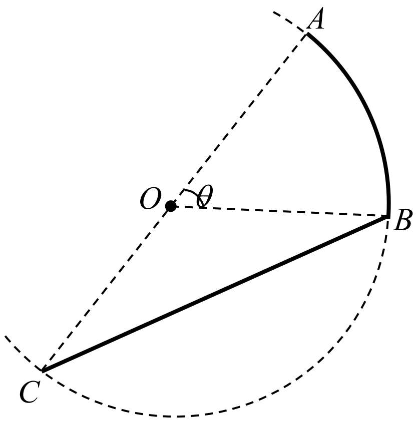
(b) A is now split into n vertical strips of equal width. Using calculus, find the exact value of

$$\lim_{n\to\infty}\frac{1}{n}\left(e^{\frac{3}{n}}+e^{\frac{6}{n}}+e^{\frac{9}{n}}+\ldots+e^{\frac{3n-3}{n}}+e^{3}+n\right).$$ [3]

	Suggested Solutions
(a)	$h=\frac{1}{5}$  $\sum_{k=1}^{5}\left[hf(kh)\right]$ $=\frac{1}{5}\left[f\left(\frac{1}{5}\right)+f\left(\frac{2}{5}\right)+f\left(\frac{3}{5}\right)+f\left(\frac{4}{5}\right)+f\left(\frac{5}{5}\right)\right]$ = Area of 5 rectangles > Area of A

(b)	$\lim\limits_{n\to\infty}\dfrac{1}{n}\left(e^{\frac{3}{n}}+e^{\frac{6}{n}}+e^{\frac{9}{n}}+\ldots+e^{\frac{3n-3}{n}}+e^{3}+n\right)$ $=\lim\limits_{n\to\infty}\dfrac{1}{n}\left[\left(e^{\frac{3}{n}}+1\right)+\left(e^{\frac{6}{n}}+1\right)+\left(e^{\frac{9}{n}}+1\right)+\ldots+\left(e^{\frac{3n-3}{n}}+1\right)+\left(e^{3}+1\right)\right]$ $=\int_0^1 e^{3x}+1\,dx$ $=\left[\dfrac{e^{3x}}{3}+x\right]_0^1$ $=\dfrac{e^3}{3}+1-\dfrac{e^0}{3}-0$ $=\dfrac{e^3}{3}+\dfrac{2}{3}$

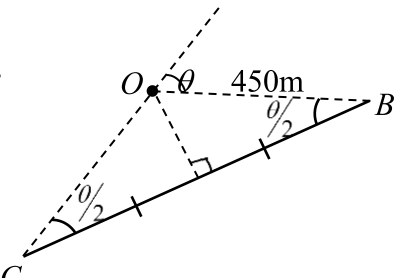
5



The tourism board plans to construct a tram track around a tourist attraction. The diagram above shows part of the blueprint of the track, where the tram will run along a circular path with centre O and radius 450 m, from a fixed point A to a variable point B, and then straight across to a fixed point C. The angle AOB is denoted as θ, where $0 \le \theta \le \pi$. The speed of the tram along the circular path will be maintained at 8 m/s and the speed along the straight path BC will be maintained at $\sqrt{6}$ m/s.

(a) Show that the time taken for the tram to travel from point A to C is $\frac{225}{4}\theta + 150\sqrt{6}\cos\frac{\theta}{2}$. [3]

(b) Use calculus to find the maximum time taken for the tram to travel from point A to point C.
(You need not show that your answer gives a maximum.) [3]

	Suggested Solutions
(a)	Arc AB $= r\theta = 450\theta$ Time taken along arc AB $= \frac{450\theta}{8} = \frac{225\theta}{4}$  $\angle OBC = \frac{1}{2}\theta$ ($\angle$ at centre = 2 $\angle$ at circumference / ext. $\angle$ of a Δ = 2 int. $\angle$ in Δ) $BC = = 2\left(450\cos\frac{\theta}{2}\right) = 900\cos\frac{\theta}{2}$

