NANYANG JUNIOR COLLEGE

JC2 COMMON TEST

Higher 2

FURTHER MATHEMATICS **9649**

26th June 2024

3 hours

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

An answer booklet will be provided with this question paper. You should follow the instructions on the front cover of the answer booklet. If you need additional answer paper ask the invigilator for a continuation booklet.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

Section A: Pure Mathematics [75 marks]

1 Let l be a line in the xy-plane that passes through the origin and makes an angle θ with the positive x-axis where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Let $\mathbf{A}$ be the matrix representation of the transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ that maps every vector in $\mathbb{R}^2$ onto its vector projection on l.

(a) By considering the standard basis vectors for $\mathbb{R}^2$ under the transformation T, show clearly that $\mathbf{A} = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}$. [3]

(b) Find a basis for the nullspace of T. [2]

(c) Find, in exact form, the angle θ such that the vector projection of $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$ onto l is $\begin{pmatrix} \frac{3+5\sqrt{3}}{4} \\ \frac{\sqrt{3}+5}{4} \end{pmatrix}$. [3]

2 Let $\mathbf{A}$ be the matrix representation of the transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$.

A basis for the nullspace of T is given by $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

(a) State, with a reason, the value of an eigenvalue of $\mathbf{A}$ and a corresponding eigenvector. [1]

It is also given that $\mathbf{A} = \begin{pmatrix} -5 & 2 & 3 \\ -2 & 0 & 2 \\ -4 & 2 & 2 \end{pmatrix}$ and the other eigenvalues of $\mathbf{A}$ are -2 and -1 with corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ respectively.

(b) Without performing row operations on $\mathbf{A}$, explain why these two eigenvectors form a basis of the range space of T. [2]

(c) Hence, express $\mathbf{A}\begin{pmatrix} p \\ q \\ r \end{pmatrix}$ in the form of $\lambda\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, where λ and μ are to be expressed in terms of p, q and r. Deduce a similar form for $\mathbf{A}^n\begin{pmatrix} p \\ q \\ r \end{pmatrix}$. [4]

The matrix $\mathbf{B}$ is such that $\mathbf{B} = 4\mathbf{A}(\mathbf{A}-\mathbf{I})^{-1}$.

(d) Without evaluating $\mathbf{B}$, find the eigenvalues of $\mathbf{B}$. [3]

3 Let $\mathrm{f}(x, y) = xy - 3y\ln(x+1)$.

(a) Find all the first and second partial derivatives of $\mathrm{f}(x, y)$. [2]

(b) Find all the stationary point(s) of $\mathrm{f}(x, y)$ for $x > 0$. For each of the stationary point(s), determine whether it is a local maximum, local minimum, or saddle point. [5]

(c) Using local linearisation of $\mathrm{f}(x, y)$ at the point $(2, 1)$, find an approximate value of $\mathrm{f}(2.05,\ 0.98)$. [3]

4 Given that $|z+3-2\mathrm{i}| \le \sqrt{13}$, $|z+5+2\mathrm{i}| \ge |z+1-4\mathrm{i}|$ and $\arg(z+6) \ge 0$, illustrate the locus of the points representing the complex number z in the Argand diagram. [3]

(a) Find the exact complex number z, in cartesian form $x + y\mathrm{i}$, which gives the minimum value of $|z+7-\mathrm{i}|$. [3]

(b) Determine the maximum value of $\arg\left(z+8-3\mathrm{i}\right)$, giving your answer in the form of $\tan^{-1} p$, for some rational number p. [4]

5 A large tank initially contains 100 gallons of fresh water. Brine flows in at a constant rate of 2 gallons/minute containing 1 kg of salt/gallon. The mixture is constantly stirred and flows out at a constant rate of w gallon/minute. Let x be the amount of salt (in kg) in the tank at time t (in minute).

It is given that $w = 2$.

(a) Find x in terms of t. and sketch the graph of x against t. [4]

It is given instead that $w = 1$.

(b) Find x when the tank contains 150 gallons of mixture, giving your answer to 2 decimal places.[6]

(c) Find the time taken for the salt concentration in the mixture to reach 0.9 kg/gallon. [2]

6 Alan downloaded a new mobile game FM: Battlegrounds which was developed by No Sine Studios. In the game, players form alliances, grow a population of a certain species and invade the territories of other alliances on a map with finite area.

The species in Alan's alliance are classified as 'children', 'adult' or 'ageing adult'. The population of the species in the alliance after k months is given by P_k where

$$P_k = A_k + B_k + C_k.$$

In this equation,

A_k is the number of 'ageing adults' (that is, species that are too old to breed),

B_k is the number of 'adults' of breeding age, and

C_k is the number of 'children' in the alliance (that is, those too young to breed).

