

2025 ASRJC H23 Math Preliminary Exam Solutions

1	Solution:	
	Consider the vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \left\ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\ \left\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\ \cos \theta$ where θ is the angle between vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ [M1] . $(ax + by + cz) = \sqrt{a^2 + b^2 + c^2} \sqrt{x^2 + y^2 + z^2} \cos \theta$ $(ax + by + cz)^2 = (a^2 + b^2 + c^2)(x^2 + y^2 + z^2) \cos^2 \theta$ Since $\cos^2 \theta \leq 1$, Therefore $(ax + by + cz)^2 \leq (a^2 + b^2 + c^2)(x^2 + y^2 + z^2)$ [B1]	
	For equality to hold, $\cos^2 \theta = 1 \Rightarrow \cos \theta = \pm 1 \Rightarrow \theta = 0$ or π. Therefore a necessary and sufficient condition is $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is parallel to $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. [A1]	
	(i) $(a^2(1) + b^2(1) + c^2(1))^2 \leq (a^4 + b^4 + c^4)(1^2 + 1^2 + 1^2)$ [B1] $(a^2 + b^2 + c^2)^2 \leq 3[(a^2 + b^2 + c^2)^2 - 2((ab)^2 + (bc)^2 + (ca)^2)]$ [M1] $\Rightarrow 2(a^2 + b^2 + c^2)^2 \geq 6((ab)^2 + (bc)^2 + (ca)^2)$ [M1] $\Rightarrow \frac{(a^2 + b^2 + c^2)^2}{3} \geq (ab)^2 + (bc)^2 + (ca)^2$	
	(ii) Let $x = p, y = 2q, z = r$ and $a = 8, b = 4, c = 1$. [B1] LHS = $8p + 8q + r^2 = 243^2 = 59049$ RHS = $8^2 + 4^2 + 1^2 \cdot p^2 + 4q^2 + r^2 = 81 \cdot 729 = 59049$ [M1] Therefore these assigned values and variables satisfy $ax + by + cz^2 = a^2 + b^2 + c^2 \cdot x^2 + y^2 + z^2$ Therefore $\begin{pmatrix} p \\ 2q \\ r \end{pmatrix}$ is parallel to $\begin{pmatrix} 8 \\ 4 \\ 1 \end{pmatrix}$ [B1]	

$$\text{ie } \begin{pmatrix} p \\ 2q \\ r \end{pmatrix} = k \begin{pmatrix} 8 \\ 4 \\ 1 \end{pmatrix}$$

ie $p = 8k, 2q = 4k, r = k$

Substitute into $8p + 8q + r = 243$,

$$64k + 16k + k = 243$$

$$81k = 243$$

$$k = \frac{243}{81} = 3$$

$$\therefore p = 24, q = 6, r = 3$$

[M1]

[A1]

	$\text{ie } \begin{pmatrix} p \\ 2q \\ r \end{pmatrix} = k \begin{pmatrix} 8 \\ 4 \\ 1 \end{pmatrix}$ $\text{ie } p = 8k, 2q = 4k, r = k$ Substitute into $8p + 8q + r = 243$, $64k + 16k + k = 243$ $81k = 243$ $k = \frac{243}{81} = 3$ $\therefore p = 24, q = 6, r = 3$	[M1] [A1]

Solution:

$$f(r) > f(r+1), \quad r=1, 2, 3, \dots$$

So for $r \in [k^n, k^{n+1}-1]$, the largest value $= f(k^n)$ and the smallest value $= f(k^{n+1})$ [M1]

$$\begin{aligned} \text{No of terms in } [k^n, k^{n+1}-1] &= (k^{n+1}-1) - k^n + 1 \\ &= k^{n+1} - k^n \\ &= k^n(k-1) \end{aligned} \quad [B]$$

$$\therefore k^n(k-1)f(k^{n+1}) \leq \sum_{r=k^n}^{k^{n+1}-1} f(r) \leq k^n(k-1)f(k^n) \quad \text{--- (*)} \quad [AG1]$$

