Hwa Chong Institution 华侨中学
Mathematics (SMTP)

Name: Wan Rm Huo (27) Class: 3L1 Date: 17th July

MA303.1.2 – Perpendicular Lines

At the end of this unit, you should be able to

- **obtain gradient of perpendicular lines**
- **find the equation of perpendicular bisector of two points**

E1. Using Pythagoras' Theorem, show that the three points $A(2,-1)$, $B(5,4)$ and $C(15,-2)$ forms a right-angled triangle. Name the vertex with the right angle.

Length of AB : $\sqrt{(-3)^2 + (-5)^2}$ units
Length of AC : $\sqrt{(-3)^2 + (-15)^2}$ units
Length of BC : $\sqrt{(-10)^2 + (-6)^2}$ units.

Since $(AB)^2 + (BC)^2 = (AC)^2$,
ABC forms a right-angle triangle

(a) Find the equations of the lines AB and BC.

Gradient of AB $= \frac{4-(-1)}{5-2}$
$= \frac{5}{3}$

$y+1 = \frac{5}{3}(x-2)$
$y = \frac{5}{3}x - \frac{13}{3}$

Gradient of BC $= \frac{-2-4}{15-5}$
$= -\frac{5}{3}$

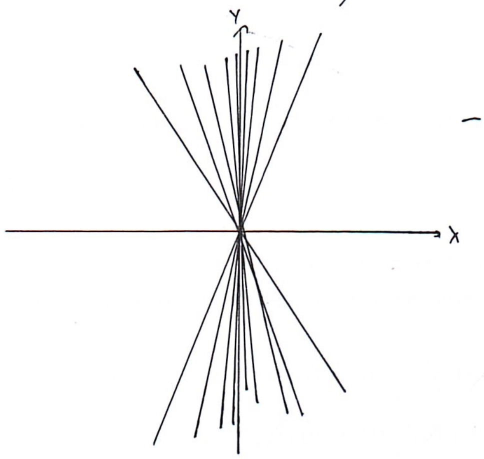
$y-5 = -\frac{5}{3}(x-4)$
$y = -\frac{5}{3}x + \frac{35}{3}$

(b) If m_1 is the gradient of the line AB and m_2 is the gradient of the line BC, show that $m_1 m_2 = -1$.

$\frac{5}{3} \cdot (-\frac{5}{3}) = -1$

E2. If $m_1 m_2 = -1$, where m_1 and m_2 are the gradients of two lines, then the two lines are perpendicular. Is the converse true? If not, state a condition whereby the converse will always be true.

No. The line must not be perpendicular.



$-\infty$ or ∞ (undefined)

P1. Show that the lines formed by AB and BC where $A = (t, 1)$ and $B = (3, 4)$ and $C = (6, t+1)$ are perpendicular for all values of t.

Let gradient of AB be M_{AB}.
Let gradient of BC be M_{BC}.

$$M_{AB} = \frac{4-1}{3-t}, \quad t \neq 3$$

$$M_{BC} = \frac{(t+1)-4}{6-3} = \frac{t-3}{3}$$

$$M_{AB} M_{BC} = \frac{3}{3-t} \cdot \frac{t-3}{3} = \frac{t-3}{3-t} = -1 \text{ (shown)}$$

t cannot be 3 as when $t=3$,

A: (3, 1)
B: (3, 4)
C: (6, 4)

∴ AB is vertical, BC is horizontal

E3. Let $A(-2, 3)$ and $B(3, -4)$ be two points on the Cartesian plane.

(a) Find the coordinates of the midpoint C.

Mid point $= \left(\frac{-2+3}{2}, \frac{3-4}{2}\right)$

$= \left(\frac{1}{2}, \frac{1}{2}\right)$

(b) Find the gradient of AB, and hence deduce the gradient of a perpendicular line to AB.

Gradient of $AB = \frac{-4-3}{3-(-2)}$

$= -\frac{7}{5}$

Gradient of perpendicular line to $AB = -1 \div -\frac{7}{5}$

$= \frac{5}{7}$

(c) Find the equation of the perpendicular line to AB, passing through the point C. This line is called the **perpendicular bisector** to the line AB.

