

4

SEQUENCES AND SERIES

SYLLABUS

- Concepts of sequence and series for finite and infinite cases
- Sequence as function $y = f(n)$ where n is a positive integer
- Relationship between u_n (the n th term) and S_n (the sum to n terms)
- Sequence given by a formula for the n th term
- Sequence generated by a relation $u_{n+1} = f(u_n)$, including the use of a graphing calculator or a computer to generate the sequence
- Use of Σ notation
- Sum and difference of two series
- Convergence of a series and the sum to infinity
- Formula for the n th term and the sum of a finite arithmetic series
- Formula for the n th term and the sum of a finite geometric series
- Condition for convergence of an infinite geometric series
- Formula for the sum to infinity of a convergent geometric series

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LECTURE 1

Lesson Outline

- Introduction to the concepts & notations of sequences and series
 - Arithmetic Progression:
 - Definition and formula for the n th term of an Arithmetic Progression
 - Formula for the sum of an Arithmetic Progression
-

1 INTRODUCTION

1.1 Sequence

A **sequence** refers to a set of numbers written consecutively and is usually written as $u_1, u_2, u_3, \dots, u_n, \dots$ where u_n is called the **n th term** of the sequence.

Consider the following sequence of numbers. Fill in the missing numbers and give the general term.

Sequence	n th term
1, 2, 4, 8, 16, __, __, 128	$u_n = 2^{n-1}, n = 1, 2, 3, \dots, 8$
$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \dots$	$u_n =$
1, 1, 2, 3, 5, 8, 13, __, __, ...	Relationship between terms is

The first sequence is a **finite sequence** since it has a finite number of terms, whereas the other two are **infinite sequences**.

A sequence can also be called a **progression**.

1.2 Series

A **series** is formed when the terms of a sequence are added, i.e., $u_1 + u_2 + u_3 + \dots + u_n + \dots$.

Examples

1. $2 + 4 + 8 + 16 + 32$ is a **finite series**
2. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$ is an **infinite series**

We denote the sum of the first n terms of a sequence by $S_n = u_1 + u_2 + u_3 + \cdots + u_n$.

Example

The sum of the first 4 terms of the sequence 2, 4, 8, 16, 32, ... is $S_4 = 2 + 4 + 8 + 16 = 30$.

The n th term of a sequence u_n is related to the sum of the first n terms of the sequence by

$$u_n = S_n - S_{n-1}.$$

Example

For the sequence 2, 4, 8, 16, 32, ...,

$$S_4 = 2 + 4 + 8 + 16, S_5 = 2 + 4 + 8 + 16 + 32.$$

Therefore, the 5th term of the sequence is $u_5 = S_5 - S_4 = 32$.

In this chapter, we will study two special types of sequences and their series, namely, the arithmetic progression and the geometric progression.

2 ARITHMETIC PROGRESSION

2.1 Introduction to Arithmetic Progression

The following three sequences follow a simple pattern. What are the next two terms?

Self-Practice
Write down the
general term u_n

- 1, 4, 7, 10, __, __, ... $u_n =$ __, $n = 1, 2, 3, \dots$
- -15, -11, -7, -3, __, __, ... $u_n =$ __, $n = 1, 2, 3, \dots$
- $29, 27\frac{1}{2}, 26, 24\frac{1}{2},$ __, __, ... $u_n =$ __, $n = 1, 2, 3, \dots$

2.2 Definition and Formula for the n th term of an Arithmetic Progression

An **arithmetic progression (AP)** is a sequence of numbers in which each term other than the first is obtained from the preceding one by the addition of a **constant**, called the **common difference**, d .

The first number is called the **first term**, a .

1st term	2nd term	3rd term	4th term	...	n th term
u_1	u_2	u_3	u_4	...	u_n
a	$a + d$	$a + 2d$	$a + 3d$	...	$a + (n-1)d$

1. The formula for the n th term (u_n) of an arithmetic progression is

$$u_n = a + (n-1)d .$$

2. If $u_1, u_2, \dots, u_n$ are consecutive terms of an arithmetic progression, then $u_n - u_{n-1}$ is a constant which is the common difference, d . Therefore

$$d = u_n - u_{n-1} .$$

3. To show that a sequence $u_1, u_2, \dots, u_n$ is in arithmetic progression, we need to find u_n and show that

$$\begin{aligned} &u_n - u_{n-1} \text{ is a constant} \\ &(\text{i.e. } u_n - u_{n-1} \text{ is independent of } n). \end{aligned}$$

It is insufficient to show that $u_2 - u_1$ is a constant.

Example 1

Find the n th term of the arithmetic progression 15, 9, 3, ... and hence find the 30th term.

Solution

AP with first term $a = 15$ and common difference $d =$

n th term =

30th term $u_{30} =$ ■

Example 2

The first four terms u_1, u_2, u_3, u_4 of an arithmetic progression are such that

$$u_4 - u_2 = 15 \text{ and } 4u_3 = 9u_1 .$$

Find the value of u_1 .

Solution

$$u_4 - u_2 = 15$$

$$4u_3 = 9u_1$$

Example 3

The sum of the first n terms of a series is $S_n = 2n^2 - 3n$.

- (a) Write down the first term.
- (b) Find the n th term and show that the terms are in arithmetic progression.

Solution

- (a) The first term u_1 is
- (b) The n th term

To check if it is an
AP, check if
 $u_n - u_{n-1}$ is a
constant.