	Time taken along BC $=\dfrac{900\cos\frac{\theta}{2}}{\sqrt{6}}$ $=150\sqrt{6}\cos\dfrac{\theta}{2}$ Total time $=\dfrac{225}{4}\theta+150\sqrt{6}\cos\dfrac{\theta}{2}$ (Shown) <u>Method 2 (To find BC)</u> $BC^2=450^2+450^2-2(450)(450)\cos(\pi-\theta)]$ $=2(450^2)-2(450^2)(-\cos\theta)$ $=2(450^2)(1+\cos\theta)$ $=2(450^2)(2\cos^2\dfrac{\theta}{2})$ $=900^2\cos^2\dfrac{\theta}{2}$ $BC=900\cos\dfrac{\theta}{2}$
(b)	Let $T=\dfrac{225}{4}\theta+150\sqrt{6}\cos\dfrac{\theta}{2}$. $\dfrac{\mathrm{d}T}{\mathrm{d}\theta}=\dfrac{225}{4}-75\sqrt{6}\sin\dfrac{\theta}{2}$ For longest time, $\dfrac{\mathrm{d}T}{\mathrm{d}\theta}=\dfrac{225}{4}-75\sqrt{6}\sin\dfrac{\theta}{2}=0$ $\sin\dfrac{\theta}{2}=\dfrac{3}{4\sqrt{6}}$ $\theta=0.622368$ $\therefore T=\dfrac{225}{4}\left(0.622368\right)+150\sqrt{6}\cos\dfrac{0.622368}{2}$ $=384.784942$ seconds $=6.41308$ minutes ≈ 6.41 minutes (to 3.f.)

6 Given that b is a real constant such that $0 < b < 4$, describe fully a sequence of transformations that transforms the curve $y = x^2$ to the curve $y = 4x^2 + bx + 1$. [4]

Sketch the curve $y = \dfrac{1}{4x^2 + bx + 1}$. Give the equation of any asymptotes and the coordinates of any axial intercepts and turning points, in terms of b where appropriate. [3]

Suggested Solutions

$$y = 4x^2 + bx + 1$$
$$= 4\left(x^2 + \frac{b}{4}x + \frac{1}{4}\right)$$
$$= 4\left[\left(x + \frac{b}{8}\right)^2 + \frac{1}{4} - \frac{b^2}{64}\right]$$
$$= 4\left(x + \frac{b}{8}\right)^2 + 1 - \frac{b^2}{16}$$

Method 1

(1) Translate the graph $\frac{b}{8}$ units in the negative x-direction.

(2) Scale the graph parallel to the y-axis by a scale factor of 4.

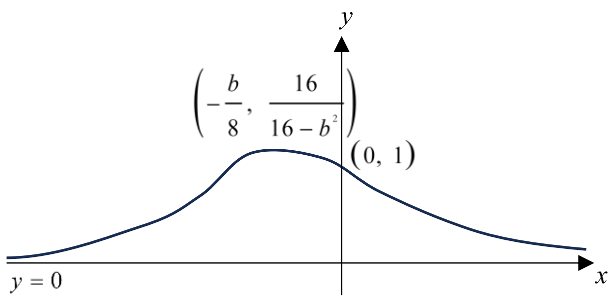
(3) Translate the graph $1 - \frac{b^2}{16}$ units in the positive y-direction

Method 2

(I) Translate the graph $\frac{b}{4}$ units in the negative x-direction.

(II) Scale the graph parallel to the x-axis by a scale factor of $\frac{1}{2}$.

(III) Translate the graph $1 - \frac{b^2}{16}$ units in the positive y-direction



7 **(a)** Use the formula listed in the List of Formulae (MF27) to explain why $\sum_{r=0}^{\infty} \frac{1}{r!} = \mathrm{e}$. [1]

(b) Show that $\sum_{r=0}^{\infty} \frac{r+1}{r!} = 2\mathrm{e}$. [3]

(c) Find the value, in terms of e, of $\sum_{r=6}^{\infty} \frac{r+1}{r!}$. [3]

	Suggested Solutions
(a)	Given in MF27 $\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots$ $= \sum_{r=0}^{\infty} \frac{x^r}{r!}$ Letting $x = 1$, $\mathrm{e}^1 = \sum_{r=0}^{\infty} \frac{1^r}{r!}$ $\Rightarrow \sum_{r=0}^{\infty} \frac{1}{r!} = \mathrm{e}$
(b)	$\sum_{r=0}^{\infty} \frac{r+1}{r!} = \sum_{r=0}^{\infty} \frac{r}{r!} + \sum_{r=0}^{\infty} \frac{1}{r!}$ $= \sum_{r=1}^{\infty} \frac{r}{r!} + \mathrm{e}$ $= \sum_{r=1}^{\infty} \frac{1}{(r-1)!} + \mathrm{e}$ $= \sum_{r=0}^{\infty} \frac{1}{r!} + \mathrm{e}$ $= \mathrm{e} + \mathrm{e} = 2\mathrm{e}$
(c)	$\sum_{r=6}^{\infty} \frac{r+1}{r!} = \sum_{r=0}^{\infty} \frac{r+1}{r!} - \sum_{r=0}^{5} \frac{r+1}{r!}$ $= 2\mathrm{e} - \frac{217}{40}$

8 The function f is defined by

$$\mathrm{f}: x \mapsto \frac{ax-3}{5x-4}, \quad \text{for } x \in \mathbb{R},\ x \neq \frac{4}{5}, \text{ and } a \in \mathbb{R}.$$

(a) Find the range of f in terms of a. [1]

The function f is such that $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ for all x in the domain of f.

