2022 RVHS_ACJC_EJC_NJC Prelim Paper 1_Modified

1. It is given that the function $\mathrm{f}(x) = \dfrac{\mathrm{e}^x - 1}{\mathrm{e}^x + 1}$.

(i) By substituting $u = \mathrm{e}^x$ or otherwise, show that $\displaystyle\int_0^1 \mathrm{f}(x)\,\mathrm{d}x = a\ln\left(\frac{\mathrm{e}+1}{2}\right) + b$, where a and b are constants to be determined. [3]

(ii) Use Simpson's rule with 5 ordinates to find an approximate value of $\displaystyle\int_0^1 \mathrm{f}(x)\,\mathrm{d}x$, leaving your answer to 5 decimal places and hence find an approximate value for $\ln\left(\dfrac{\mathrm{e}+1}{2}\right)$. [3]

2. The sequence $\{P_n\}$ is given by $P_0 = 0$, $P_1 = \dfrac{7}{5}$ and $P_{n+1} = \dfrac{1}{3}\left(5P_n + 2P_{n-1}\right) + 3^{n-1}$ for $n \geq 1$. By considering the sequence $\{Q_n\}$, where $P_n = Q_n + \dfrac{1}{10}\left(3^{n+1}\right)$ for $n \geq 0$, find an expression for P_n as a function of n. [7]

3. A curve C has polar equation

$$r = 2 + \cos 5\theta \text{ for } 0 \leq \theta \leq \frac{4\pi}{5}.$$

(i) Find the polar coordinates of the points on C for which the value of r is a maximum or a minimum. [3]

(ii) Sketch C, stating the line of symmetry and indicating clearly the polar coordinates of the points found in part **(i)**. [3]

(iii) Find the area of the region bounded by C, the line $y = x$ and the initial line. [2]

4.

(a) Show by integration that $\displaystyle\int \frac{\sqrt{1+x^2}}{x^4}\,\mathrm{d}x = -\frac{\left(1+x^2\right)^{\frac{3}{2}}}{3x^3} + c$. [2]

(b) **(i)** Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{xy}{1+x^2} = \frac{1}{x^4}$$

for $x > 0$, given that $y = 0$ when $x = 1$. [4]

(ii) Use one step of the Euler method to calculate an approximation to the value of y when $x = 1.5$. [1]

(iii) Show that there is a large discrepancy between the approximation found in part **(ii)** and the actual value of y when $x = 1.5$. Give a reason why the discrepancy occurred. [3]

5. The linear transformation $\mathrm{T}:\mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix

$$\mathbf{M} = \begin{pmatrix} 2 & -3 & 6 & 2 \\ -2 & 3 & -3 & -3 \\ 4 & -6 & 9 & 5 \\ 2 & -3 & 3 & a \end{pmatrix}.$$

The range space of T is denoted by R.

Suppose that $a \neq 3$.

(i) Explain why $\left\{ \begin{pmatrix} 2 \\ -2 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ -3 \\ 9 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 5 \\ a \end{pmatrix} \right\}$ is a basis of R. [4]

(ii) Given that $\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$ is an element of R, show that $w = x + y$. [3]

It is now given that $a = 3$.

(iii) State a basis for the null space of T. Hence, find the set of vectors $\mathbf{x}$ such that

$$\mathbf{Mx} = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix}.$$ [3]

6. The equation $e^{\frac{x}{3}} - 2x + 1 = 0$ has two positive roots. Show that the value of one of the roots lies between 1 and 6 while that of the other root lies between 6 and 9. [2]

An attempt to estimate the larger root α is made by using the Newton-Raphson method applied to the equation $e^{\frac{x}{3}} - 2x + 1 = 0$ with the initial approximation $x_0 = 6$. Calculate the next two approximations, x_1 and x_2, and explain why x_1 is a worse approximation to α than x_0 is. [4]

Two iterative formulae are proposed to find the value of α:

$$x_{n+1} = \frac{e^{\frac{x_n}{3}} + 1}{2} \quad \text{and} \quad x_{n+1} = 3\ln(2x_n - 1).$$

Only one of the formulae is feasible.

