RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 5

Chapter 6A: Differentiation Techniques

SYLLABUS INCLUDES

- Differentiation of simple functions defined implicitly or parametrically.
- Finding the approximate value of a derivative at a given point using a graphing calculator.

CONTENTS

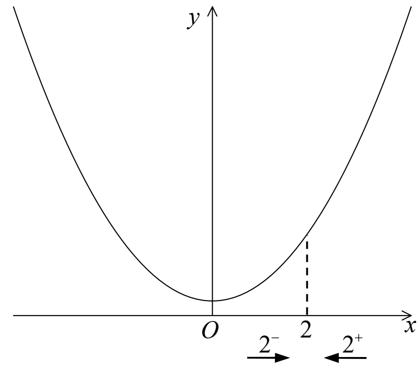
INTRODUCTION

Differentiation deals with the concept of change. It is useful for understanding phenomena in many fields of study, including business, economics and social sciences. This topic reinforces the ideas of derivatives that students learnt in 'O'-Level Additional Mathematics and further develops the idea of derivatives as a limit. In this chapter, students will learn more techniques of differentiation such as implicit differentiation and parametric differentiation. Students will also learn to use the graphing calculator to calculate derivatives.

1 Limits and Derivatives

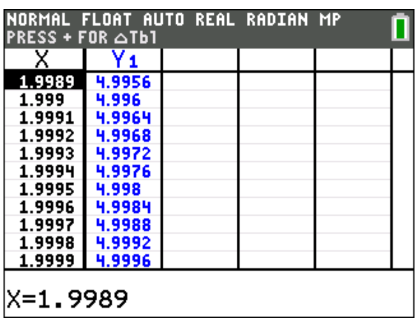
1.1 Limits

Consider the expression x^2+1.



Key in **Y1 = X² + 1** and go to [2nd][tablset] and put **TblStart = 1.9989** or **2.0011** and **ΔTbl = 0.0001 or –0.0001 respectively.**

Let x approach 2 from the left.
We write $x \to 2^-$.



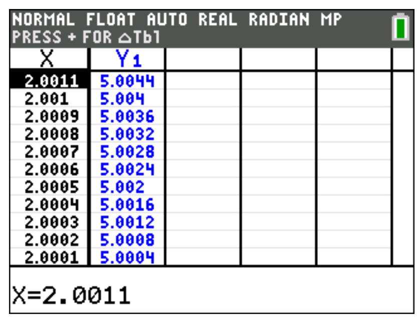
NORMAL FLOAT AUTO REAL RADIAN MP
PRESS + FOR ΔTbl

X	Y1
1.9989	4.9956
1.999	4.996
1.9991	4.9964
1.9992	4.9968
1.9993	4.9972
1.9994	4.9976
1.9995	4.998
1.9996	4.9984
1.9997	4.9988
1.9998	4.9992
1.9999	4.9996

X=1.9989

From the table above, we see that as $x \to 2^-$, x^2+1 approaches 5,
i.e. the limiting value of x^2+1 is 5.
We write $\lim_{x\to 2^-}\left(x^2+1\right)=5$.

Let x approach 2 from the right.
We write $x \to 2^+$.



NORMAL FLOAT AUTO REAL RADIAN MP
PRESS + FOR ΔTbl

X	Y1
2.0011	5.0044
2.001	5.004
2.0009	5.0036
2.0008	5.0032
2.0007	5.0028
2.0006	5.0024
2.0005	5.002
2.0004	5.0016
2.0003	5.0012
2.0002	5.0008
2.0001	5.0004

X=2.0011

From the table above, we see that as $x \to 2^+$, x^2+1 approaches 5,
i.e. the limiting value of x^2+1 is 5.
We write $\lim_{x\to 2^+}\left(x^2+1\right)=5$.

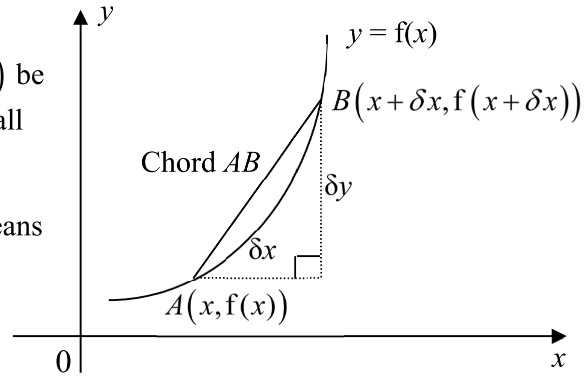
Since $\lim_{x\to 2^-}\left(x^2+1\right)=\lim_{x\to 2^+}\left(x^2+1\right)=5$, the limit of x^2+1 as $x\to 2$ exists and is given by the value 5. We write $\lim_{x\to 2}\left(x^2+1\right)=5$.

$$\boxed{\text{If } \lim_{x\to a^-}\mathrm{f}(x)=\lim_{x\to a^+}\mathrm{f}(x)=k\text{, where } k \text{ is real, then } \lim_{x\to a}\mathrm{f}(x)=k.}$$

(Please refer to Appendix for more information and results on Limits.)

1.2 Derivative as a Limit

Consider a curve with equation $y = \mathrm{f}(x)$.
Let $A\left(x,\ \mathrm{f}\left(x\right)\right)$ and $B\ \left(x+\delta x, \mathrm{f}\left(x+\delta x\right)\right)$ be two points on the curve, where δx is a small increment in x.



