Section A: Pure Mathematics [40 marks]

1 The region A is bounded by the curves $y=\sqrt{x+1}$, $y=\sqrt{7-2x}$, the x-axis and the y-axis.

(a) Find the exact area of A. [4]

(b) Find the volume of the solid obtained when A is rotated through 2π radians about the y-axis. [3]

2 It is given that $y=\ln\left[\sin\left(x+\frac{\pi}{4}\right)\right]$, where $-\frac{\pi}{4}<x<\frac{3\pi}{4}$.

(a) Show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+1=0$. Hence find the first four non-zero terms of the Maclaurin expansion of y, leaving your answer in exact form. [6]

(b) Verify the result obtained in part **(a)** using standard series from the List of Formulae (MF27). [5]

3 The parametric equations of the curve C are

$$x=1-3\operatorname{cosec}\theta \text{ and } y=2\cot\theta-3, \text{ where } 0\le\theta\le\pi.$$

(a) Show $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2}{3}\sec\theta$. Hence find the equation of the normal to C at the point where $\theta=\frac{\pi}{4}$. Give the equation in the form $y=Ax+B$, where A and B are exact constants to be found. [4]

(b) Show that the normal found in part **(a)** will cut C again. [2]

(c) Find the Cartesian equation of C. [2]

(d) Sketch C, indicating clearly its key features. [3]

(e) Find the range of values of m such that there is no intersection between the line $y=m(x-1)-3$ and C. [2]

4 Let A, B and C be the points on the same plane with position vectors **a**, **b** and **c** respectively and $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$. It is given that vectors **a**, **b** and **c** are unit vectors.

(a) **(i)** By considering $\mathbf{c}\bullet\mathbf{c}$, find the value of $\mathbf{a}\bullet\mathbf{b}$. [3]

(ii) Find the angle $\angle AOB$. [2]

(iii) Draw the position vectors **a**, **b** and **c** on a single diagram. Using your diagram, identify the type of triangle ABC. [2]

(b) The point D has position vector $\mathbf{a}+\mathbf{b}$. Find the area of the quadrilateral $ACBD$. [2]

Section B: Probability and Statistics [60 marks]

5 The eleven letters in the word INSPIRATION are each printed on separate, identical cards.

(a) Find the number of ways in which the cards can be arranged in a row if,

(i) there are no restrictions, [1]

(ii) the letters N are together or the letters I must all be separated, but not both. [3]

(b) Three of the eleven cards are removed at random. Find the probability that the letters on the eight cards left behind are all distinct. [2]

6 A basketball free throw game involves a team of two players, Ben and John. The game consists of at most three throws and the moment a player makes a successful shot, the team wins, and the game will end.

The game uses the following rules.

- Only one player is selected for each game and the probability that Ben is selected in a game is 0.7.
- The probabilities that Ben and John make a successful shot in any single attempt are 0.1 and 0.07 respectively.
- The shots are independent of each other.

(a) Find the probability that the team wins the game. [3]

(b) The team did not win the game. Find the probability that John was the one who was selected to shoot. [3]

(c) The team attempts the game repeatedly until the first game is won. Find the least number of attempts required such that the probability of winning within n games is at least 0.95. [2]

7 An ice-cream seller records the monthly ice cream sales, s thousands dollars for different temperature, t degrees Celsius during the winter season. The recorded values are shown in the table below.

t	1	4	5	6	7	8	9
s	14	15	15	16	18	21	23

(a) It is given that the regression line of s on t is $s = 1.125t + 11$. Using this regression line, find the sum of the squares of the residuals. [1]

(b) State the coordinates of an additional data point such that, with all 8 data points, the regression line remains the same as in part **(a)**. [1]

(c) Sketch a scatter diagram of s against t for the data given in the table. [1]

The following three models are proposed, where a, b, c, d, f and h are positive constants.

(A) $s = at^2 + b$

(B) $s = -ce^t + d$

(C) $s = f\ln(t+h)$

(d) Explain which of these models give the best fit to the data. State the values of the constants for the chosen model. [2]

A temperature of F degrees Fahrenheit is equivalent to a temperature of C degrees Celsius, where $F = \frac{9}{5}C + 32$.