For the formula that is not feasible, demonstrate graphically why the iterative process fails. For the formula that is feasible, use it with first iteration $x_1 = 7.5$ to obtain α correct to 3 decimal places. [4]

7. The urban development authority of a metropolitan region is studying the population movements between three cities within the region, namely A, B and C. Every year, it is observed that the populations of these cities move concurrently in the following way:

- 20% of the population of A moves to C;
- 10% of the population of B moves to A while another 20% moves to C;
- 30% of the population of C moves to A while another 10% moves to B.

Let a_n, b_n and c_n denote the population (in millions) of A, B and C respectively at the end of year n.

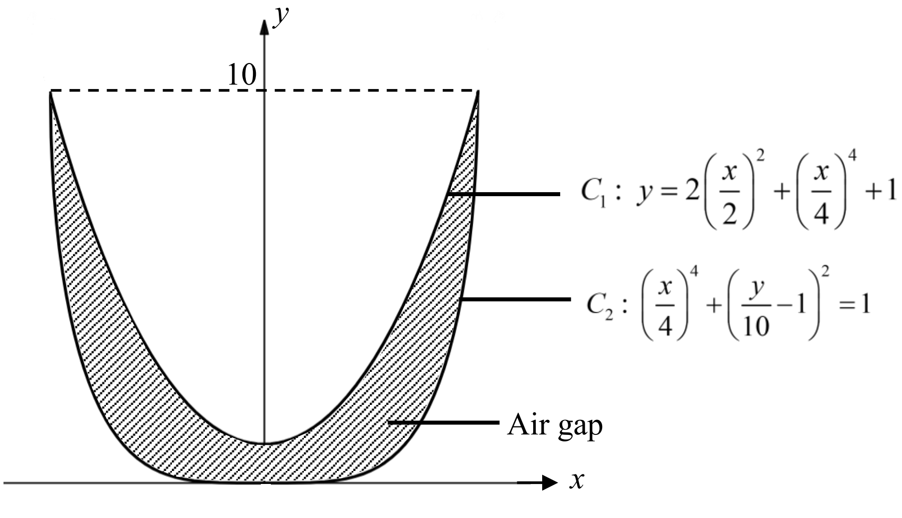
The population movements may be modelled by the recurrence relation $\mathbf{v}_{n+1} = \mathbf{M}\mathbf{v}_n$, where $\mathbf{M} = \begin{pmatrix} \frac{4}{5} & \frac{1}{10} & \frac{3}{10} \\ p & \frac{7}{10} & \frac{1}{10} \\ \frac{1}{5} & \frac{1}{5} & q \end{pmatrix}$ and $\mathbf{v}_n = \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}$. Assume that the total population of the three cities remains constant at any time. Initially, the populations of A, B and C are $7\ 600\ 000$, $4\ 800\ 000$ and $5\ 600\ 000$ respectively.

(i) State the values of p and q. [1]

(ii) With the values of p and q found in part **(i)**, find the eigenvalues and eigenvectors of the matrix **M**. [5]

(iii) Hence by diagonalizing the matrix **M**, determine the steady-state populations of A, B and C in the long run. [4]

8. A company manufactures double-walled glass mugs that contain an air gap between two layers of glass. The diagram below shows the cross-section of a double-walled glass mug with height 10 cm. The inner glass layer is represented by the curve C_1 with equation $y=2\left(\frac{x}{2}\right)^2+\left(\frac{x}{4}\right)^4+1$. The outer glass layer is represented by the curve C_2 with equation $\left(\frac{x}{4}\right)^4+\left(\frac{y}{10}-1\right)^2=1$. The shaded region indicates the air gap between the inner and outer glass layers. The glass mug is formed by rotating the shaded region π radians about y-axis.



Cross section of a double-walled glass mug

(i) Using the substitution $x=4\sqrt{\sin\theta}$ where $0<\theta\le\frac{\pi}{2}$, find $\int x\sqrt{1-\left(\frac{x}{4}\right)^4}\,\mathrm{d}x$. [4]

(ii) Hence, find the exact volume of the air gap. [4]

(iii) Find the maximum volume of liquid that the mug can hold. [2]

(iv) Find the surface area of the inner glass layer. [2]

9. The Solow model of economic growth was developed in 1956 by economist and Nobel Memorial Prize in Economic Sciences winner Robert Solow. A differential equation central to this model is given by

$$\frac{\mathrm{d}k}{\mathrm{d}t} = sq - \delta k \qquad (*)$$

where

- k denotes the amount of capital per unit of labour where $k > 0$,
- t denotes time in years,
- s denotes the savings rate where $0 < s < 1$,
- q denotes the production function per unit of labour and is a function of k, and
- δ denotes the constant depreciation rate where $\delta > 0$.