■

2.3 Formula for the Sum of an Arithmetic Progression

Reference: <https://nrich.maths.org/2478>



Carl Friedrich Gauss (1777-1855) is recognised as being one of the greatest mathematicians of all time. During his lifetime he made significant contributions to almost every area of mathematics, as well as physics, astronomy and statistics. Like many of the great mathematicians, Gauss showed amazing mathematical skill from an early age, and there are many stories which show how clever he could be.

The most well-known story is a tale from when Gauss was still at primary school. One day Gauss' teacher asked his class to add together all the numbers from 1 to 100, assuming that this task would occupy them for quite a while. He was shocked when young Gauss, after a few seconds of thought, wrote down the answer 5050. The teacher couldn't understand how his pupil had calculated the sum so quickly in his head, but the eight year old Gauss pointed out that the problem was actually quite simple.

He had added the numbers in pairs – the first and the last, the second and the second to last and so on, observing that $1 + 100 = 101$, $2 + 99 = 101$, $3 + 98 = 101$, and so on, so $50 \times 101 = 5050$.

This shows that Gauss had a natural insight into mathematics. Rather than performing a great feat of mental arithmetic, Gauss had seen the structure of the problem and used it to find a short cut to a solution.

One way of presenting Gauss' method is to write out the sum twice, the second time reversing it as shown.

$$\begin{array}{ccccccc} \textcircled{1} & + & \textcircled{2} & + & \textcircled{3} & + \cdots + & \textcircled{n} \\ \textcircled{n} & + & \textcircled{(n-1)} & + & \textcircled{(n-2)} & + \cdots + & \textcircled{1} \end{array}$$

Gauss added the rows pairwise – each pair adds up to $n+1$ – and there are n pairs, so the sum of the rows is also $n \times (n+1)$. It follows that $2 \times (1 + 2 + \cdots + n) = n \times (n+1)$, from which we obtain the formula

$$1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2}.$$

In this section, we will derive a formula to add the terms of an arithmetic progression. Recall that the general term of an arithmetic progression can be written as $u_n = a + (n-1)d$, where a is the first term and d the common difference.

Let S_n be the sum of the first n terms of an arithmetic progression. Then

$$S_n = (a) + (a+d) + (a+2d) + \cdots + (a+(n-3)d) + (a+(n-2)d) + (a+(n-1)d)$$

Reversing the order, we have

$$S_n = (a+(n-1)d) + (a+(n-2)d) + (a+(n-3)d) + \cdots + (a+2d) + (a+d) + (a)$$

Adding the two expressions gives

$$\begin{aligned} 2S_n &= (a) + (a+d) + (a+2d) + \cdots + (a+(n-3)d) + (a+(n-2)d) + (a+(n-1)d) \\ &\quad + (a+(n-1)d) + (a+(n-2)d) + (a+(n-3)d) + \cdots + (a+2d) + (a+d) + (a) \\ &= n(2a + (n-1)d) \end{aligned}$$

$$\therefore S_n = \frac{n}{2}(2a + (n-1)d)$$

Alternatively,

$$S_n = \frac{n}{2}(a+l),$$

where last term, $l = a + (n-1)d$, since $S_n = \frac{n}{2}(a + a + (n-1)d)$.

These are the formulae for the sum of the first n terms for an AP.

Example 4

This gives you a hint
that it is an AP with
 $a = 5000$ and
 $d = 800$.

A canoeist in ACJC is undergoing a training programme in preparation for a competition. He canoes 5000 m on day 1, and on each successive day, the distance covered is increased by 800 m. If he wishes to cover a total distance of 200 000 m on days 1 to 15 inclusive, will he be able to meet his target? If so, give the exact total distance covered. If not, find the least number of days for his target to be met.

Solution

We can solve $2n^2 + 23n - 1000 \geq 0$ using **[TABLE]** in GC because n is an integer.

Press **[2ND]**, followed by **[GRAPH]** to show the values in **[TABLE]**.

NORMAL FLOAT DEC REAL RADIAN MP PRESS + FOR Δ Tb1					NORMAL FLOAT DEC REAL RADIAN MP PRESS + FOR Δ Tb1				
X	Y1				X	Y1			
-32	312				14	-286			
-31	209				15	-205			
-30	110				16	-120			
-29	15				17	-31			
-28	-76				18	62			
-27	-163				19	159			
-26	-246				20	260			
-25	-325				21	365			
-24	-400				22	474			
-23	-471				23	587			
-22	-538				24	704			
X = -29					X = 18				

Extra Practice

Annex A: Practice Question 1

LECTURE 2

Lesson Outline

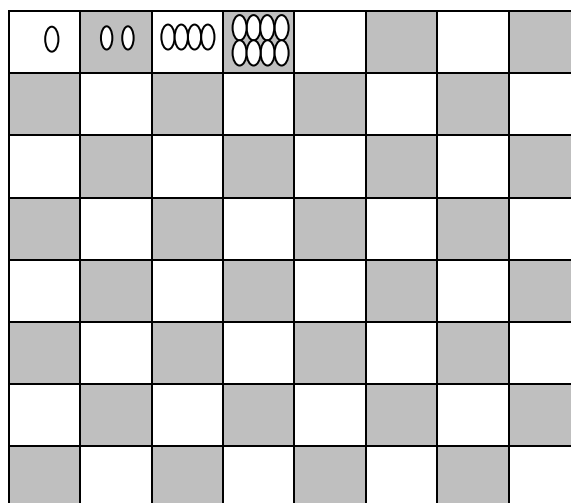
- Geometric Progression:
 - Definition and formula for the n th term of a Geometric Progression
 - Formula for the sum of a Geometric Progression
 - Sum to infinity of a Geometric Progression
-