For the positive constants α, β, γ and δ, each between 0 and 1, the numbers in each group are modelled by the following system of recurrence relations.

$$\begin{aligned} A_{k+1} &= A_k + \alpha B_k - \beta A_k \\ B_{k+1} &= B_k + \gamma C_k - \alpha B_k \\ C_{k+1} &= C_k + \delta B_k - \gamma C_k \end{aligned}$$

(a) **(i)** Give an interpretation of each of α, β, γ and δ. [4]

(ii) Hence give one criticism of this population model. [1]

In a beta version of the game, in Alan's alliance invasion to other territories, it has to attack in the north direction to advance its position. Similarly, other alliances can also attack his alliance and his alliance can retreat in the south direction. The distance due North of Alan's alliance from the East-West latitude line, in units, after n weeks is denoted by x_n. Initially, his alliance is 2 units due North from the latitude line. After 1 week, there is no change in the position of his alliance. It is given that x_n satisfies the recurrence relation

$$x_n = 4x_{n-1} - 5x_{n-2}, \ n \geq 2.$$

(b) Solve the recurrence relation for x_n, giving your answer in the form of $A(x+y\mathrm{i})^n + B(x-y\mathrm{i})^n$ where $A, B \in \mathbb{C}$ and $x, y \in \mathbb{R}$ are to be determined. Hence show that $x_n = 2^{\frac{3}{2}} 5^{\frac{n}{2}} \cos\left(\frac{\pi}{4} + n\theta\right)$, where $\theta = \tan^{-1}\frac{1}{2}$. [6]

(c) Give one criticism of modelling x_n in this way. [1]

7 A curve D is given by the equation

$$x = 2\cos t - \cos 2t - 1,$$
$$y = 2\sin t - \sin 2t, \qquad 0 \le t \le 2\pi.$$

(a) The arc-length between two points on D, where $t = \alpha$ and $t = \beta$, is given by the formula

$$\int_{\alpha}^{\beta} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

Using integration, show that the length of D is 16. [3]

(b) The surface area generated when the arc between two points on D, where $t = \alpha$ and $t = \beta$, is rotated fully about the x-axis is given by the formula

$$\int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

When D is rotated through π radians about the x-axis, a surface of revolution is formed with area A. Using calculus, determine the exact value of A. [4]

(c) Show that the equation of D can be expressed in polar form $r = 2(1 - \cos\theta)$. [3]

(d) Hence, using calculus, find the exact area enclosed by the curve D. [3]

Section B: Probability and Statistics [25 marks]

8 The number of thunderstorms, x, reported in a particular month by 100 meteorological stations are as follows:

x	0	1	2	3	4	5
Number of stations	22	37	20	13	6	2

By considering the mean number of thunderstorms for the month and its variance, explain why a Poisson distribution would be a suitable model for the above data.

Test at the 10% significance level whether the number of thunderstorms in the particular month has a Poisson distribution. [6]

9 The Society for the Prevention of Cruelty to Animals (SPCA) is investigating the survival times (in days) of guinea pigs reared by a particular farm. The farmer, having some statistical knowledge, decides to conduct a test on his own to show that the average survival times of guinea pigs in his farm is at least 60 days. He picks a random sample of 20 guinea pigs, which yielded the following survival times (in days).

40	42	42	43	43	44	44	48	48	48
52	53	55	55	58	60	65	65	75	120

With the assumption that the survival times of guinea pigs in his farm is normally distributed, the farmer conducted a significance test at 10% level and reported his findings to SPCA.

(a) Calculate the unbiased estimates of the population mean and variances of the survival times of guinea pigs from the farm. [1]

(b) Replicate the significance test that the farmer conducted, stating clearly the conclusion of his report to SPCA. [4]

While going through the farmer's report, SPCA noticed that the assumption of the survival times of guinea pigs in the farm being normally distributed may not be valid.

(c) With reference to the data from the random sample of 20 guinea pigs, explain why SPCA disagreed with the farmer's normal distribution assumption. [1]

10 The continuous random variable X has a uniform distribution on the interval $0 < x < 1$ and the random variable T is defined by $T = \dfrac{-\ln(1-X)}{\lambda}$, $\lambda > 0$.

(a) Find the cumulative distribution function of T, and hence state the distribution of T. [3]

A factory manufactures products made of ceramics. Every product made is subjected to quality inspection. A batch of a large number of products is sent for inspection. The time taken, T, to inspect each product follows an exponential distribution with mean 0.5 minutes.

(b) A random sample of 60 products are inspected. Find the approximate probability that the total time taken to inspect them is at most 40 minutes. [3]

Let E be the event that the time to inspect a product is less than one minute. A sequence of independent observations of T is made until E occurs. Let the random variable Y be the value of T observed when E occurs.

(c) Show that $\mathrm{P}(Y \le y) = \dfrac{1-\mathrm{e}^{-2y}}{1-\mathrm{e}^{-2}}$. [3]

The next part of question is printed on the next page.

The quality of a product is measured by the continuous random variable V with cumulative distribution function given by

$$F(v) = \begin{cases} 0, & v < 0, \\ 1 - \dfrac{(k-v)^2}{k^2}, & 0 \le v \le k, \\ 1, & v > k \end{cases}$$

where k is a positive integer.

The related discrete random variable W, which takes on values $\frac{1}{2}, \frac{3}{2}, \ldots, \frac{2k-1}{2}$, is the profit made when the product is sold. The probability distribution function of W is defined by

$$P\left(W = \frac{2r-1}{2}\right) = P(r-1 \le V \le r) \text{ for } r = 1, 2, \ldots, k.$$

(d) Show that the expected profit per product, $E(W) = \frac{1}{2} + \frac{1}{k^2}\sum_{r=1}^{k-1} r^2$. [4]

———END OF PAPER———