Note: A variable quantity, prefix by δ, means a small change in that quantity.

Let δy be the corresponding small increment in y due to δx.

Then $\delta y = \mathrm{f}(x+\delta x) - \mathrm{f}(x)$ and the gradient of the chord AB is given by $\dfrac{\delta y}{\delta x}$.

Notice that the gradient of the curve at A can be approximated by the gradient of the chord AB. The closer B is to A, the better is the approximation.

Thus, $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at $A =$ gradient of the curve at A

$= $ gradient of the tangent to the curve at A

$= \lim\limits_{B \to A}$ (gradient of chord AB)

$= \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$ (since $\delta x \to 0$ *as* $B \to A$)

$= \lim\limits_{\delta x \to 0} \dfrac{\mathrm{f}(x+\delta x) - \mathrm{f}(x)}{\delta x}$.

Note: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}}{\mathrm{d}x}(y)$ **is not a fraction**.

Definition of derivative: If $y = \mathrm{f}(x)$, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \lim\limits_{\delta x \to 0} \dfrac{\mathrm{f}(x+\delta x) - \mathrm{f}(x)}{\delta x}$.

Using the definition of derivative, we can show that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^n\right) = nx^{n-1}$, where $n \in \mathbb{Z}^+$.

(Please refer to Appendix for the proof.)

Derivatives of Elementary Functions

In Secondary School, you learnt the derivatives of the following elementary functions:

y	$\dfrac{\mathrm{d}y}{\mathrm{d}x}$
x^n for **fixed** n	nx^{n-1}
e^x	e^x
$\ln x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$

2 Basic Rules of Differentiation

Let f and g be two functions of x.

We denote $\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{f}(x)\right]$ and $\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{g}(x)\right]$ by $\mathrm{f}'(x)$ and $\mathrm{g}'(x)$.

The following are some basic rules of differentiation:

(1) $\frac{\mathrm{d}}{\mathrm{d}x}\left[a\mathrm{f}(x) \pm b\,\mathrm{g}(x)\right] = a\mathrm{f}'(x) \pm b\mathrm{g}'(x)$, where a and b are constants.

(2) $\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{f}(x)\mathrm{g}(x)\right] = \mathrm{g}(x)\mathrm{f}'(x) + \mathrm{f}(x)\mathrm{g}'(x)$ **(Product Rule)**

(3) $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{f}(x)}{\mathrm{g}(x)}\right] = \frac{\mathrm{g}(x)\mathrm{f}'(x) - \mathrm{f}(x)\mathrm{g}'(x)}{\left(\mathrm{g}(x)\right)^2}$ **(Quotient Rule)**

(4) $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{fg}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{f}\left(\mathrm{g}(x)\right)\right) = \mathrm{f}'\left(\mathrm{g}(x)\right)\mathrm{g}'(x)$ **(Chain Rule)**

If we write $y = \mathrm{f}(u)$ and $u = \mathrm{g}(x)$, then the **chain rule** can be written as $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$.

(5) $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}$

Applying Chain Rule, we have the following results:

y	$\frac{\mathrm{d}y}{\mathrm{d}x}$
$\left[\mathrm{f}(x)\right]^n$	$n\left[\mathrm{f}(x)\right]^{n-1}\mathrm{f}'(x)$
$\mathrm{e}^{\mathrm{f}(x)}$	$\mathrm{e}^{\mathrm{f}(x)}\mathrm{f}'(x)$
$\ln\left(\mathrm{f}(x)\right),\ \mathrm{f}(x) > 0$	$\frac{1}{\mathrm{f}(x)}\mathrm{f}'(x)$
$\sin\left(\mathrm{f}(x)\right)$	$\left[\cos\left(\mathrm{f}(x)\right)\right]\ \mathrm{f}'(x)$
$\cos\left(\mathrm{f}(x)\right)$	$\left[-\sin\left(\mathrm{f}(x)\right)\right]\ \mathrm{f}'(x)$
$\tan\left(\mathrm{f}(x)\right)$	$\left[\sec^2\left(\mathrm{f}(x)\right)\right]\mathrm{f}'(x)$

Example 1

Differentiate the following with respect to x.

(a) $(x^5+1)^3$, **(b)** 2^{x+3}, **(c)** $\ln\sqrt{x^2+1}$, **(d)** $\sin\sqrt{x}$.

Solution

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x^5+1\right)^3\right)$ $=3\left(x^5+1\right)^2\left(5x^4\right)$ $=15x^4\left(x^5+1\right)^2$	**(b)** $\frac{\mathrm{d}}{\mathrm{d}x}\left(2^{x+3}\right)$ $=\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\ln 2^{x+3}}\right)$, see below **Note** $=\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{(x+3)\ln 2}\right)$ $=\mathrm{e}^{(x+3)\ln 2}\ln 2$ $=2^{x+3}\ln 2$
(c) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\sqrt{x^2+1}\right)$ $=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\ln\left(x^2+1\right)\right)$ $=\frac{1}{2}\cdot\frac{1}{x^2+1}\cdot 2x=\frac{x}{x^2+1}$	**(d)** $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin\sqrt{x}\right)$ $=\cos\sqrt{x}\;\times\;\frac{1}{2\sqrt{x}}$ $=\frac{1}{2\sqrt{x}}\cos\sqrt{x}$

Example 2

Differentiate the following with respect to x.