(b) Find the value of a. [2]

(c) Hence find $\mathrm{f}^{2025}(2)$. [2]

Another function g is defined by

$$\mathrm{g}: x \mapsto \left(x - \frac{1}{5}\right)^2 + \frac{2}{5}, \text{ for } x \in \mathbb{R},\ 0 < x < \frac{3}{5}.$$

(d) Find the range of fg. [2]

	Suggested Solutions
(a)	Horizontal asymptote: $y = \frac{a}{5}$ $R_\mathrm{f} = \left(-\infty, \frac{a}{5}\right) \cup \left(\frac{a}{5}, \infty\right)$ Or $R_\mathrm{f} = \mathbb{R} \backslash \left\{\frac{a}{5}\right\}$
(b)	Let $y = \frac{ax-3}{5x-4}, \quad x \neq -\frac{4}{5}$ $5xy - 4y = ax - 3$ $5xy - ax = 4y - 3$ $x = \frac{4y-3}{5y-a}$ $\therefore\ \mathrm{f}^{-1}: x \mapsto \frac{4x-3}{5x-a}$ for $x \in \mathbb{R},\ x \neq \frac{a}{5}$ Since $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ for all x in the domain of $\mathrm{f}(x)$, $a = 4$ <u>Method 2</u> $\mathrm{f}(x) = \mathrm{f}^{-1}(x) \Rightarrow D_\mathrm{f} = D_{\mathrm{f}^{-1}} = R_\mathrm{f}$ $D_\mathrm{f} = \left(-\infty, \frac{4}{5}\right) \cup \left(\frac{4}{5}, \infty\right) = \left(-\infty, \frac{a}{5}\right) \cup \left(\frac{a}{5}, \infty\right) = R_\mathrm{f}$ $a = 4$ <u>Method 3</u> $\mathrm{f}^{-1}(x) = \mathrm{f}(x) \Rightarrow \mathrm{ff}^{-1}(x) = \mathrm{ff}(x) \Rightarrow x = \mathrm{ff}(x)$ Sub $x = 0$

$\mathrm{ff}(0)=0$

$\mathrm{f}\left(\frac{3}{4}\right)=0$

$$\frac{\frac{3a}{4}-3}{-\frac{1}{4}}=0$$

$a=4$

(c) $\mathrm{f}^{-1}(x)=\mathrm{f}(x)$

$\Rightarrow \mathrm{ff}^{-1}(x)=\mathrm{ff}(x) \Rightarrow x=\mathrm{f}^2(x)$

$\Rightarrow \mathrm{f}(x)=\mathrm{f}^3(x)$

$$\begin{aligned}\mathrm{f}^{2025}(2)&=\mathrm{f}^2\left(\mathrm{f}^{2023}(2)\right)\\&=\mathrm{f}^2\left(\mathrm{f}^{2021}(2)\right)\\&=\ldots\\&=\mathrm{f}^2\left(\mathrm{f}(2)\right)\\&=\mathrm{f}(2)\\&=\frac{5}{6}\end{aligned}$$