(i) Simplify (*)

$$k^n(k-1) \frac{1}{k^{k+1}} \leq \sum_{r=k^n}^{k^{n+1}-1} f(r) \leq k^n(k-1) \cdot \frac{1}{k^n}$$

$$\frac{k-1}{k} \leq \sum_{r=k^n}^{k^{n+1}-1} f(r) \leq k-1 \quad [M1]$$

$$\text{Let } k=2, \quad \frac{1}{2} \leq \sum_{r=2^n}^{2^{n+1}-1} f(r) \leq 1 \quad [M1]$$

$$\sum_{r=1}^{2^{N+1}-1} \frac{1}{r} = \sum_{n=0}^N \left(\sum_{r=2^n}^{2^{n+1}-1} f(r) \right)$$

$$\sum_{n=0}^N \left(\frac{1}{2} \right) \leq \sum_{r=1}^{2^{N+1}-1} \frac{1}{r} \leq \sum_{n=0}^N (1) \quad [M1]$$

$$\frac{1}{2}(N+1) \leq \sum_{r=1}^{2^{N+1}-1} \frac{1}{r} \leq N+1 \quad [AG1]$$

As $N \rightarrow \infty$, the lower bound $\left(\frac{N+1}{2}\right) \rightarrow \infty$.

Hence $\sum_{r=1}^{\infty} \frac{1}{r}$ does not converge. [A1]

(i) let $k=2$,

$$\sum_{k=2}^{\infty} \frac{1}{r^k} \leq 2^n \cdot \frac{1}{(2^n)^2} = \frac{1}{2^n}$$

[u1]

$$\therefore \sum_{k=1}^{\infty} \frac{1}{r^k} \leq \sum_{n=0}^{\infty} \left(\sum_{k=2}^{\infty} \frac{1}{r^k} \right) = \sum_{n=0}^{\infty} \left(\frac{1}{2^n} \right)$$

[u1]

$$= \frac{1}{1 - \frac{1}{2}}$$

$$= \frac{1}{\frac{1}{2}}$$

$$= \frac{2}{1}$$

$$= 2$$

[AGG]

(ii) For 1 digit no, $\boxed{1}$

For 2 digit no, $\boxed{1 \ 1}$

For 3 digit no, $\boxed{1 \ 1 \ 1}$

[m1]

$\therefore$ SC(1000) has $8 + 8 \cdot 9 + 8 \cdot 9^2$

$$= 8(1 + 9 + 9^2)$$

[m1]

$$= 8 \left(\frac{1(1-9^3)}{1-9} \right)$$

$$= 9^3 - 1 \text{ distinct numbers}$$

For any n ,
there is a
m digit no
such that
 $n \leq m$

For m digit no, no of distinct numbers is $8 \cdot 9^{m-1}$

[u1]

$$\sigma(n) \leq O(m) = \sum_{i=1}^{10^m-1} \frac{1}{r^i} \leq 8 \cdot 9 \left(\frac{1}{10} \right) + 8 \cdot 9^2 \left(\frac{1}{10^2} \right) + \dots + 8 \cdot 9^m \left(\frac{1}{10^m} \right)$$

[u1]

$$= 8 \left(1 + \frac{9}{10} + \left(\frac{9}{10} \right)^2 + \dots + \left(\frac{9}{10} \right)^{m-1} \right)$$

$$= 8 \left(\frac{1 - \left(\frac{9}{10} \right)^m}{1 - \frac{9}{10}} \right)$$

$$= 80 \left(1 - \left(\frac{9}{10} \right)^m \right)$$

[m1]

≤ 80 for all m i.e. n as well.