(a) $(2x-3)\sqrt{x-5}$, **(b)** $\frac{x+1}{x^2-3}$, **(c)** $\ln\left(\ln x\right)$, $x>1$, **(d)** $\log_5(x^2+1)$

Solution

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\left((2x-3)\sqrt{x-5}\right)$ $=2\sqrt{x-5}+(2x-3)\frac{1}{2\sqrt{x-5}}$ $=\frac{4(x-5)+2x-3}{2\sqrt{x-5}}=\frac{6x-23}{2\sqrt{x-5}}$	**(b)** $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x+1}{x^2-3}\right)$ $=\frac{\left(x^2-3\right)(1)-(x+1)(2x)}{\left(x^2-3\right)^2}$ $=\frac{x^2-3-2x^2-2x}{\left(x^2-3\right)^2}=-\frac{x^2+2x+3}{\left(x^2-3\right)^2}$
(c) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(\ln x)\right)$ $=\frac{1}{\ln x}\left(\frac{1}{x}\right)$ $=\frac{1}{x\ln x}$	**(d)** $\frac{\mathrm{d}}{\mathrm{d}x}\left(\log_5\left(x^2+1\right)\right)$ $=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\ln\left(x^2+1\right)}{\ln 5}\right)$ $=\frac{1}{\ln 5}\times\frac{1}{x^2+1}\times 2x=\frac{2x}{\left(x^2+1\right)\ln 5}$

Note: In Examples **1(b)**, **1(c)** and **2(d)** we have used the following properties of logarithm:

- $x^y=\mathrm{e}^{\ln\left(x^y\right)}$, $\quad x^y>0$
- $\log_x y=\frac{\log_z y}{\log_z x}$, $\quad x,y,z>0,\quad x,z\neq 1$
- $\ln\left(x^y\right)=y\ln x$, $\quad x>0$

3 Derivatives of Standard Functions

3.1 Trigonometric Functions

y	$\frac{\mathrm{d}y}{\mathrm{d}x}$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

y	$\frac{\mathrm{d}y}{\mathrm{d}x}$
$\cot(\mathrm{f}(x))$	$\left[-\operatorname{cosec}^2(\mathrm{f}(x))\right] \mathrm{f}'(x)$
$\sec(\mathrm{f}(x))$	$\left[\sec(\mathrm{f}(x))\tan(\mathrm{f}(x))\right] \mathrm{f}'(x)$
$\operatorname{cosec}(\mathrm{f}(x))$	$\left[-\operatorname{cosec}(\mathrm{f}(x))\cot(\mathrm{f}(x))\right] \mathrm{f}'(x)$

Note: **1.** The above results hold for x measured in radians only.
If x is measured in degrees, then convert $x°$ to $\frac{\pi x}{180}$ radians.

2. The derivative of $\sec x$ and $\operatorname{cosec} x$ can be found in the Formula List.

We can verify the formula easily. For example,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\cos x}\right)\\ &= \frac{-1}{\cos^2 x}(-\sin x)\\ &= \frac{1}{\cos x}\left(\frac{\sin x}{\cos x}\right) = \sec x \tan x\end{aligned}$$

Example 3

Differentiate the following with respect to x.

(a) $\tan 2x°$, **(b)** $\cos^3 x \ \sin\frac{x}{2}$, **(c)** $\mathrm{e}^{2+\sec^2 x}$.

Solution

(a) $\frac{\mathrm{d}}{\mathrm{d}x}(\tan 2x°)$ $= \frac{\mathrm{d}}{\mathrm{d}x}\left(\tan\frac{2\pi x}{180}\right)$ $= \sec^2\frac{2\pi x}{180}\times\frac{2\pi}{180}$ $= \frac{\pi}{90}\sec^2 2x°$	**(b)** $\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos^3 x \ \sin\frac{x}{2}\right)$ $= \left(\sin\frac{x}{2}\right)\left(3\cos^2 x\right)(-\sin x) + \left(\cos^3 x\right)\left(\frac{1}{2}\cos\frac{x}{2}\right)$ $= \frac{1}{2}\cos^2 x\left(\cos x\cos\frac{x}{2} - 6\sin x\sin\frac{x}{2}\right)$
(c) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{2+\sec^2 x}\right)$ $= \mathrm{e}^{2+\sec^2 x} \times 2\sec x \times \sec x\tan x$ $= 2\,\mathrm{e}^{2+\sec^2 x}\sec^2 x\tan x$	

3.2 Inverse Trigonometric Functions

y	$\dfrac{\mathrm{d}y}{\mathrm{d}x}$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}, \quad \lvert x\rvert<1$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}, \quad \lvert x\rvert<1$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}, \quad x\in\mathbb{R}$

y	$\dfrac{\mathrm{d}y}{\mathrm{d}x}$
$\sin^{-1}(\mathrm{f}(x))$	$\dfrac{\mathrm{f}'(x)}{\sqrt{1-[\mathrm{f}(x)]^2}}, \quad \lvert \mathrm{f}(x)\rvert<1$
$\cos^{-1}(\mathrm{f}(x))$	$-\dfrac{\mathrm{f}'(x)}{\sqrt{1-[\mathrm{f}(x)]^2}}, \quad \lvert \mathrm{f}(x)\rvert<1$
$\tan^{-1}(\mathrm{f}(x))$	$\dfrac{\mathrm{f}'(x)}{1+[\mathrm{f}(x)]^2}, \quad \mathrm{f}(x)\in\mathbb{R}$

Note: **1.** $\sin^{-1} x \neq (\sin x)^{-1}$. Recall that $(\sin x)^{-1} = \dfrac{1}{\sin x} = \operatorname{cosec} x$.