3 GEOMETRIC PROGRESSION

3.1 Introduction to Geometric Progression

There once lived an Indian king, Iadava, whose son died in combat. Heartbroken, he locked himself in his castle. One day, a young man named Sessa came up to the castle from his village, claiming he had a game called 'chess' for his bereaved king. Iadava was taken by the game and it helped him come to terms with his son's sacrifice. He exclaimed to Sessa, "Ask me what you want and I will give it immediately!" Sessa replied, "Your Highness, seeing that it was the game of chess that brightened your life again, I would like to receive a grain of wheat for the first square of the chessboard, and two grains of wheat for the second, four for the third, eight for the fourth, and so on, until the sixty-fourth square." Iadava roared with laughter, amused by this humble request. "I would send my servant to have it counted out immediately!"

If you were the servant to the king, how would you feel?



The number of grains of wheat on each square of the chessboard follows a pattern: 1, 2, 4, 8,

Can you identify the pattern?

The following three sequences follow a similar pattern. What are their next two terms?

Self-Practice
Write down the
general term u_n

- 3, 6, 12, 24, __, __, ... $u_n =$, $n = 1, 2, 3, \dots$
- 12, -4, $1\frac{1}{3}$, __, __, ... $u_n =$, $n = 1, 2, 3, \dots$
- $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{9}$, __, __, ... $u_n =$, $n = 1, 2, 3, \dots$

3.2 Definition and Formula for the n th term of a Geometric Progression

A **geometric progression (GP)** is a sequence of numbers in which each term other than the first is obtained from the preceding one by the multiplication of a non-zero constant, called the **common ratio**, r .

1st term	2nd term	3rd term	4th term	...	n th term
u_1	u_2	u_3	u_4	...	u_n
a	ar	ar^2	ar^3	...	ar^{n-1}

For a geometric progression with first term, a , and non-zero common ratio, r :

- The formula for the n th term (u_n) of the geometric progression is:

$$u_n = ar^{n-1}$$

- If $u_1, u_2, \dots, u_n$ are the consecutive terms of the geometric progression, then the common ratio, r , which is a constant, is:

$$r = \frac{u_n}{u_{n-1}}$$

- Hence, to show that a sequence $u_1, u_2, \dots, u_n$ is in geometric progression, we need to find u_n in terms of n , and show that

$$\frac{u_n}{u_{n-1}} \text{ is a constant (i.e. independent of } n\text{).}$$

It is insufficient to show that $\frac{u_2}{u_1}$ is a constant.

Example 5

The 4th and 8th terms of a geometric progression are 3 and $\frac{1}{27}$ respectively. Find the possible values of the first term and common ratio.

Solution

■

Example 6

The sum of the first n terms of a series is given by the expression $6 - \frac{2^{n+1}}{3^{n-1}}$.

By finding an expression for the n th term of the series, or otherwise, show that this is a geometric series, and state the values of the first term and the common ratio. [J1981/I/1]

Solution

To check if GP, we
use $\frac{u_n}{u_{n-1}}$.

■

3.3 Formula for the sum of a Geometric Progression

For a geometric progression with first term, a , and non-zero common ratio r , the formula for the sum of the first n terms of the geometric progression, S_n , is given by:

$$S_n = \frac{a(1-r^n)}{1-r}, r \neq 1.$$

Proof

$$S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} \quad (1)$$

Multiplying (1) by r , we have

$$rS_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n \quad (2)$$

(1) – (2):

$$\begin{aligned} S_n - rS_n &= a - ar^n \\ S_n(1-r) &= a(1-r^n) \\ S_n &= \frac{a(1-r^n)}{(1-r)}, r \neq 1. \end{aligned}$$

■

Example 7

The first and second terms of a geometric progression are 10 and 11 respectively. Find the least number of terms such that their sum exceeds 8000. [AM/1998/II/2c]

Solution

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Example 8

John deposited \$1000 in a bank account on 1st January 2005. Compound interest at a rate of 4% per year is paid out on the last day of each year. How much will he have in his account on 1st January 2016?

John's wife, Mary, deposited \$1000 in a bank account on 1st January 2005, and the same amount on 1st January in each succeeding year. Compound interest at a rate of 4% per year is paid out on the last day of each year. How much does she have in her account on 2nd January 2016?

Solution

	Amount in account on 1st of January
2005 (1st yr.)	1000
2006 (2nd yr.)	$1000(1.04)$
2007 (3rd yr.)	
2008 (4th yr.)	
2016 (12th yr.)	

It is not necessary to list out so many rows. You can stop as long as you can identify the patterns (with at least 3 initial rows).

This is the term of a geometric progression with $a = \underline{\hspace{1cm}}$ and $r = \underline{\hspace{1cm}}$.

On the 1st of January 2016, John has .

	Amount in account on 2nd of January
2005 (1st yr.)	<u>1000</u>
2006 (2nd yr.)	<u>1000</u> + $1000(1.04)$
2007 (3rd yr.)	<u>1000</u> + $(1000(1.04) + 1000(1.04)^2)$
2008 (4th yr.)	<u>1000</u> + $(1000(1.04) + 1000(1.04)^2 + 1000(1.04)^3)$
2016 (12th yr.)	

This is a geometric series with terms and a common ratio of .

On the 2nd of January 2016, Mary has .

■

3.4 Sum to Infinity of a Geometric Progression

Case 1: $|r| > 1$

Consider the infinite series
 $2 + 4 + 8 + 16 + 32 + \dots$

What do you think is the sum of this series?