3	Solution:	
	(i) For P_2, the train has 2 seats. $P_2 = P$ T_2 sits in his assigned seat $= P$ T_1 sits in his assigned seat $= \frac{1}{2}$ $P_3 = P$ T_3 sits in his assigned seat $= P$ T_1 sits in his assigned seat $+ P$ T_1 sits in $S_2 \cap T_2$ sits in S_1 $= \frac{1}{3} + \left(\frac{1}{3} \times \frac{1}{2} \right)$ $= \frac{1}{2}$	[A1] [A1]
	(ii) If passenger T_1 sits in S_k, then $T_2, T_3, \dots, T_{k-1}$ will sit in their assigned seats as they are all available for them. For T_k, since his assigned seat S_k has been taken, he would have to choose randomly from the remaining $n - k - 1 = n - k + 1$ seats. The situation is the same as a train having $n - k + 1$ seats and the first passenger choosing his seat randomly. So we have P T_n sits in S_n T_1 sits in $S_k = P_{n-k+1}$ $P_n = \sum_{k=1}^n P$ T_n sits in $S_n \cap T_1$ sits in S_k $= \sum_{k=1}^n P$ T_n sits in S_n T_1 sits in $S_k \times P$ T_1 sits in S_k $= \frac{1}{n} \sum_{k=1}^n P$ T_n sits in S_n T_1 sits in S_k $= \frac{1}{n} \left[P$ T_n sits in S_n T_1 sits in $S_1 + \sum_{k=2}^{n-1} P$ T_n sits in S_n T_1 sits in $S_k \right.$ $\left. + P$ T_n sits in S_n T_1 sits in $S_n \right]$ $= \frac{1}{n} \left[1 + \sum_{k=2}^{n-1} P_{n-k+1} + 0 \right]$ Let $r = n - k + 1$, then when $k = 2, r = n - 1$ and when $k = n - 1, r = 2$. $P_n = \frac{1}{n} \left[1 + \sum_{k=2}^{n-1} P_{n-k+1} \right]$ $= \frac{1}{n} \left[1 + \sum_{r=2}^{n-1} P_r \right]$	[A2] [B1] [B1] [M1] [AG1]

	(iii) $P_4 = \frac{1}{4} \left[1 + \sum_{r=2}^3 P_r \right] = \frac{1}{4} 1 + P_2 + P_3 = \frac{1}{4} \left[1 + \frac{1}{2} + \frac{1}{2} \right] = \frac{1}{2}$ Conjecture: $P_n = \frac{1}{2}$ We have already proven $P_1 = \frac{1}{2}$ ie the result is true for $n = 1$. Assuming the result is true for $n = 2, \dots, k$. ie $P_1 = P_2 = \dots = P_k = \frac{1}{2}$.	[A1] [B1] [B1]
	$P_{k+1} = \frac{1}{k+1} \left[1 + \sum_{r=2}^k P_r \right]$ $= \frac{1}{k+1} \left[1 + \sum_{r=2}^k \left(\frac{1}{2} \right) \right]$ $= \frac{1}{k+1} \left[1 + \frac{1}{2} k - 1 \right]$ $= \frac{1}{k+1} \left[\frac{k+1}{2} \right]$ $= \frac{1}{2}$ Hence by Mathematical Induction, $P_n = \frac{1}{2}$ is true for all positive integer n.	[M1] [A1]
4	Solution	
	(a) Since $n-1 = mp-1 = m-1 p + p-1$ $x^{n-1} \equiv 1 \pmod{p}$ $\Leftrightarrow x^{m-1 p + (p-1)} \equiv 1 \pmod{p}$ $\Leftrightarrow x^{m-1 p} x^{p-1} \equiv 1 \pmod{p}$ $\Leftrightarrow x^{m-1 p} \equiv 1 \pmod{p} \text{ since } x^{p-1} \equiv 1 \pmod{p} \text{ by Fermat's Little Theorem}$ $\Leftrightarrow x^{p^{m-1}} \equiv 1 \pmod{p}$ $\Leftrightarrow x^{m-1} \equiv 1 \pmod{p} \text{ since } x^{p-1} \equiv 1 \pmod{p} \Rightarrow x^p \equiv x \pmod{p}$	[B1] [B1] [B1]
	(b) $n = p_1 p_2 \dots p_k$ $\Rightarrow \text{If } x^{n-1} \equiv 1 \pmod{n}$ then $x^{n-1} = 1 + sn$ for some integer n	