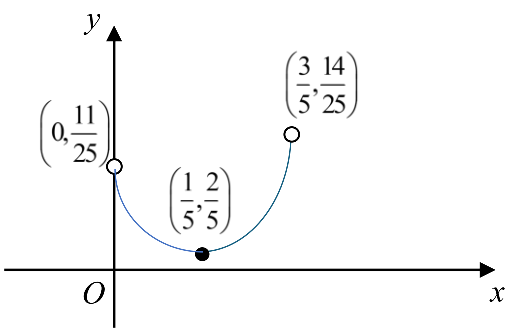
<u>Alternative presentation</u>

$\mathrm{f}^{-1}(x)=\mathrm{f}(x)$

$\Rightarrow \mathrm{f}^2(x)=\mathrm{ff}^{-1}(x)=x$

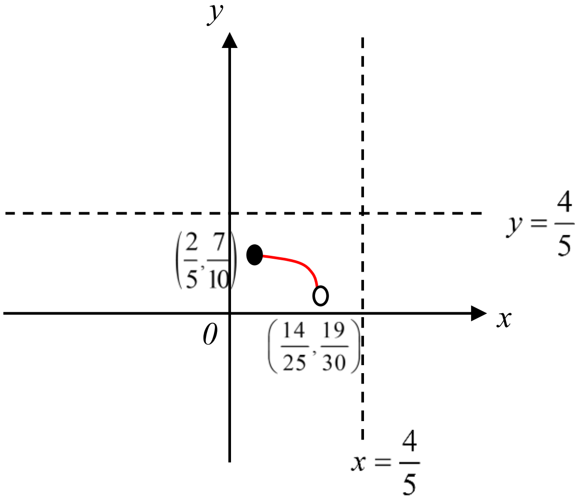
$$\begin{aligned}\mathrm{f}^{2025}(2)&=\underbrace{\mathrm{ff}^{-1}\ldots\mathrm{ff}^{-1}}_{1012\text{ pairs of }\mathrm{ff}^{-1}}\mathrm{f}(2)\\&=\mathrm{f}(2)\\&=\frac{5}{6}\end{aligned}$$

(d) $y=\mathrm{g}(x)$



$R_{\mathrm{g}}=\left[\frac{2}{5},\ \frac{14}{25}\right)$

$\mathrm{f}(x)=\dfrac{4x-3}{5x-4}$

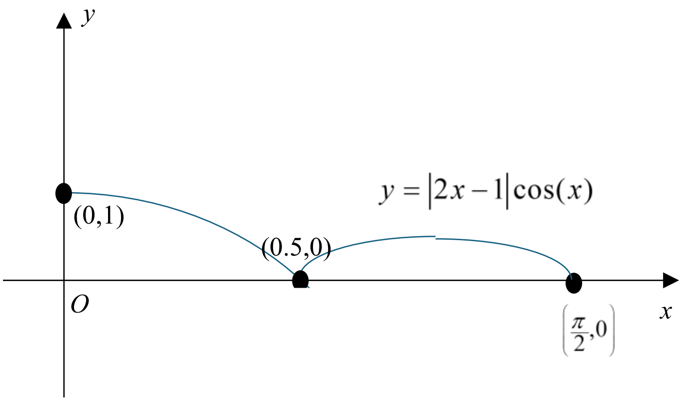


$$\left(0, \frac{3}{5}\right) \xrightarrow{g} \left[\frac{2}{5}, \frac{14}{25}\right) \xrightarrow{f} \left(\frac{19}{30}, \frac{7}{10}\right]$$

$$R_{fg} = \left(\frac{19}{30}, \frac{7}{10}\right]$$

9 **(a)** Find $\int (2x-1)\cos x \, dx$. [3]

(b) Hence find the value of $\int_0^{\frac{\pi}{2}} |2x-1|\cos x \, dx$. Give your answer in the form $A - 4\cos B$, where A and B are exact constants to be determined. [4]

	Suggested Solutions
(a)	Method 1 $\int (2x-1)\cos x \, dx$ $= (2x-1)\sin x - \int 2\sin x \, dx$ $= (2x-1)\sin x + 2\cos x + C$ Method 2 $\int (2x-1)\cos x \, dx$ $= \int 2x\cos x - \cos x \, dx$ $= 2x\sin x + \int 2\sin x \, dx - \sin x \, dx$ $= 2x\sin x - 2\cos x - \sin x \, dx + C$ $= (2x-1)\sin x - 2\cos x + C$
(b)	 $\int_0^{\frac{\pi}{2}} \|(2x-1)\|\cos x \, dx$ $= -\int_0^{0.5} (2x-1)\cos x \, dx + \int_{0.5}^{\frac{\pi}{2}} (2x-1)\cos 2x \, dx$ $= -\left[(2x-1)\sin x + 2\cos x\right]_0^{0.5}$ $+ \left[(2x-1)\sin x + 2\cos x\right]_{0.5}^{\frac{\pi}{2}}$

	$= -\left[2\cos(0.5) - 2\right] + \left[(\pi - 1)(1) + 2(0) - 2\cos(0.5)\right]$ $= \pi + 1 - 4\cos(0.5)$ $A = \pi + 1, \;\; B = 0.5$

10 **(a)** Express $\dfrac{1}{Q(500-Q)}$ in partial fractions. [2]

(b) A group of scientists is monitoring the population of a particular species of birds on an island. At time t years after the start of the monitoring, the number of birds on the island is Q. The scientists observe that the birth rate is proportional to the bird's population and the birds are dying at a rate proportional to the square of the bird's population. There were 1000 birds on the island when the scientists first started monitoring the population. They discover that the population remains unchanged when there are 500 birds.