(e) Using the model you chose in part **(d)**, re-write the equation so that it can be used to estimate the monthly sales when the temperature, T, is given in degrees Fahrenheit. [2]

8 A food producer claims that the mean mass of a can of beans it produces is 425 g. Following customer feedback, the production manager wishes to test if the mean mass of a can of beans is indeed 425 g.

The production manager took a random sample of size 50 and the mass of each can, in x g, is recorded and the results are shown below:

$$\sum x = 21209,\ \sum (x - 424.18)^2 = 522$$

(a) State what it means for a sample to be random in this context. [1]

(b) Find the unbiased estimates for the population mean and variance. [2]

(c) State the hypotheses for the manager's test, defining any parameters you use. Carry out the test at the 5% level of significance, giving your conclusion in the context of the question. [5]

The production manager wishes to test whether the mean mass of a can of beans has increased using the alternative manufacturing process. He finds that the mean mass of 55 randomly chosen cans is 426.5 g. He carries out a hypothesis test at 10% level of significance.

(d) Explain, with justification, how the population standard deviation of the mass of a can produced under the alternative process will affect the conclusion made by the production manager. [3]

9 In each round of a treasure hunt game, a player randomly selects a spot from a large number of predefined treasure locations on an island. Each spot uncovers one outcome, and a score x is awarded based on the outcome. The table below shows all the possible outcomes in a round and the corresponding scores. Each round is independent and the game resets after every round, so that the probabilities remain unchanged.

Outcome	Cursed trap	Small trap	Mystery box	Gold chest
Score, x	-3	-0.3	p	5
$\mathrm{P}(X=x)$	0.2	p	q	0.1

(a) Show that $\mathrm{E}(X)=-0.1+0.4p-p^2$. [2]

(b) Hence find the maximum and minimum possible values of $\mathrm{E}(X)$. [2]

(c) The treasure hunt game is played for 30 rounds. If $p=0.4$, find the probability that the player's mean score exceeds 0. [3]

(d) The treasure hunt game is played for 10 rounds. Given that the probability of finding more than 3 Small traps in these 10 rounds is 10%, find the value of p. [2]

(e) Over a long period of time, it is observed that the number of Small traps found in 10 rounds of the game follows a bimodal distribution, with one of the modes being 3. Find the two exact possible values of p, showing your working clearly. [3]

10 **In this question you should state the parameters of any distributions you use.**

At a burger shop, the wait time, W (in minutes), is defined to be the time from when a customer places an order at the counter until the food is collected. It was proposed that W is modelled by $\mathrm{N}\left(2, 1.5^2\right)$.

(a) Give a reason why this model is not suitable. [1]

A new model for W is given by $\mathrm{N}\left(5, 1^2\right)$.

(b) Find the range of values of k such that at least 90% of customers experience a wait time longer than k minutes. [2]

A customer bought burgers from the shop on three independent occasions, with wait times denoted by W_1, W_2 and W_3 respectively. Let $\bar{W} = \dfrac{W_1 + W_2 + W_3}{3}$.

(c) Find the values of $\mathrm{Var}\left(\bar{W} - W_1\right)$ and $\mathrm{Var}\left(\bar{W}\right) + \mathrm{Var}\left(W_1\right)$ and hence, determine whether $\mathrm{Var}\left(\bar{W} - W_1\right) = \mathrm{Var}\left(\bar{W}\right) + \mathrm{Var}\left(W_1\right)$. [2]

(d) Find the probability that the mean wait time is within one minute of the wait time on the first occasion. [3]

On the 4$^{\text{th}}$ occasion, the customer went to the restroom immediately after the order was placed. The time spent in the restroom, in T minutes, is modelled by $T \sim N\left(7, 1.5^2\right)$.

Assume that the time taken to walk to the restroom is negligible and W and T follow independent normal distributions.

(e) The customer is in a rush and will not wait for more than 3 minutes after leaving the restroom. Given that the burger is not ready for collection after the customer leaves the restroom, find the probability that the customer will leave the shop without collecting the burger. [4]

To reduce the wait time for customers, the shop later installs self-order kiosks. The wait time from when a customer places an order at the kiosk until the food is collected is modelled as being reduced by 20% compared to that at the counter.

(f) 8 customers who ordered at the counter and 8 customers who ordered at the kiosk are randomly selected and their wait times are recorded. The wait time of each customer is independent of the others. Find the probability that exactly 2 of these 16 customers have a wait time of at least 5 minutes. [4]