	$x^{n-1} = 1 + s \ p_1 p_2 \dots p_n$ $x^{n-1} \equiv 1 \pmod{p_i}$ for every p_i $x^{\left(\frac{n}{p_i}\right)_{p_i-1}} \equiv 1 \pmod{p_i}$ From (a), we have $x^{\left(\frac{n}{p_i}\right)-1} \equiv 1 \pmod{p_i}$ $\Leftarrow x^{\left(\frac{n}{p_i}\right)-1} \equiv 1 \pmod{p_i}$ for every p_i From part (a), $x^{\left(\frac{n}{p_i}\right)_{p_i-1}} \equiv 1 \pmod{p_i}$ $x^{n-1} \equiv 1 \pmod{p_i}$ for every p_i By Chinese Remainder Theorem, $x^{n-1} \equiv 1 \pmod{n}$	[B1] [B1] [B1] [B1]
	(c) Take any integer x with $\gcd(x, n) = 1$. Then for each $p_i n$, Fermat's Little theorem gives $x^{p_i-1} \equiv 1 \pmod{p_i}$. Since $p-1 n-1$, we have $n-1 = k \ p-1$ for some integer k . Then $x^{n-1} = x^{p_i-1 \cdot k} \equiv 1^k \pmod{p_i} \equiv 1 \pmod{p_i}$. This holds for every prime divisor p_i of n . By Chinese Remainder Theorem, $x^{n-1} \equiv 1 \pmod{n}$.	[B1] [B1] [B1]
5	Solution:	
	(ii) There are n ways to pick a square from the 1 st row. Suppose the square is in the j th column. "Block off" the squares in the 1 st row and the squares in the j th column. The remaining $(n-1)$ squares can be picked from the $(n-1)$ unblocked rows and the $(n-1)$ unblocked columns in S_{n-1} ways. Hence $S_n = n S_{n-1}$.	[A1]
	(ii) If the square in row i and column j is selected, then map the integer i to the integer j . Conversely, if the integer i is mapped to the integer j , then select the square in row i and column j . $S_n = n!$	[B1] [A1]

	(iii) This is equivalent to a permutation of the integers $\{1, 2, 3, \dots, n\}$ except that no integer is mapped to itself, e.g. the integer 1 cannot be mapped to itself (i.e. the 1st slot) but to the other $(n - 1)$ slots. There are $(n - 1)$ slots to choose from. Suppose 1 is mapped to the rth slot. There are $(n - 1)$ position that 1 can occupy. <u>Case 1</u> The integer r is mapped to 1st slot (i.e. 1 and r swap places) We just need to do a derangement of all the integers other than 1 and r, which can be done in D_{n-2} ways. <u>Case 2</u> The integer r is not mapped to 1st slot (i.e. 1 and r don't swap places). The rth slot is now occupied by "1". For the time being, put "r" in the 1st slot. Now do a derangement of all the slots other than the rth slot. Since the "r" in the 1st slot cannot occupy the 1st slot as every integer must be mapped to another position, this ensures that "r" is forced out of the 1st slot. There are D_{n-1} ways to do this derangement. Combining cases 1 and 2, we have $D_n = (n - 1)(D_{n-1} + D_{n-2})$. $D_1 = 0$ and $D_2 = 1$.	[B1] [B1] [B1] [A1]
	(iv) Let S be the set of all permutations. $S = n!$ Let B_k be the set containing all permutations of n objects where the k^{th} object is in its original position (k^{th} position). Since $B_j = (n - 1)!$ for $1 \leq j \leq n$, we can choose any of the n integers to be fixed, $\sum_{j=1}^n B_j = \binom{n}{1} (n - 1)!$. We can do the same for $j \neq k$ and $1 \leq j, k \leq n$ and similarly when three, four etc. objects are fixed. $D_n = S - B_1 \cup B_2 \cup \dots \cup B_n $ $= S - \left(\sum_{j=1}^n B_j - \sum_{j \neq k} B_j \cap B_k + \sum_{j \neq k \neq l} B_j \cap B_k \cap B_l \right. \\ \left. + \dots + (-1)^{n+1} B_1 \cap B_2 \cap \dots \cap B_n \right)$	[M1] [M1] [M1]