Equation of line: $y = \frac{5}{7}x + c$

Sub $\left(\frac{1}{2}, \frac{1}{2}\right)$ into equation

$\frac{1}{2} = \frac{5}{7} \cdot \frac{1}{2} + c$

$c = \frac{1}{7}$

Equation of line: $y = \frac{5}{7}x + \frac{1}{7}$

To find the perpendicular bisector of the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$:

- **Find the midpoint C.**
- **Find the gradient of AB.**
- **Deduce the gradient of the perpendicular.**
- **Obtain the equation using the gradient of the perpendicular with the midpoint C.**

P2. Find the equation of the perpendicular bisector to the following pairs of points:

(a) $(-5,-2)$ and $(3,14)$

Mid point : $\left(\frac{-5+3}{2}, \frac{-2+14}{2}\right)$

$= (-1, 6)$

Gradient of bisector $= -\frac{3-(-5)}{14-(-2)}$

$= -\frac{1}{2}$

$\therefore$ Equation of bisector :

$y-6 = -\frac{1}{2}(x-(-1))$

$y = -\frac{1}{2}x + \frac{11}{2}$

(b) (p,q) and $(3p,q)$

Midpoint $= \left(\frac{q+q}{2}, \frac{3p+p}{2}\right)$

$= (q, 2p)$

Gradient of bisector $= -\frac{3p-p}{q-q}$

$= -\frac{2p}{0}$

Equation of bisector : $y = 2p$

P3. Given the equation of a line $3y+2x-5=0$.

(a) Show that $A(1,1)$ and $B(4,-1)$ lie on the line.

Sub $(1,1)$ into $3y+2x-5=0$

$3+2-5=0$ (shown)

$\therefore$ A lies on the equation.

Sub $(4,-1)$ into $3y+2x-5=0$

$-3+13-5=0$ (shown)

$\therefore$ B lies on the equation.

(b) The perpendicular bisector of AB cuts another line whose equation is given by $2y+5x+3=0$ at C. Find the coordinates of C.

$3y+2x-5=0$

$3y=-2x+5$

$y=-\frac{2}{3}x+\frac{5}{3}$

Gradient of perpendicular : $-1 \div \left(-\frac{2}{3}\right)$

$= \frac{3}{2}$

Midpoint : $\left(\frac{1+4}{2}, \frac{1+(-1)}{2}\right)$

$= \left(\frac{5}{2}, 0\right)$

Equation of perpendicular :

$y-0 = \frac{3}{2}\left(x-\frac{5}{2}\right)$

$y = \frac{3}{2}x - \frac{15}{4}$

Finding C,

$2y+5x+3=0$ — ①

$y = \frac{3}{2}x - \frac{15}{4}$ — ②

Sub ② into ①

$2\left(\frac{3}{2}x - \frac{15}{4}\right) + 5x + 3 = 0$

$8x = \frac{9}{2}$

$x = \frac{9}{16}$

Sub $x = \frac{9}{16}$ into ②

$y = \frac{3}{2} \times \frac{9}{16} - \frac{15}{4}$

$= -\frac{93}{32}$

C is at $\left(\frac{9}{16}, -\frac{93}{32}\right)$

P4. Given two lines whose equations are $5y = 2x + 8$ and $4y = 10 - px$ where p is a real number.

(a) What is the value of p if the two lines are

i. parallel?

Gradient of 1st line : $\frac{2}{5}$

Gradient of 2nd line : $-\frac{p}{4}$

$\therefore -\frac{p}{4} = \frac{2}{5}$

$5p = -8$

$p = -\frac{8}{5}$

ii. perpendicular?

Gradient of perpendicular of 1st line $= \frac{-1}{\frac{2}{5}}$

$= -\frac{5}{2}$

$\therefore -\frac{p}{4} = -\frac{5}{2}$

$2p = 20$

$p = 10$

(b) Find the distance between the two lines if they are parallel.

$5y = 2x + 8$

$y = \frac{2}{5}x + \frac{8}{5}$ — ①

$4y = 10 - 10x$ — ②

Sub ① into ②

$\frac{8}{5}x + \frac{22}{5} = 10 - 10x$

$\frac{55}{9}x = \frac{18}{5}$

$x = \frac{9}{29}$

Sub $x = \frac{9}{29}$ into ①:

$y = \frac{18}{29} + \frac{8}{5}$

Distance $= \sqrt{\left(\frac{9}{29} - 0\right)^2 + \left(\frac{50}{29} - \frac{5}{2}\right)^2}$

$= 0.836$ units (3s.f.)