2. Students must be able to obtain the derivatives of inverse trigonometric functions using implicit differentiation (see Section 5).

3. The derivatives of $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$ can be found in the Formula List.

Example 4

Differentiate the following with respect to x.

(a) $\sin^{-1}\left(\dfrac{x}{3}\right)$, **(b)** $\cos^{-1}\left(x^2\right)$, **(c)** $\sin^{-1}(2\sin x)$, **(d)** $\tan^{-1}(\mathrm{e}^x)$.

Solution

(a)
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1}\left(\frac{x}{3}\right)\right) = \frac{1}{\sqrt{1-\left(\frac{x}{3}\right)^2}} \times \frac{1}{3} = \frac{1}{3\sqrt{\frac{9-x^2}{9}}} = \frac{1}{\sqrt{9-x^2}}$$

(b)
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos^{-1} x^2\right) = -\frac{1}{\sqrt{1-\left(x^2\right)^2}} \times 2x = -\frac{2x}{\sqrt{1-x^4}}$$

(c)
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1}(2\sin x)\right) = \frac{1}{\sqrt{1-(2\sin x)^2}} \times 2\cos x = \frac{2\cos x}{\sqrt{1-4\sin^2 x}}$$

(d)
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1}\left(\mathrm{e}^x\right)\right) = \frac{1}{1+\left(\mathrm{e}^x\right)^2} \times \mathrm{e}^x = \frac{\mathrm{e}^x}{1+\mathrm{e}^{2x}}$$

Example 5

Using a graphing calculator, find the value of the derivative of

(a) $f(x) = 2x(x-4)$ at the point on the graph where $x = 5.2$,

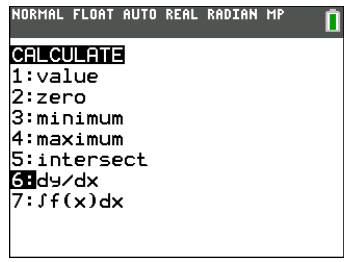
(b) $g(x) = \sin^3\left(e^x + 4\right)$ at the point on the graph where $x = \dfrac{\pi}{8}$.

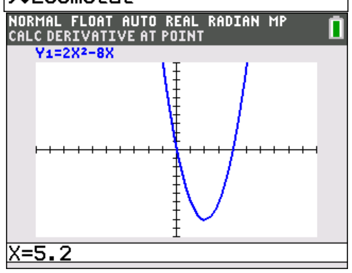
Solution

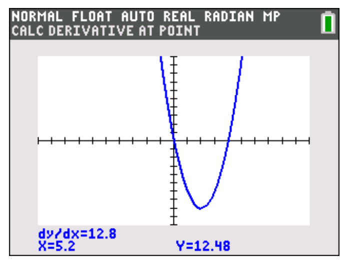
(a) $f(x) = 2x(x-4)$

Method 1:

In the $\boxed{\text{y=}}$ screen, enter **Y₁ = 2X² - 8X**, and choose **6:Zstandard** from the $\boxed{\text{zoom}}$ menu. Choose **6:dy/dx** from the $\boxed{\text{2nd}}\boxed{\text{trace}}$ menu. Type **5.2** and press $\boxed{\text{enter}}$.





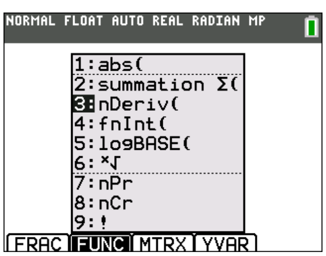


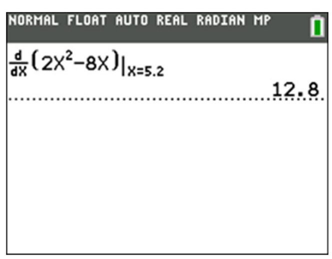
Note: The point on the curve at which we wish to find the gradient at must be in the window.

Method 2:

On the Home Screen, press $\boxed{\text{alpha}}\boxed{\text{window}}\boxed{3}$, to select **3:nDeriv(** . Key in as shown below, then press $\boxed{\text{enter}}$.

Alternatively, on the Home screen, choose **8:nDeriv(** from $\boxed{\text{math}}$ menu.





So $f'(5.2) = 12.8$

(b) $g(x) = \sin^3\left(e^x + 4\right)$

Using GC, $g'\left(\dfrac{\pi}{8}\right) = 1.60$ (3sf).

4 Higher Order Derivatives

Let $y = \mathrm{f}(x)$. The following are some definitions and notations:

(1) The **first derivative** of y with respect to x, denoted by $\frac{\mathrm{d}y}{\mathrm{d}x}$ (or $\mathrm{f}'(x)$), is obtained by differentiating y with respect to x.

(2) The **second derivative** of y with respect to x, denoted by $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ (or $\mathrm{f}''(x)$), can be obtained by differentiating $\frac{\mathrm{d}y}{\mathrm{d}x}$ with respect to x, i.e. $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \mathrm{f}''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$.

(3) The **third derivative** of y with respect to x, denoted by $\frac{\mathrm{d}^3 y}{\mathrm{d}x^3}$ (or $\mathrm{f}^{(3)}(x)$), can be obtained by differentiating $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ with respect to x, i.e. $\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = \mathrm{f}^{(3)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)$.