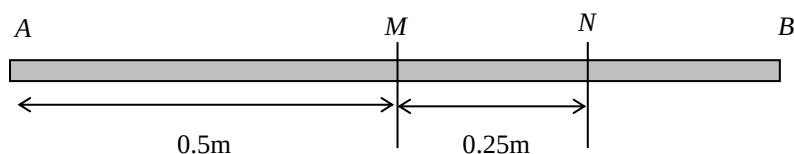
Case 2: $|r| < 1$

Consider the infinite series
 $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$

What do you think is the sum of this series?

Let us now consider a ruler of length AB where $AB = 1$ m.

Let M be the mid-point of AB and N be the mid-point of MB .



What can we say about the value of the infinite series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots?$$

Suppose S_n is the sum of a geometric progression with first term a and common ratio r . Then

$$S_n = \frac{a(1-r^n)}{1-r}.$$

Case 1: $|r| > 1$

If $|r| > 1$, then as $n \rightarrow \infty$,

$$r^n \rightarrow \infty.$$

Therefore, $S_n \rightarrow \infty$.

We say that S_n **does not converge**.

Case 2: $|r| < 1$

If $|r| < 1$, then as $n \rightarrow \infty$, $r^n \rightarrow 0$.

$$\frac{a(1-r^n)}{1-r} \rightarrow \frac{a(1-0)}{1-r} = \frac{a}{1-r}.$$

Therefore $S_n \rightarrow \frac{a}{1-r}$.

We say that S_n **converges** to

$$\frac{a}{1-r}, \text{ or that the sum to}$$

infinity, S_∞ , is $\frac{a}{1-r}$.

1. Formula for sum to infinity:

For $|r| < 1$, the sum to infinity of a geometric progression, or S_∞

(or $\lim_{n \rightarrow \infty} S_n$), is

$$S_\infty = \frac{a}{1-r}.$$

- 2 Condition for the existence of sum to infinity:

As $n \rightarrow \infty$, $S_n \rightarrow S_\infty$. This occurs when $|r| < 1$. Hence to show that a geometric progression converges (or that its sum to infinity exists), we must show that

$$|r| < 1.$$

Example 9

The n th term of a series is given by $\frac{15(2^{n-1})}{5^n}$. Show that the terms form a geometric series. Explain why this series converges and find its sum to infinity.

Solution

|

■

Example 10

An infinite geometric progression is such that the sum of all the terms after the n th term is equal to twice the value of the n th term.

Show that the sum to infinity of the whole progression is thrice the value of the first term.

Given that the first term is one, find also the least value of n so that the sum of the first n terms differs from the **limiting sum** by less than 0.01.

Solution

■

Extra Practice

Annex A: Practice Question 2, 3 and 4

LECTURE 3

Lesson Outline

- Summary of Arithmetic and Geometric Progressions
 - Sequences
 - Sequences defined by a formula for the n th term
 - Sequences generated by a relation $u_{n+1} = f(u_n)$
-

4 SUMMARY OF ARITHMETIC AND GEOMETRIC PROGRESSIONS

	Arithmetic Progression (AP)	Geometric Progression (GP)
How to distinguish between an AP and a GP?	$u_n - u_{n-1} = \text{constant } (= d),$ $d = \text{common difference}$	$\frac{u_n}{u_{n-1}} = \text{constant } (= r),$ $r = \text{common ratio}$
Express the n th term as the difference between two sums	$u_n = S_n - S_{n-1}$	
General term, u_n (first term, a)	$u_n = a + (n-1)d$	$u_n = ar^{n-1}$
Sum of first n terms, S_n	$S_n = \frac{n}{2}(2a + (n-1)d)$ $= \frac{n}{2}(a + l)$ $l = \text{last term}$	$S_n = \frac{a(1-r^n)}{1-r}$ $= \frac{a(r^n-1)}{r-1}, r \neq 1$
Sum to infinity, S_∞	Not applicable	$S_\infty = \frac{a}{1-r}, r < 1$
To show S_n converges	Not applicable	Show $ r < 1$

Example 11

If the first, fifth and tenth terms of an arithmetic progression are consecutive terms of a geometric progression and the sum of the second and eighth terms of the arithmetic progression is 20, find

- (i) the first term and the (non-zero) common difference of the arithmetic progression;
- (ii) the common ratio of the geometric progression.

[J/1973/I/1 (modified)]

Solution

- (i) Let first term and common difference of AP be a and d respectively.

■

Example 12

In 2003, a man earned 20 thousand dollars and spent it all. In the next seven years, his salary increased by 2 thousand dollars per annum but inflation caused his expenditure to rise by 5% of the previous year's expenditure. Find the amount he will save by the end of 2010, giving your answer to 4 significant figures. [ACJC/2010/Promo]

Solution

Number of years from 2003 to 2010 inclusive = 8

Salary follows arithmetic progression, $u_1 = 20\ 000$, $d = 2000$, $n = 8$

$$\text{Total salary for 8 years} = \frac{8}{2}(2(20\ 000) + (8-1)(2000))$$

Expenditure follows geometric progression, $u_1 = 20\ 000$, $r = 1.05$, $n = 8$

$$\text{Total expenditure for 8 years} = 20\ 000 \left(\frac{1 - (1.05)^8}{1 - 1.05} \right)$$

$$\text{Amount saved} = \frac{8}{2}(2(20\ 000) + (8-1)(2000)) - 20\ 000 \left(\frac{1 - (1.05)^8}{1 - 1.05} \right)$$

$$= \$25\ 017.82 \text{ (nearest cent)} = \$25\ 020 \text{ (4 sf)}$$

■

Example 13

Given that the first, third and fourth terms of a certain geometric series are consecutive terms in an arithmetic progression and that the sum to infinity is $\sqrt{5} - 1$, find the first term of the geometric series. [CJC/2002]