(i) Show that the differential equation relating Q and t is given by

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = kQ(500-Q),$$

where k is a positive constant. [2]

(ii) Hence, solve the differential equation, expressing Q in terms of k and t. [5]

	Suggested Solutions
10(a)	$\dfrac{1}{Q(500-Q)} = \dfrac{A}{Q} + \dfrac{B}{500-Q} = \dfrac{(500-Q)+BQ}{Q(500-Q)}$ $\Rightarrow 1 = A(500-Q)+BQ$ For $Q=0$, $1=500A \Rightarrow A=\dfrac{1}{500}$. For $Q=500$, $1=500B \Rightarrow B=\dfrac{1}{500}$. $\therefore \dfrac{1}{Q(500-Q)} = \dfrac{1}{500Q} + \dfrac{1}{500(500-Q)}$
10 (b)(i)	Increase rate α Q $\Rightarrow$ Increase rate $= aQ$ (a is the constant of proportionality for increase rate) Decrease rate α Q^2 $\Rightarrow$ Increase rate $= bQ^2$ (b is the constant of proportionality for decrease rate)

	$\therefore \dfrac{\mathrm{d}Q}{\mathrm{d}t} = aQ - bQ^2$ (Increase rate – Decrease rate) $\dfrac{\mathrm{d}Q}{\mathrm{d}t} = 0$ when $Q = 500$ $\Rightarrow 500a - 250000b = 0$ $\Rightarrow a = 500b$ $\therefore \dfrac{\mathrm{d}Q}{\mathrm{d}t} = 500bQ - bQ^2$ $\Rightarrow \dfrac{\mathrm{d}Q}{\mathrm{d}t} = kQ\left(500 - Q\right)$ where $k = b$
10 (b)(ii)	$\displaystyle\int \frac{1}{Q\left(500-Q\right)}\,\mathrm{d}Q = \int k\ \mathrm{d}t$ $\displaystyle\frac{1}{500}\int \frac{1}{Q} + \frac{1}{500-Q}\,\mathrm{d}Q = \int k\ \mathrm{d}t$ (from part (a)) $\dfrac{1}{500}\left[\ln Q - \ln\left\|500-Q\right\|\right] = kt + C$ $\ln\left\|\dfrac{Q}{500-Q}\right\| = 500kt + 500C$ **Resolving the modulus**: **<u>Method 1 (using arbitrary constant):</u>** $\ln\left\|\dfrac{Q}{500-Q}\right\| = 500kt + 500C$ $\Rightarrow \left\|\dfrac{Q}{500-Q}\right\| = \mathrm{e}^{500kt+500C}$ $\Rightarrow \dfrac{Q}{500-Q} = \pm\mathrm{e}^{500C}\ \mathrm{e}^{500kt}$ $\Rightarrow \dfrac{Q}{500-Q} = H\mathrm{e}^{500kt}$, where $H = \pm\mathrm{e}^{500C}$ is an arbitrary constant Therefore, we have $Q = 500H\mathrm{e}^{500kt} - QH\mathrm{e}^{500kt}$ $Q = \dfrac{500H\mathrm{e}^{500kt}}{1+H\mathrm{e}^{500kt}}$ When $t = 0$, $Q = 1000$

$$\frac{1000}{500-1000} = H \Rightarrow H = -2$$

$$\therefore Q = \frac{-1000e^{500kt}}{1-2e^{500kt}}$$
$$\Rightarrow Q = \frac{1000e^{500kt}}{2e^{500kt}-1}$$

Method 2 (using the initial condition):

When $t = 0$, $Q = 1000$, and Q stabilized at 500 given in the question.

Therefore $Q \geq 500 \Rightarrow Q - 500 \geq 0$
$$\Rightarrow \ln\left|\frac{Q}{500-Q}\right| = \ln\left(\frac{Q}{Q-500}\right)$$

Hence we have

$$\ln\left(\frac{Q}{Q-500}\right) = 500kt + 500C = 500kt + D$$

When $t = 0$, $Q = 1000$

$$\ln\left(\frac{1000}{1000-500}\right) = 0 + D$$
$$\Rightarrow D = \ln 2$$

$$\ln\left(\frac{Q}{Q-500}\right) = 500kt + \ln 2$$
$$\frac{Q}{Q-500} = e^{500kt+\ln 2}$$
$$\frac{Q}{Q-500} = e^{\ln 2}e^{500kt} = 2e^{500kt}$$
$$Q = 2e^{500kt}\left(Q-500\right)$$
$$Q - 2e^{500kt}Q = -1000e^{500kt}$$
$$Q = \frac{-1000e^{500kt}}{1-2e^{500kt}}$$
$$Q = \frac{1000e^{500kt}}{2e^{500kt}-1}$$

11 The points A and B have coordinates $(15,3,0)$ and $(5,9,5)$ respectively. Two lines l_1 and l_2, which are perpendicular to each other, have the following equations.

$$l_1 : \mathbf{r} = (15-\lambda)\mathbf{i} + (3+2\lambda)\mathbf{j} + 4\lambda\mathbf{k},$$

$$l_2 : \mathbf{r} = (5+8\mu)\mathbf{i} + (9-2\mu)\mathbf{j} + (5+m\mu)\mathbf{k},$$

where λ and μ are parameters and m is a constant.