(4) In general, the ***n*th derivative** of y with respect to x is denoted by $\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$ (or $\mathrm{f}^{(n)}(x)$), where $n \in \mathbb{Z}^+$ and $\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = \mathrm{f}^{(n)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}}\right)$ for $n = 2, 3, 4, \ldots$.

Example 6

By listing the first 4 derivatives, write down an expression for the nth derivative of $y = 3\mathrm{e}^{-2x}$, $n \in \mathbb{Z}^+$.

Solution

Given $y = 3\mathrm{e}^{-2x}$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\mathrm{e}^{-2x} \times (-2)$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 3(-2)\mathrm{e}^{-2x} \times (-2) = 3(-2)^2 \mathrm{e}^{-2x}$$

$$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = 3(-2)^2 \mathrm{e}^{-2x} \times (-2) = 3(-2)^3 \mathrm{e}^{-2x}$$

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} = 3(-2)^3 \mathrm{e}^{-2x} \times (-2) = 3(-2)^4 \mathrm{e}^{-2x}$$

$$\vdots$$

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = 3(-2)^n \mathrm{e}^{-2x}$$

5 Implicit Differentiation

For a relation involving x and y, for example $y^3 - x + y = 0$, which cannot be expressed in the form $y = \mathrm{f}(x)$, we can obtain $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ by differentiating every term in the equation with respect to x. This method is known as **implicit differentiation**.

To illustrate, we differentiate $y^3 - x + y = 0$ implicitly w.r.t. x,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y^3 - x + y\right) = 0$$

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(y^3\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(x\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(y\right) = 0 \qquad --- \quad (*)$$

We use the Chain Rule to find

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y^3\right) = \frac{\mathrm{d}}{\mathrm{d}y}\left(y^3\right) \times \frac{\mathrm{d}y}{\mathrm{d}x} = 3y^2\frac{\mathrm{d}y}{\mathrm{d}x}$$

Thus, from (*), we get

$$3y^2\frac{\mathrm{d}y}{\mathrm{d}x} - 1 + \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\left(3y^2 + 1\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3y^2 + 1}$$

Example 7

Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for each of the following curve.

(a) $x^2 - xy + y^2 = 4$, **(b)** $x\cos y = 3y^2$, **(c)** $\ln\left(x + y\right) = 2\cos^2 y$,

(d) $y = x^x$, $x > 0$, **(e)** $y = \sin^{-1} x$, for $|x| < 1$.

Solution

(a)	(b)
$x^2 - xy + y^2 = 4$ Differentiating w.r.t. x, $2x - \left(x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y\right) + 2y\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ $\left(-x + 2y\right)\dfrac{\mathrm{d}y}{\mathrm{d}x} = -2x + y$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{-2x + y}{-x + 2y}$	$x\cos y = 3y^2$ Differentiating w.r.t. x, $x\left(-\sin y\right)\dfrac{\mathrm{d}y}{\mathrm{d}x} + \cos y = 6y\dfrac{\mathrm{d}y}{\mathrm{d}x}$ $\cos y = \left(6y + x\sin y\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\cos y}{6y + x\sin y}$

(c) $\ln(x+y)=2\cos^2 y$

Differentiating w.r.t. x,

$$\frac{1}{x+y}\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)=4\cos y(-\sin y)\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$4\frac{\mathrm{d}y}{\mathrm{d}x}\sin y\cos y+\frac{1}{x+y}\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{x+y}$$

$$4\frac{\mathrm{d}y}{\mathrm{d}x}(x+y)\sin y\cos y+\frac{\mathrm{d}y}{\mathrm{d}x}=-1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{1+4(x+y)\sin y\cos y}$$

(d) $y=x^x,\quad x>0$

$\ln y=x\ln x$

Differentiating w.r.t. x,

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=x\times\frac{1}{x}+1\times\ln x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=y(1+\ln x)=x^x(1+\ln x)$$

Alternatively,

$y=x^x,\ x>0$

$y=\mathrm{e}^{\ln x^x}=\mathrm{e}^{x\ln x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^{x\ln x}\left[x\left(\frac{1}{x}\right)+\ln x\right]$$

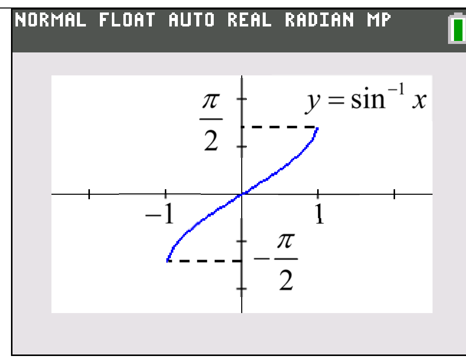
$$\frac{\mathrm{d}y}{\mathrm{d}x}=x^x(1+\ln x)$$

(e) $y=\sin^{-1}x,\ |x|<1$

$x=\sin y$

$$1=\cos y\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\cos y}$$



If $\frac{\mathrm{d}y}{\mathrm{d}x}$ is to be expressed in terms of x …

$$\cos^2 y+\sin^2 y=1\quad\Rightarrow\quad\cos y=\pm\sqrt{1-\sin^2 y}=\pm\sqrt{1-x^2}$$

From the graph of $y=\sin^{-1}x$, $|x|<1$, the gradient of the graph is always positive. Thus $\frac{\mathrm{d}y}{\mathrm{d}x}>0\Rightarrow\cos y>0$. Hence $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-x^2}}$.