Solution

**Example 14**

- (i) A company is drilling for oil. Using machine A, the depth drilled on the first day is 256 metres. On each subsequent day, the depth drilled is 7 metres less than on the previous day. Drilling continues daily up to and including the day when a depth of less than 10 metres is drilled. What depth is drilled on the 10th day, and what is the total depth when drilling is completed? [6]
- (ii) Using machine B, the depth drilled on the first day is also 256 metres. On each subsequent day, the depth drilled is $\frac{8}{9}$ of the depth drilled on the previous day. How many days does it take for the depth drilled to exceed 99% of the theoretical maximum total depth? [4]
- [9740/N2011/I/9]

Solution

- (i) AP with $a = 256$, $d = -7$

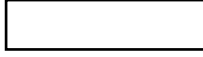
$$\begin{aligned} u_{10} &= a + (10-1)d \\ &= 256 + (9)(-7) = 193 \end{aligned}$$

193m is drilled on the 10th day.

For a depth of less than 10m to be drilled on the n th day,

(ii) GP with $a = 256$, $r = \frac{8}{9}$

Theoretical max depth, $S_{\infty} = \frac{256}{1 - \frac{8}{9}} = 2304$



$$\frac{256 \left[1 - \left(\frac{8}{9} \right)^n \right]}{1 - \frac{8}{9}} > 0.99(2304)$$

$$\left(\frac{8}{9} \right)^n < 0.01$$

$$n > \frac{\lg 0.01}{\lg \left(\frac{8}{9} \right)}$$

$$n > 39.1$$

$$\text{least } n = 40$$

It takes 40 days.

■

Extra Practice

Annex A: Practice Question 5 and 6

5 SEQUENCES

A **sequence** refers to a set of numbers written consecutively (or “an ordered set of mathematical objects” or “a set of elements in some definite order”) and is usually written as $u_1, u_2, u_3, \dots, u_n, \dots$ where u_n is called the n th term of the sequence.

In Sections 2 and 3, we have already discussed two special kinds of sequences, namely arithmetic progression and geometric progression. In this section, we will look at sequences defined more generally.

5.1 Sequences defined by a formula for the n th term

A sequence can be expressed as a function $f(n)$ where n is a positive integer. The table below shows some commonly used sequences and their corresponding functions.

Sequence	u_n	Function	Domain	Finite/ Infinite
$\{1, 2, 3, 4, \dots\}$	n	$f(n) = n$	$\{1, 2, 3, 4, \dots\}$	Infinite
$\{2, 4, 6, 8, \dots\}$	$2n$	$f(n) = 2n$	$\{1, 2, 3, 4, \dots\}$	Infinite
$\{1, 3, 5, 7, \dots\}$	$2n-1$	$f(n) = 2n-1$	$\{1, 2, 3, 4, \dots\}$	Infinite
$\{1, 3, 5, 7\}$	$2n-1$	$f(n) = 2n-1$	$\{1, 2, 3, 4\}$	Finite
$\{-1, 1, -1, 1, \dots\}$	$(-1)^n$	$f(n) = (-1)^n$	$\{1, 2, 3, 4, \dots\}$	Infinite
$\{1, -1, 1, -1, \dots\}$	$(-1)^{n+1}$	$f(n) = (-1)^{n+1}$	$\{1, 2, 3, 4, \dots\}$	Infinite
$\{1, 4, 9, 16, \dots\}$			$\{1, 2, 3, 4, \dots\}$	
$\{4, 8, 16, 32, \dots, 2^{n+1}\}$			$\{1, 2, 3, 4, \dots, n\}$	
$\{1, 2, 6, 24, 120, \dots\}$			$\{1, 2, 3, 4, 5, \dots\}$	
$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\right\}$			$\{1, 2, 3, 4, \dots\}$	

5.2 Sequences generated by a relation $u_{n+1} = f(u_n)$

Some sequences $\{u_n\}$ may also be defined by its first term u_1 (or first few terms) and a relation between the general term u_{n+1} and the preceding term u_n (or more preceding terms).

For example, consider the sequence 1, 4, 7, 10, 13, ... It can be defined by $u_1 = 1$ and $u_{n+1} = u_n + 3$ for $n \geq 1$.

The equation $u_{n+1} = u_n + 3$ is also called the **recurrence relation** for the sequence, and $u_1 = 1$ is the **initial condition** of the sequence.

This sequence is also an arithmetic progression with first term $a = 1$ and common difference $d = 3$.

For example, the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, ... is a sequence in which each term is the sum of the two preceding terms.

For $\{u_n\} = \{1, 1, 2, 3, 5, 8, 13, \dots\}$, we can define it with:

- the first two terms, $u_1 = 1$ and $u_2 = 1$, and
- $u_n = u_{n-1} + u_{n-2}$ for $n \geq 3$.

The Fibonacci sequence occurs not just in Mathematics, but in Biology as well.

In our syllabus, we will only deal with sequences generated by recurrence relations of the form $u_{n+1} = f(u_n)$.