	$= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \dots + (-1)^n (n-n)!$ $= n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!} \right)$	[B1]
	(v) $D_6 = 6! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} \right) = 360 - 120 + 30 - 6 + 1 = 265$	[A2]
	Answer the following questions In this question you will investigate these generalisations of the product rule further. (a) Find an expression for $D^3(fg)$, the third derivative of the product fg, showing your workings clearly.	[2]
	The general form of $D^n(fg)$, the n^{th} derivative of the product fg is as follows: For $n \in \mathbb{N}$, $D^n(fg) = \sum_{i=0}^n \binom{n}{i} (D^i f)(D^{n-i} g)$ where $D^0 f = f$. (b) (i) Show that $\binom{k}{i} + \binom{k}{i-1} = \binom{k+1}{i}$ for positive integers k and i satisfying $k \geq i$. (ii) Prove the result for general form of $D^n(fg)$, the n^{th} derivative of the product fg.	[2] [9]
	The cubic polynomial $p(x)$ has three distinct roots r_1, r_2 and r_3. (c) (i) Find an expression for $D(p(x))$. (ii) Prove that the tangent to the curve $y = p(x)$ at the midpoint of any two of the roots intersects the curve at the third root.	[2] [5]

Question	Answer	Marks
(a)	$D^3(fg) = D(D^2(fg))$ $= D(f D^2g + 2Df Dg + g D^2f)$ $= f D^3g + Df D^2g + 2D^2f Dg + 2Df D^2g + g D^3f + Dg D^2f$ $= f D^3g + 3Df D^2g + 3D^2f Dg + g D^3f$	M1 A1
(b)(i)	$\binom{k}{i} + \binom{k}{i-1} = \frac{k!}{(k-i)!i!} + \frac{k!}{(k-i+1)!(i-1)!}$ $= \frac{k![k-i+1+i]}{(k-i+1)!i!}$ $= \frac{k!(k+1)}{(k+1-i)!i!}$ $= \frac{(k+1)!}{((k+1)-i)!i!}$ $= \binom{k+1}{i}$	M1 B1
(ii)	Base case $n = 1$ $\sum_{i=0}^1 \binom{1}{i} (D^i f)(D^{1-i} g) = \binom{1}{0} (D^0 f)(D^{1-0} g) + \binom{1}{1} (D^1 f)(D^{1-1} g)$ $= f Dg + g Df$ $= D(fg)$ The result is true for $n = 1$	B1
	Assume the result is true for $n = k, k \in \mathbb{Z}^+$, ie $D^k(fg) = \sum_{i=0}^k \binom{k}{i} (D^i f)(D^{k-i} g)$	M1