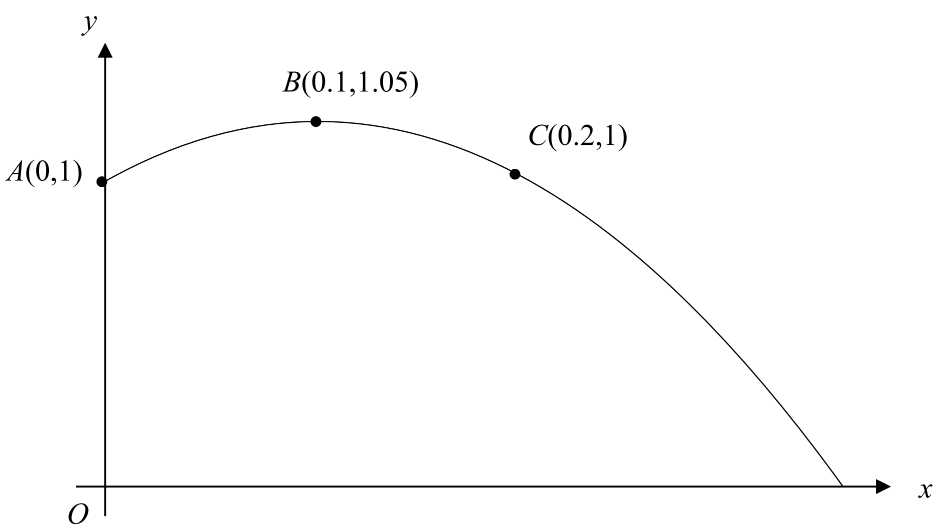
6 Parametric Differentiation

6.1 Parametric Equations

The coordinates of a point (x, y) on a curve can sometimes be meaningfully expressed in terms of a third independent variable t, which is called a **parameter**.
In other words, we can describe the curve by the equations: $x = \mathrm{f}(t)$, $y = \mathrm{g}(t)$, where $t \in \mathbb{R}$.
Such equations are called **parametric equations**.

Consider a particle moving in the Cartesian plane and suppose at a certain time $t\,(\geq 0)$, it is at the point (x, y) where $x = t, y = 1 + t - 5t^2$. What is the path (or curve) traced out by the particle? The table below shows some points on the curve.

t	0	0.1	0.2	0.3	0.4
x	0	0.1	0.2	0.3	0.4
y	1	1.05	1	0.85	0.6



The equations $x = t, y = 1 + t - 5t^2$ are called **parametric equations** where the x and y co-ordinates are expressed in terms of a parameter, in this case, t.

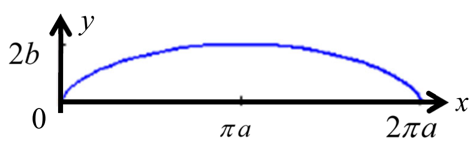
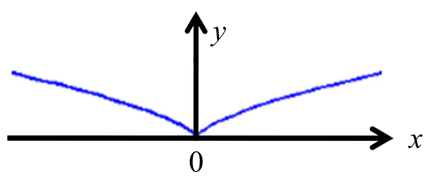
Can we obtain the Cartesian equation of this curve?

By eliminating the parameter t, it is clear that the Cartesian equation is $y = 1 + x - 5x^2$, $x \geq 0$. We observe that the particle follows a parabolic path or trajectory.

What is the advantage of expressing the equation of a curve in parametric form?

In the above example, we can track (trace) the position of the particle as it moves along a path of motion. In many cases, especially for applications in the Physical Sciences it is more appropriate to express the curve in parametric form. For example, planetary motion is related to the graph of conic sections.

Examples of curves that can be expressed in parametric form:

	Parametric Equations	**Cartesian Equation**	**Remark**
1	$x = 1+t$, $y = 3-t$ $t \in \mathbb{R}$	$t = x-1 = 3-y$ $\Rightarrow y = -x+4$	Straight line
2	$x = r\cos\theta$, $y = r\sin\theta$ where r is a positive constant and $0 \le \theta \le 2\pi$	$\sin^2\theta + \cos^2\theta = 1$ $\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$ $x^2 + y^2 = r^2$	Circle centred at the origin and radius r
3	$x = a(\theta - \sin\theta)$, $y = b(1-\cos\theta)$ where a, b are positive constants and $0 \le \theta \le 2\pi$ 	Cannot be expressed neatly	The curve is called a cycloid. The parametric equations are better in representing this curve as compared to cartesian.
4	$x = t^3$, $y = t^2$, $t \in \mathbb{R}$ 	$x^2 = (t^3)^2 = t^6$, $y^3 = (t^2)^3 = t^6$ $y^3 = x^2$	The curve is called a semi-cubical parabola.

We will study how to sketch parametric curves in the chapter on *Graphing Techniques*. For now, let us first look into how we can obtain the gradient of the curve at any given parameter.

6.2 Differentiation of Parametric Equations

A curve C is defined by the equations: $x = \mathrm{f}(t)$, $y = \mathrm{g}(t)$, where $t \in \mathbb{R}$. If the Cartesian form cannot be obtained, then we are unable to express $\frac{\mathrm{d}y}{\mathrm{d}x}$ as a function of the variable x. However, we are still able to obtain $\frac{\mathrm{d}y}{\mathrm{d}x}$ as a function of the parameter t.