Example 15

Write down the 2nd, 3rd, and 4th terms of the following sequences:

- (a) $u_1 = 2, u_{n+1} = 3u_n$ for $n \geq 1$ (b) $u_1 = -2, u_{n+1} = -u_n + 3$ for $n \geq 1$

Solution

(a) $u_1 = 2$

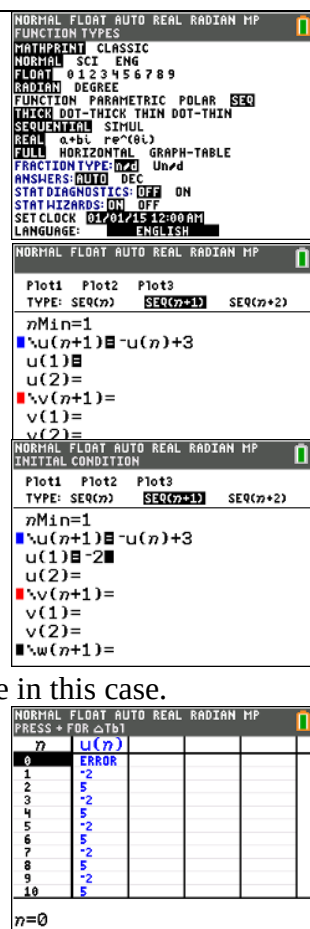
$$u_2 = 3u_1 = 3(2) = 6$$

(b) $u_1 = -2$

$$u_2 = -u_1 + 3 = -(-2) + 3 = 5$$

You may also use the GC to find the terms of a sequence generated by a recurrence relation. Using the GC to solve **(b)**,

- 1) Set GC in **SEQ** mode.
Press **[MODE]**, select **[SEQ]** and press **[ENTER]**.
- 2) Type in the recurrence relation.
Press **[Y=]**. Under TYPE, select **SEQ(n+1)**.
nMin should be 1.
nMin is the starting value of n , or the index of the first term of the sequence.
For **u(n+1)**, enter $u(n+1) = -u(n) + 3$.
This is the recurrence relation.
To type **u(n)**, press **[ALPHA]** **[TRACE]** **[1]**.
- 3) Type in the initial condition.
For **u(1)**, enter **[(-)]** then **[2]**.
This value is the first term in the sequence in this case.
- 4) View the terms in the sequence from the **TABLE**.
Press **[2ND]** then **[GRAPH]**.
From the **TABLE**, you can see that $u_2 = 5$, $u_3 = -2$, and $u_4 = 5$.



Example 16

- (a) An arithmetic sequence is one where each term is obtained from the previous term by adding a constant d . The first term of the arithmetic sequence is denoted by a . Express this as a recurrence relation.
- (b) A geometric sequence is one where each term is obtained from the previous term by multiplying a constant r . The first term of the geometric sequence is denoted by a . Express this as a recurrence relation.

Solution

(a) $u_1 = a$

(b) $u_1 = a$

Notice that as n increases, u_n approaches $\frac{1}{2}$. We say that the sequence $\{u_n\}$ **converges** to $\frac{1}{2}$ as n approaches infinity. We write $u_n \rightarrow \frac{1}{2}$ as $n \rightarrow \infty$, or $\lim_{n \rightarrow \infty} u_n = \frac{1}{2}$. The **limit** of the sequence is $\frac{1}{2}$.

A sequence $\{u_n\}$ is said to be **convergent** if $u_n \rightarrow L$ as $n \rightarrow \infty$ where L is a constant, also known as the **limit** of the sequence. Otherwise, we say the sequence is **not convergent** or **divergent**.

Example 17

Determine if the following sequences are convergent. If they are, find their limits.

1. $u_n = 4\left(\frac{1}{3}\right)^{n-1} - 2$

When $n \rightarrow \infty$, $4\left(\frac{1}{3}\right)^{n-1} \rightarrow 0$,
 $u_n \rightarrow -2$

2. $u_n = \frac{n^2 - 5n + 6}{2n^2 + 7n - 2}$

Divide both numerator and denominator by n^2 .

3. $u_n = 2^{3n-1}$

Since $(-1)^n = \begin{cases} 1, & n \text{ is even} \\ -1, & n \text{ is odd} \end{cases}$,

When $n \rightarrow \infty$, $(-1)^n$ may be -1 or 1 .

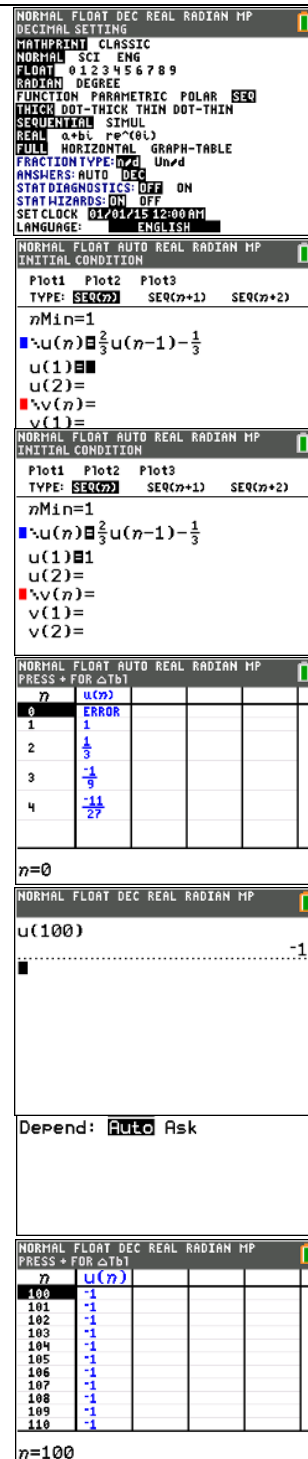
■

Graphing Calculator Activity 2

If $3u_n = 2u_{n-1} - 1$ for all $n \geq 2$ and $u_1 = 1$, find u_2 , u_3 and u_{100} .