	For $n = k + 1$, $ \begin{aligned} D^{k+1}(fg) &= D(D^k(fg)) \\ &= D\left(\sum_{i=0}^k \binom{k}{i} (D^i f)(D^{k-i} g)\right) \\ &= \sum_{i=0}^k \binom{k}{i} (D^i f)(D^{k-i+1} g) + \sum_{i=0}^k \binom{k}{i} (D^{i+1} f)(D^{k-i} g) \\ &= \sum_{i=0}^k \binom{k}{i} (D^i f)(D^{k-i+1} g) + \sum_{i=1}^{k+1} \binom{k}{i-1} (D^i f)(D^{k-i+1} g) \\ &= f D^{k+1} g + \sum_{i=1}^k \binom{k}{i} (D^i f)(D^{k-i+1} g) \\ &\quad + \sum_{i=1}^k \binom{k}{i-1} (D^i f)(D^{k-i+1} g) + \binom{k}{k} (D^{k+1} f)(D^0 g) \\ &= f D^{k+1} g + \sum_{i=1}^k \left(\binom{k}{i} + \binom{k}{i-1}\right) (D^i f)(D^{k-i+1} g) + g D^{k+1} f \\ &= \binom{k+1}{0} D^0 f D^{k+1} g + \sum_{i=1}^k \binom{k+1}{i} (D^i f)(D^{k-i+1} g) + \binom{k+1}{k+1} D^{k+1} f D^{k-(k+1)+1} g \\ &= \sum_{i=0}^{k+1} \binom{k+1}{i} (D^i f)(D^{k+1-i} g) \end{aligned} $	M1 M1 M1 M1 M1 M1 A1
	So the result is true for $n = 1$ and if true for $n = k$ is also true for $n = k + 1$. Hence by induction, the result is proven true for $n \in \mathbb{N}$.	A1
(c) (i)	$p(x) = a(x - r_1)(x - r_2)(x - r_3)$ $(Dp)(x) = a((x - r_2)(x - r_3) + (x - r_1)(x - r_3) + (x - r_1)(x - r_2))$	M1 A1
(ii)	$ \begin{aligned} p\left(\frac{r_1 + r_2}{2}\right) &= a\left(\frac{r_2}{2} - \frac{r_1}{2}\right)\left(\frac{r_1}{2} - \frac{r_2}{2}\right)\left(\frac{r_1 + r_2}{2} - r_3\right) \\ &= -\frac{a}{8}(r_2 - r_1)^2(r_1 + r_2 - 2r_3) \end{aligned} $	M1
	Gradient = $ \begin{aligned} &(Dp)\left(\frac{r_1 + r_2}{2}\right) \\ &= a\left(\left(\frac{r_1 + r_2}{2} - r_2\right)\left(\frac{r_1 + r_2}{2} - r_3\right) + \left(\frac{r_1 + r_2}{2} - r_1\right)\left(\frac{r_1 + r_2}{2} - r_3\right) + \left(\frac{r_1 + r_2}{2} - r_1\right)\left(\frac{r_1 + r_2}{2} - r_2\right)\right) \\ &= a\left(\left(\frac{r_1}{2} - \frac{r_2}{2}\right)\left(\frac{r_1 + r_2 - 2r_3}{2}\right) + \left(\frac{r_2}{2} - \frac{r_1}{2}\right)\left(\frac{r_1 + r_2 - 2r_3}{2}\right) + \left(\frac{r_2}{2} - \frac{r_1}{2}\right)\left(\frac{r_1}{2} - \frac{r_2}{2}\right)\right) \\ &= a\left(\frac{r_2}{2} - \frac{r_1}{2}\right)\left(\frac{r_1}{2} - \frac{r_2}{2}\right) \\ &= -\frac{a}{4}(r_2 - r_1)^2 \end{aligned} $	M1

	Eq of tangent at $x = \frac{r_1 + r_2}{2}$ is $y = -\frac{a}{4}(r_2 - r_1)^2 x + c$. $-\frac{a}{8}(r_2 - r_1)^2 (r_1 + r_2 - 2r_3) = -\frac{a}{4}(r_2 - r_1)^2 \left(\frac{r_1 + r_2}{2}\right) + c$ $\Rightarrow c = \frac{a}{4}(r_2 - r_1)^2 \left(\frac{r_1 + r_2}{2}\right) - \frac{a}{8}(r_2 - r_1)^2 (r_1 + r_2 - 2r_3)$ $= \frac{a}{8}(r_2 - r_1)^2 ((r_1 + r_2) - (r_1 + r_2 - 2r_3))$ $= \frac{a}{4}(r_2 - r_1)^2 r_3$	M1 M1
	Eq of tangent at $x = \frac{r_1 + r_2}{2}$ is $y = -\frac{a}{4}(r_2 - r_1)^2 x + \frac{a}{4}(r_2 - r_1)^2 r_3$. When $y = 0$, $x = r_3$	A1