Suppose $x = \mathrm{f}(t)$ and $y = \mathrm{g}(t)$, where $t \in \mathbb{R}$.

By Chain rule : $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x}$ Recall $\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}t}}$.

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \Big/ \frac{\mathrm{d}x}{\mathrm{d}t}$$

Example 8

Find $\frac{dy}{dx}$ for each of the following parametric equations.

(a) $x=\frac{1}{1+t},\ y=\frac{t^2}{1+t}$ **(b)** $x=\sin 2\theta,\ y=\cos 4\theta$ **(c)** $x=\ln t,\ y=\sin^{-1} t$

(d) $x=a(\theta-\sin\theta)$, $y=b(1-\cos\theta)$ where a, $b>0$

Solution **(a)** $x=\frac{1}{1+t} \Rightarrow \frac{dx}{dt}=-\frac{1}{(1+t)^2}$ $y=\frac{t^2}{1+t} \Rightarrow \frac{dy}{dt}=\frac{(1+t)(2t)-t^2(1)}{(1+t)^2}=\frac{2t+t^2}{(1+t)^2}$	$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}$ $=\frac{2t+t^2}{(1+t)^2}\times\frac{(1+t)^2}{-1}$ $=-2t-t^2$
(b) $x=\sin 2\theta \Rightarrow \frac{dx}{d\theta}=2\cos 2\theta.$ $y=\cos 4\theta \Rightarrow \frac{dy}{d\theta}=-4\sin 4\theta.$	$\frac{dy}{dx}=\frac{dy}{d\theta}\Big/\frac{dx}{d\theta}$ $=\frac{-4\sin 4\theta}{2\cos 2\theta}$ $=\frac{-2\times 2\sin 2\theta\cos 2\theta}{\cos 2\theta}$ $=-4\sin 2\theta$
(c) $x=\ln t \Rightarrow \frac{dx}{dt}=\frac{1}{t}$ $y=\sin^{-1} t \Rightarrow \frac{dy}{dt}=\frac{1}{\sqrt{1-t^2}}$	$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{t}{\sqrt{1-t^2}}$
(d) $x=a(\theta-\sin\theta) \Rightarrow \frac{dx}{d\theta}=a(1-\cos\theta)$, $y=b(1-\cos\theta) \Rightarrow \frac{dy}{d\theta}=b(\sin\theta)$	$\frac{dy}{dx}=\frac{dy}{d\theta}\Big/\frac{dx}{d\theta}$ $=\frac{b\sin\theta}{a(1-\cos\theta)}$

Example 9

A curve has parametric equations $x = 2t - \frac{1}{t}$, $y = t + \frac{4}{t}$, $t \neq 0$.

Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\left(1 - \frac{9}{2t^2+1}\right)$.

Deduce that the gradient at any point on the curve is less than $\frac{1}{2}$.

Solution

$$x = 2t - \frac{1}{t} \quad \Rightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}t} = 2 + \frac{1}{t^2} = \frac{2t^2+1}{t^2}$$

$$y = t + \frac{4}{t} \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 1 - \frac{4}{t^2} = \frac{t^2-4}{t^2}$$

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t} \bigg/ \frac{\mathrm{d}x}{\mathrm{d}t} \\
&= \frac{t^2-4}{2t^2+1} \\
&= \frac{\frac{1}{2}\left(2t^2+1\right) - \frac{1}{2} - 4}{2t^2+1} \\
&= \frac{1}{2} - \frac{9}{2\left(2t^2+1\right)} \text{ (or by long division)} \\
&= \frac{1}{2}\left(1 - \frac{9}{2t^2+1}\right)
\end{aligned}$$

For $t \in \mathbb{R} \backslash \{0\}$,

$$\begin{aligned}
\frac{9}{2t^2+1} &> 0 \\
1 - \frac{9}{2t^2+1} &< 1 \\
\frac{1}{2}\left(1 - \frac{9}{2t^2+1}\right) &< \frac{1}{2} \times 1 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &< \frac{1}{2}
\end{aligned}$$

$\therefore$ Gradient at any point on the curve is less than $\frac{1}{2}$.

CONCLUSION

This chapter has covered the essential techniques to differentiating functions of various types (e.g. polynomial, exponential, logarithmic, and trigonometric). We will deal with the applications of differentiation in solving many practical problems in the next chapter.

DO YOU KNOW???

What is CALCULUS?

Calculus is a central branch of mathematics that is built on two complementary ideas: differentiation and integration. It is a very important branch of mathematics that deals with motion and change and is applied widely in the industrial and commercial fields. Where there is motion or growth, where variable forces are at work producing acceleration, calculus is the right tool to apply. The discovery of calculus by Newton & Leibniz independently was motivated by the mathematical needs of the scientists of the 17th century. Differential calculus dealt with the problem of calculating rates of change. It enabled people to define slopes of curves, to calculate the velocities and accelerations of moving bodies, to find the firing angle that gave a cannon its greatest range, and to predict the times when planets would be closest together or farthest apart. Today, calculus and its extensions in mathematical analysis are far reaching, and the physicists, mathematicians and astronomers who first developed the subject would surely be amazed and delighted.