Describe the behaviour of the sequence as n tends to infinity.

- 1) Set GC in **SEQ** mode.
Press **[MODE]**, select **[SEQ]** and press **[ENTER]**.
- 2) Type in the recurrence relation.
Press **[Y=]**. Under TYPE, select **SEQ(n)**.
nMin should be 1.
nMin is the starting value of n , or the index of the first term of the sequence.
For **u(n)**, enter $u(n) = \frac{2}{3}u(n-1) - \frac{1}{3}$.
This is the recurrence relation.
To type **u(n-1)**, press **[ALPHA]** **[TRACE]** **[1]**.
- 3) Type in the initial condition.
For **u(1)**, enter 1.
This value is the first term in the sequence in this case.
- 4) View the terms in the sequence from the **TABLE**.
Press **[2ND]** then **[GRAPH]**.
From the **TABLE**, you can see that $u_2 = \quad$ and $u_3 = \quad$.
- 5) Find u_{100} and describing the behaviour of the sequence as n tends to infinity.
From the home screen, enter $u(100)$.
Hence $u(100) = -1$.
You may also press **[2ND]** then **[WINDOW]** to enter **[TBL SET]**.
From there, you can adjust **TblStart** to be 100.



Hence, as n tends to infinity, the sequence _____.

Example 18

A patient is put on medication which is taken once a day. The dose is 35 mg and each day the patient's metabolism burns off 70% of the drug in the body. It is known that if the level of the drug in the body reaches 54 mg, then the consequences could be fatal.

Let u_n be the amount of drug in the patient's body (in mg) on the n th day, right after taking the drug for the day.

- (i) Write down the amount of drug in the patient's body as a recurrence relation.
- (ii) How much drug is there in the patient's body on day 4?
- (iii) Is it safe for the patient to continue taking the medication indefinitely?

Solution

(i) $u_1 = 35$

(ii) $u_4 =$

- (iii) Yes, it is safe, as the amount of drug in the patient's body will increase and approach 50 mg in the long run, never reaching 54 mg.

NORMAL FLOAT AUTO REAL RADIAN MP			
SECOND CONDITION IF NEEDED			
Plot1	Plot2	Plot3	
TYPE: SEQ(n)	SEQ(n+1)	SEQ(n+2)	
nMin=1			
u(n+1)=0.3u(n)+35			
u(1)=35			
u(2)=			
v(n+1)=			
v(1)=			
v(2)=			
w(n+1)=			

NORMAL FLOAT AUTO REAL RADIAN MP			
PRESS + FOR Δ tbl			
n	u		
1	35		
2	45.5		
3	48.65		
4	49.595		
5	49.879		
6	49.964		
7	49.989		
8	49.997		
9	49.999		
10	50		
11	50		
n=1			

NORMAL FLOAT AUTO REAL RADIAN MP			
PRESS + FOR Δ tbl			
n	u		
100	50		
101	50		
102	50		
103	50		
104	50		
105	50		
106	50		
107	50		
108	50		
109	50		
110	50		
n=100			

Example 19

The positive numbers x_n satisfy the relation $x_{n+1} = (x_n + 5)^{\frac{1}{2}}$, for $n = 1, 2, 3, \dots$, and $x_1 = 1$. As $n \rightarrow \infty$, $x_n \rightarrow p$.

- (i) Find the value of p to 3 decimal places.
 (ii) Find the exact value of p .

Solution

- (i) Using GC,

As $n \rightarrow \infty$, $x_n \rightarrow$ _____.

$p \approx$ _____.

- (ii) As $n \rightarrow \infty$, $x_n \rightarrow p$ and $x_{n+1} \rightarrow p$.

$$p = (p + 5)^{\frac{1}{2}}.$$

$$p^2 = p + 5$$

$$p^2 - p - 5 = 0 \quad \text{--- (1)}$$

$$p = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-5)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{21}}{2}$$

Since p should be a positive number,

$$p = \frac{1 - \sqrt{21}}{2} \text{ is not applicable.}$$

$$\therefore p = \frac{1 + \sqrt{21}}{2}.$$

NORMAL FLOAT AUTO REAL RADIAN MP			
INITIAL CONDITION			
Plot1	Plot2	Plot3	
TYPE: SEQ(n)	SEQ(n+1)	SEQ(n+2)	
nMin=1			
u(n+1)=√u(n)+5			
u(1)=1			
v(n+1)=			
v(1)=			
v(2)=			

NORMAL FLOAT DEC REAL RADIAN MP			
PRESS + FOR Δ Tabl			
n	u(n)		
1	1		
2	2.4495		
3	2.7294		
4	2.7802		
5	2.7893		
6	2.7909		
7	2.7912		
8	2.7913		
9	2.7913		
10	2.7913		
11	2.7913		
n=1			

NORMAL FLOAT DEC REAL RADIAN MP			
PRESS + FOR Δ Tabl			
n	u(n)		
50	2.7913		
51	2.7913		
52	2.7913		
53	2.7913		
54	2.7913		
55	2.7913		
56	2.7913		
57	2.7913		
58	2.7913		
59	2.7913		
60	2.7913		
n=50			