One common choice for the production function in this model is the Cobb-Douglas production function which gives rise to $q = Ak^{\alpha}$, where α and A are positive constants where $0 < \alpha < 1$ and A represents total factor productivity.

For the rest of this question, take $q = Ak^{\alpha}$ where $0 < \alpha < 1$ and $A > 0$.

Suppose that s is constant.

(i) Find the equilibrium value for the differential equation **(∗)**. [1]

(ii) By using the substitution $u = Ak^{1-\alpha}$, show that the differential equation **(∗)** can be reduced to

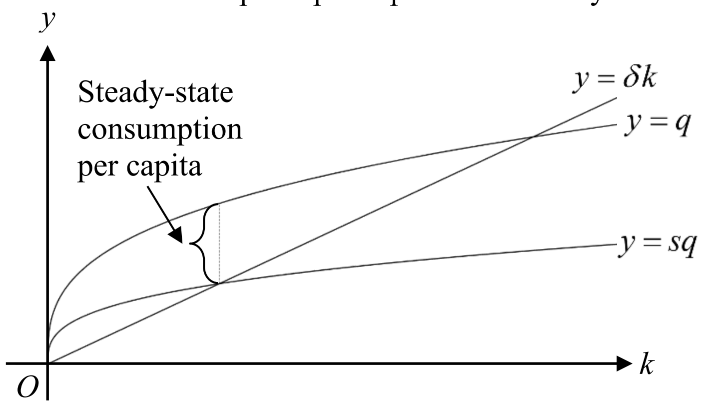
$$\frac{\mathrm{d}u}{\mathrm{d}t} = (1-\alpha)\left(sA^2 - \delta u\right).$$

Hence, given that $k = k_0$ initially where $k_0 > 0$, find k in the form $\left[\frac{sA}{\delta} + h\mathrm{e}^{\delta(\alpha-1)t}\right]^{\frac{1}{1-\alpha}}$, where h is constant in terms of k_0, α, s, A and δ to be determined. [6]

(iii) Hence, or otherwise, determine whether the equilibrium value found from part **(i)** is stable or unstable. [2]

Now, suppose instead that s varies with t.

(iv) In this model, consumption per capita is given by the difference between production per unit of labour, q, and the amount of investment, sq. Also, a steady state is said to have been achieved when $sq = \delta k$. Consider the Solow diagram below which illustrates consumption per capita at the steady state.



9. [Continued]

As consumption per capita is an indicator of the economic well-being of members of a society, it is generally in the interest of a policymaker to determine the savings rate, s, that maximises the steady-state consumption per capita c. The value of s that achieves this, s_G, is known as the Golden Rule savings rate.

Write down an expression for c and find s_G. [4]

10. A logistics company set up an online platform providing delivery services to users on a monthly paid subscription basis. The company's sales manager models the number of subscribers that the company has at the end of each month. She notes that approximately 10% of the existing subscribers leave each month, and that there will be a constant number k of new subscribers in each subsequent month after the first.

Let T_n, $n \geq 1$, denote the number of subscribers the company has at the end of the nth month after the online platform was set up.

(i) Express T_{n+1} in terms of T_n. [1]

The company has 250 subscribers at the end of the 1st month.

(ii) Find T_n in terms of n and k. [4]

(iii) Find the least number of subscribers the company needs to attract in each subsequent month after the first if it aims to have at least 350 subscribers by the end of the 12th month. [3]

Let $k = 50$ for the rest of the question.

The monthly running cost of the company is assumed to be fixed at \$4,000. The monthly subscription fee is \$10 per user which is charged at the end of each month.

(iv) Given that the mth month is the first month in which the company's revenue up to and including that month is able to cover its cost up to and including that month, find the value of m. [3]

(v) Using your answer to part **(ii)**, determine the long-term behaviour of the number of subscribers that the company has. Hence explain whether this behaviour is appropriate in terms of long-term prospects for the company's success. [3]

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