Economists use calculus to forecast global trends. Oceanographers use calculus to formulate theories about ocean current and meteorologists use it to describe the flow of air in the upper atmosphere. Biologists use calculus to forecast population size and to describe the way predators like foxes interact with their prey. Medical researchers use calculus to design ultrasound and x-ray equipment for scanning the internal organs of the body. Space scientists use calculus to design rockets and explore distant planets. Psychologists use calculus to understand optical illusions in visual perception. Physicists use calculus to design inertial navigation systems and to study the nature of time and the universe. Sports equipment manufacturers use calculus to design tennis rackets and baseball bats. Stock market analysts use calculus to predict prices and assess interest rate risk. Physiologists use calculus to describe electrical impulses in neurons in the human nervous system. Drug companies use calculus to determine profitable inventory levels and timber companies use it to decide the most profitable time to harvest trees. The list is practically endless, for almost every professional field today uses calculus in some way.

“The calculus was the first achievement of modern mathematics and it is difficult to overestimate its importance. I think it defines more unequivocally than anything else the inception of modern mathematics; and the system of mathematical analysis, which is its logical development, still constitutes the greatest technical advance in exact thinking.”

John von Neumann (1903 – 1957) - one of the great mathematicians of the century.

Appendix: Limits (For Your Information)
Useful Results

1. $\lim\limits_{x \to a} c = c$, where c is a constant.
2. $\lim\limits_{x \to a} x^n = a^n$ for $n \in \mathbb{Z}^+$.
3. $\lim\limits_{x \to a} \mathrm{f}(x) = \mathrm{f}(a)$ if $\mathrm{f}(x)$ is a polynomial.

Theorems on Limits

1. $\lim\limits_{x \to a} \left(c\mathrm{f}(x) \right) = c \lim\limits_{x \to a} \mathrm{f}(x)$, where c is a constant.
2. $\lim\limits_{x \to a} \left(\mathrm{f}(x) \pm \mathrm{g}(x) \right) = \lim\limits_{x \to a} \mathrm{f}(x) \pm \lim\limits_{x \to a} \mathrm{g}(x)$.
3. $\lim\limits_{x \to a} \left(\mathrm{f}(x) \mathrm{g}(x) \right) = \lim\limits_{x \to a} \mathrm{f}(x) \lim\limits_{x \to a} \mathrm{g}(x)$.
4. $\lim\limits_{x \to a} \dfrac{\mathrm{f}(x)}{\mathrm{g}(x)} = \dfrac{\lim\limits_{x \to a} \mathrm{f}(x)}{\lim\limits_{x \to a} \mathrm{g}(x)}$, provided $\lim\limits_{x \to a} \mathrm{g}(x) \neq 0$.

Useful Results for Limits Involving Infinity

1. $\lim\limits_{x \to \infty} \dfrac{1}{x^n} = 0$, $n \in \mathbb{R}^+$.
2. $\lim\limits_{x \to -\infty} \dfrac{1}{x^n} = 0$, $n \in \mathbb{R}^+$.

Differentiation from 1st Principles

Result 1 By considering the derivative as a limit, find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^n \right)$, $n \in \mathbb{Z}^+$

Solution:

Let $\mathrm{f}(x) = x^n$, $n \in \mathbb{Z}^+$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{f}(x) \right) = \lim_{\delta x \to 0} \frac{\mathrm{f}(x + \delta x) - \mathrm{f}(x)}{\delta x} \quad \text{(by definition)}$$

$$= \lim_{\delta x \to 0} \frac{(x + \delta x)^n - x^n}{\delta x} = \lim_{\delta x \to 0} \frac{\sum_{k=0}^{n} \binom{n}{k} x^k (\delta x)^{n-k} - x^n}{\delta x} \quad \text{using Binomial Expansion}$$

$$= \lim_{\delta x \to 0} \frac{\left(x^n + n(\delta x) x^{n-1} + \sum_{k=2}^{n} \binom{n}{k} (\delta x)^k x^{n-k} \right) - x^n}{\delta x} \quad \text{write out the 1st 2 terms}$$

$$= \lim_{\delta x \to 0} \left(n x^{n-1} + \sum_{k=2}^{n} \binom{n}{k} (\delta x)^{k-1} x^{n-k} \right)$$

$$= n x^{n-1} + \lim_{\delta x \to 0} \sum_{k=2}^{n} \binom{n}{k} (\delta x)^{k-1} x^{n-k}$$

$$= n x^{n-1}$$

Result 2 Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{1}{n}}\right)$, $n \in \mathbb{Z}^+$

Solution:

$y = x^n \Leftrightarrow x = y^{\frac{1}{n}}$ Recall: $\dfrac{\mathrm{d}x}{\mathrm{d}y} = \dfrac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}$

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(y^{\frac{1}{n}}\right) = \frac{1}{nx^{n-1}} = \frac{1}{n\left(y^{\frac{1}{n}}\right)^{n-1}} = \frac{1}{n}y^{\frac{1}{n}-1}$$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{1}{n}}\right) = \frac{1}{n}x^{\frac{1}{n}-1}$$

Result 3 Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{m}{n}}\right)$, $m, n \in \mathbb{Z}^+$

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{m}{n}}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^m\right)^{\frac{1}{n}} = \frac{1}{n}\left(x^m\right)^{\frac{1}{n}-1}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^m\right) \text{ using Chain rule}$$

$$= \frac{1}{n}x^{\frac{m}{n}-m}mx^{m-1} = \frac{m}{n}x^{\frac{m}{n}-m+m-1}$$

$$= \frac{m}{n}x^{\frac{m}{n}-1}$$

Result 3 says that $\dfrac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$ when $n \in \mathbb{Q}^+$.

<u>Summary</u>