(a) Find the value of m. [2]

(b) Given that l_1 and l_2 intersect, find the coordinates of the point of intersection E. [2]

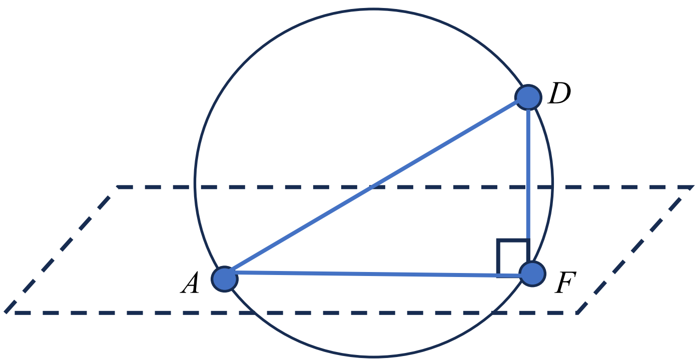
(c) Find a cartesian equation of the plane Π which contains the points A, B and E. [3]

(d) The point D has coordinates $(-1,-3,2)$. Find the position vector of the point F, the foot of perpendicular of D to Π. [3]

(e) Find the exact area of the circle that passes through A, D and F. [2]

	Suggested Solutions
(a)	$l_1 : \mathbf{r} = \begin{pmatrix}15\\3\\0\end{pmatrix} + \lambda\begin{pmatrix}-1\\2\\4\end{pmatrix}, \lambda \in \mathbb{R}$ $l_2 : \mathbf{r} = \begin{pmatrix}5\\9\\5\end{pmatrix} + \mu\begin{pmatrix}8\\-2\\m\end{pmatrix}, \mu \in \mathbb{R}$ For l_1 and l_2 to be perpendicular, $\begin{pmatrix}-1\\2\\4\end{pmatrix} \bullet \begin{pmatrix}8\\-2\\m\end{pmatrix} = 0 \Rightarrow m = 3$
(b)	To find the position vector of E, $15-\lambda = 5+8\mu \qquad \lambda+8\mu = 10$ $3+2\lambda = 9-2\mu \Rightarrow 2\lambda+2\mu = 6$ $4\lambda = 5+3\mu \qquad 4\lambda-3\mu = 5$ Solving, $\lambda = 2, \mu = 1$ $\therefore \overrightarrow{OE} = \begin{pmatrix}15-2\\3+4\\8\end{pmatrix} = \begin{pmatrix}13\\7\\8\end{pmatrix}$. Coordinates of E is $(13,7,8)$.

(c)	$\overrightarrow{AB}=\begin{pmatrix}5\\9\\5\end{pmatrix}-\begin{pmatrix}15\\3\\0\end{pmatrix}=\begin{pmatrix}-10\\6\\5\end{pmatrix}$ $\overrightarrow{AE}=\begin{pmatrix}13\\7\\8\end{pmatrix}-\begin{pmatrix}15\\3\\0\end{pmatrix}=\begin{pmatrix}-2\\4\\8\end{pmatrix}$ A normal vector to $\prod$ $=\begin{pmatrix}-10\\6\\5\end{pmatrix}\times\begin{pmatrix}-2\\4\\8\end{pmatrix}=\begin{pmatrix}28\\70\\-28\end{pmatrix}=14\begin{pmatrix}2\\5\\-2\end{pmatrix}$ A cartesian equation of $\prod$ is $\Pi:\mathbf{r}\bullet\begin{pmatrix}2\\5\\-2\end{pmatrix}=\begin{pmatrix}15\\3\\0\end{pmatrix}\bullet\begin{pmatrix}2\\5\\-2\end{pmatrix}=30+15=45$ The cartesian equation of $\prod$ is $2x+5y-2z=45$.
(d)	Method 1: Let *F* be the foot of perpendicular from *D* to $\prod$. Since *F* lies on a line perpendicular to $\prod$ and contains *D*, $\overrightarrow{OF}=\begin{pmatrix}-1+2t\\-3+5t\\2-2t\end{pmatrix}$ for some value of *t*. Since *F* lies on $\prod$, $2(-1+2t)+5(-3+5t)-2(2-2t)=45\Rightarrow t=2$ Hence $\overrightarrow{OF}=\begin{pmatrix}3\\7\\-2\end{pmatrix}$. Method 2: Projection method (More difficult) Let *F* be the foot of perpendicular from *D* to $\prod$ with a normal vector $\mathbf{n}=\begin{pmatrix}2\\5\\-2\end{pmatrix}$.

	$\overrightarrow{DF} = \left(\dfrac{\overrightarrow{DA}\bullet\mathbf{n}}{\lvert\mathbf{n}\rvert}\right)\dfrac{\mathbf{n}}{\lvert\mathbf{n}\rvert}$ $= \dfrac{1}{33}\left[\begin{pmatrix}16\\6\\-2\end{pmatrix}\bullet\begin{pmatrix}2\\5\\-2\end{pmatrix}\right]\begin{pmatrix}2\\5\\-2\end{pmatrix}$ $= \dfrac{66}{33}\begin{pmatrix}2\\5\\-2\end{pmatrix}$ $= \begin{pmatrix}4\\10\\-4\end{pmatrix}$ $\overrightarrow{OF} = \overrightarrow{OD} + \overrightarrow{DF} = \begin{pmatrix}-1\\-3\\2\end{pmatrix} + \begin{pmatrix}4\\10\\-4\end{pmatrix} = \begin{pmatrix}3\\7\\-2\end{pmatrix}$
(e)	 Since $\measuredangle DFA = 90^o$, the circle has AD as its diameter. Radius of the circle $= \dfrac{\lvert\overrightarrow{AD}\rvert}{2} = \dfrac{1}{2}\left\lvert\begin{pmatrix}-16\\-6\\2\end{pmatrix}\right\rvert = \sqrt{74}$ Area of circle $= \pi\left(\sqrt{74}\right)^2 = 74\pi$ units^2

12 **Do not use a graphing calculator for this question.**

It is given that $\mathrm{f}(z)=z^4-6z^2+k$, where k is a non-zero constant.

(a) If k is a purely imaginary number, determine, with justification, whether $\mathrm{f}(z)=0$ can have real roots. [1]

(b) Show that $\mathrm{f}(-z)=\mathrm{f}(z)$. [1]

(c) Given that $2+\mathrm{i}$ is a root of the equation $\mathrm{f}(z)=0$, determine k. Hence, or otherwise, find the remaining roots, showing your workings clearly. [6]

Use the value of k found in part **(c)** for the rest of this question.

(d) Given that the product of all the roots of $\mathrm{f}(z)=0$ is D, find the value of D, showing your workings clearly. [2]

(e) A complex number w_1 satisfies the equation $kw^4-6w^2+1=0$. Given that w_1 can be obtained from $2+\mathrm{i}$, find w_1. [2]

	Suggested Solutions
(a)	Assume that $\mathrm{f}(z)=z^4-6z^2+k=0$ has a real root x. Thus we have $x^4-6x^2=-k$ which is a contradiction as the LHS yields a real number but the RHS is a complex number. Hence $\mathrm{f}(z)=0$ cannot have real roots.
(b)	$\mathrm{f}(-z)=(-z)^4-6(-z)^2+k$ $=z^4-6z^2+k$ $=\mathrm{f}(z)$
(c)	$(2+\mathrm{i})^2=2^2+2(2)(\mathrm{i})+\mathrm{i}^2=3+4\mathrm{i}$ $(2+\mathrm{i})^4=\left[(2+\mathrm{i})^2\right]^2$ $=(3+4\mathrm{i})^2$ $=3^2+2(3)(4\mathrm{i})+(4\mathrm{i})^2$ $=-7+24\mathrm{i}$ $(2+\mathrm{i})^4-6(2+\mathrm{i})^2+k=0$ $-7+24\mathrm{i}-6(3+4\mathrm{i})+k=0$ $k=25$ Since all coefficients of $\mathrm{f}(z)=0$ are real and $2+\mathrm{i}$ is a root, then $2-\mathrm{i}$ is also a root. Since $\mathrm{f}(-z)=\mathrm{f}(z)$, then $-z$ is also a root of $\mathrm{f}(z)=0$. The remaining roots are $2-\mathrm{i}$, $-2-\mathrm{i}$ and $-2+\mathrm{i}$.

(d)	Method 1 $\left(2+\mathrm{i}\right)\left(2-\mathrm{i}\right)=\left\|2+\mathrm{i}\right\|^2=5$ $\left(-2+\mathrm{i}\right)\left(-2-\mathrm{i}\right)=\left\|-2+\mathrm{i}\right\|^2=5$ $\therefore D=5\left(5\right)=25$ Method 2 $\left(2+\mathrm{i}\right)\left(2-\mathrm{i}\right)=2^2-\mathrm{i}^2=5$ $\left(-2+\mathrm{i}\right)\left(-2-\mathrm{i}\right)=\left(-2\right)^2-\mathrm{i}^2=5$ $\therefore D=5\left(5\right)=25$
(e)	$z^4-6z^2+k=0$ $\frac{k}{z^4}-\frac{6}{z^2}+1=0$ Replace $\frac{1}{z}$ with w i.e. $w=\frac{1}{z}$ $w_1=\frac{1}{2+\mathrm{i}}$ $=\frac{2-\mathrm{i}}{2-\mathrm{i}^2}$ $=\frac{2}{5}-\frac{1}{5}\mathrm{i}$ Alternative $25w^4-6w^2+1=0$ $25^2w^4-6(25)w^2+25=0$ $(5w)^4-6(5w)^2+25=0$ $5w=2+\mathrm{i}$ $w=\frac{2}{5}+\frac{1}{5}\mathrm{i}$

13 Frederick, a social media influencer, is starting a new online account at the start of a month. From the second month onwards:

- His organic followers at the end of each month will be a times the total followers he had at the end of the previous month, where a is a positive constant.
- A company he engaged in will provide him with 1000 additional followers in the middle of each month.

Let $F(n)$ denote the total number of organic and additional followers Frederick had at the end of n months after he started his new online account, for $n \in \mathbb{Z}^+$.

(a) Write down a recurrence relationship between $F(n)$ and $F(n+1)$ for $n \in \mathbb{Z}^+$, giving your answer in terms of a. [2]

It is found that Frederick has 3500 followers at the end of the first month.

(b) Show that $F(3) = 3500a^2 + 1000a + 1000$. [2]

(c) Find an expression for $F(n)$. Hence, find the number of months required for Frederick's followers to exceed one million if $a = 1.5$. [4]

(d) Given that the number of followers exceeds 20000 by the end of the first year, determine the range of values of a. [2]

Frederick finds out that, instead of the additional 1000 followers in the middle of every month, the company can only provide him with additional b followers in the middle of every month.

(e) Given that the number of followers remains constant since the end of the first month, find the relationship between a and b. [2]

	Suggested Solutions
(a)	$F(n+1) = aF(n) + 1000$
(b)	$F(1) = 3500$ $F(3) = aF(2) + 1000$ $\quad = a\left(aF(1) + 1000\right) + 1000$ $\quad = 3500a^2 + 1000a + 1000$
(c)	$F(4) = a(3500a^2 + 1000a + 1000) + 1000$ $\quad = 3500a^3 + 1000a^2 + 1000a + 1000$ $F(5) = a(3500a^3 + 1000a^2 + 1000a + 1000) + 1000$ $\quad = 3500a^4 + 1000a^3 + 1000a^2 + 1000a + 1000$ $\vdots$

$$F(n) = 3500a^{n-1} + 1000a^{n-2} + 1000a^{n-1} + \ldots + 1000a + 1000$$

$$= 3500a^{n-1} + \frac{1000\left(1-a^{n-1}\right)}{1-a}$$

Given that $a = 1.5$, for $F(n) > 1000000$, i.e.

$$3500(1.5)^{n-1} - 2000\left(1-(1.5)^{n-1}\right) > 1000000$$

Using GC,

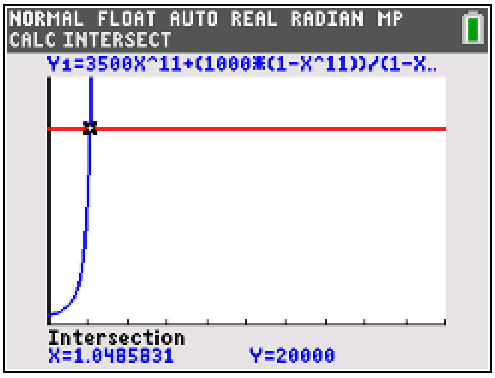
n	$3500(1.5)^{n-1} - 2000\left(1-(1.5)^{n-1}\right)$
13	711604
14	1068407
15	1603610

Frederick will need 14 months for his followers to exceed one million.

(d)

$$F(12) = 3500a^{11} + \frac{1000\left(1-a^{11}\right)}{1-a} > 20000$$

Using GC,



$a > 1.04858$

$a > 1.05$ (3 s.f.)

(e) The recurrence relation is now $F(n+1) = aF(n) + b$

Since the number of followers remains constant after the end of the first month, we have

$F(2) = F(1)$

$\Rightarrow aF(1) + b = F(1)$

$$\Rightarrow F(1) = \frac{b}{1-a}$$

	$\Rightarrow \frac{b}{1-a} = 3500$ $\Rightarrow 1-a = \frac{b}{3500}$ $\Rightarrow a = 1 - \frac{b}{3500}$