6 SERIES

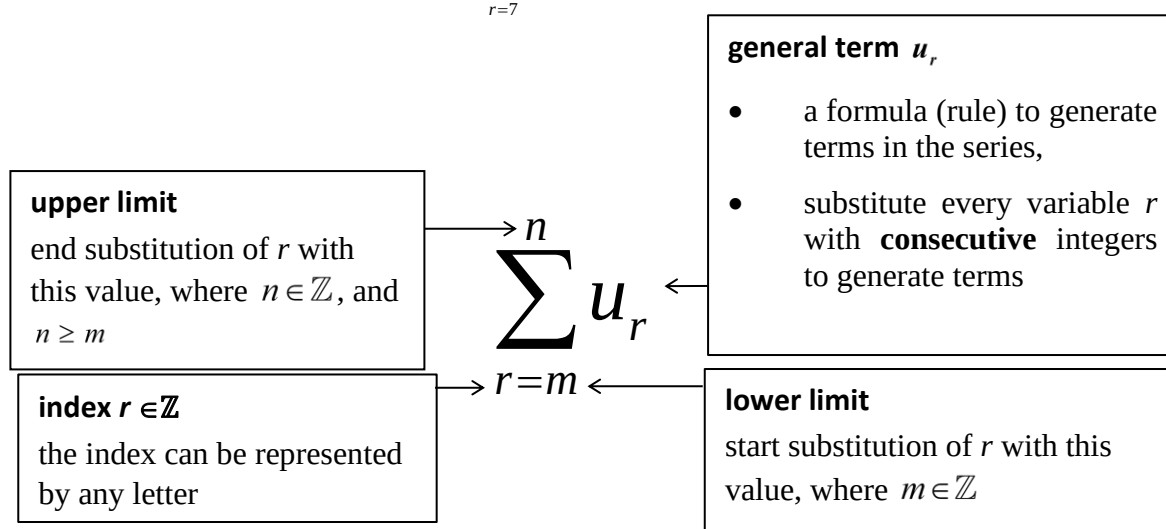
6.1 Sigma Notation Σ

The Greek letter sigma Σ is usually used to denote the sum of terms.

$$u_1 + u_2 + u_3 + \cdots + u_n = \sum_{r=1}^n u_r$$

Similarly,

- $u_5 + u_6 + u_7 + \cdots + u_{13} = \sum_{r=5}^{13} u_r$ (a finite series), and
- $u_7 + u_8 + u_9 + \cdots = \sum_{r=7}^{\infty} u_r$ (an infinite series).



Number of terms in the series = $n - m + 1$

Example 20

Expand the following expressions and write down the number of terms.

(i) $\sum_{r=1}^6 2r$

(ii) $\sum_{r=4}^8 3$

(iii) $\sum_{r=1}^6 (5 - 2r)$

(iv) $\sum_{r=0}^5 x^r$

(v) $\sum_{t=0}^4 \frac{r}{t!}$

Solution

(i) $\sum_{r=1}^6 2r =$

Number of terms =

(ii) $\sum_{r=4}^8 3 =$

Number of terms =

(iii) $\sum_{r=1}^6 (5 - 2r) =$

Number of terms =

(iv) $\sum_{r=0}^5 x^r =$

Number of terms =

(v) $\sum_{t=0}^4 \frac{r}{t!} =$

Number of terms =

■

By recognising the general term and using the $\sum$ notation, we can write a series $1 + 2 + 3 + 4 + \dots + r + \dots + 1000$ as $\sum_{r=1}^{1000} r$ or $\sum_{r=0}^{999} (r+1)$ or $\sum_{r=2}^{1001} (r-1)$, etc.

Example 21Express each of the following series in Σ notation.Can you identify the
AP and GP in this
example?

(i) $2 + 4 + 8 + 16 + \dots + 128$

(ii) $3 + 5 + 7 + 9 + \dots$

(iii) $\frac{1 \times 3}{2 \times 4} + \frac{2 \times 4}{3 \times 5} + \dots + \frac{19 \times 21}{20 \times 22}$

(iv) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \dots$

(v) $2 - 4 + 8 - 16 + \dots + 128$

(vi) $3^2 + 7^2 + 11^2 + \dots$ for 20 terms

Solution

(i) $2 + 4 + 8 + 16 + \dots + 128 =$

(ii) $3 + 5 + 7 + 9 + \dots =$

(iii) $\frac{1 \times 3}{2 \times 4} + \frac{2 \times 4}{3 \times 5} + \dots + \frac{19 \times 21}{20 \times 22} =$

(iv) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \dots =$

(v) $2 - 4 + 8 - 16 + \dots + 128$
 $=$

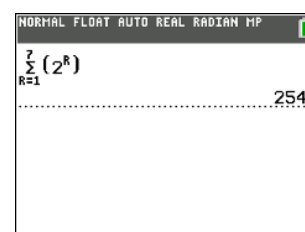
(vi) $3^2 + 7^2 + 11^2 + \dots$ for 20 terms
 $=$

■

Example 22Evaluate $2 + 4 + 8 + 16 + \dots + 128$.*Solution*

Using GC,

$2 + 4 + 8 + 16 + \dots + 128 =$

To get the summation symbol, press **[alpha]**
[window] **[1]** on the home screen of your GC.

■

LECTURE 5

Lesson Outline:

- Useful results on series
- Convergence of a series

6.2 Useful Results on Series

	General Result	Examples
1	$\sum_{r=m}^n u_r = \sum_{r=1}^n u_r - \sum_{r=1}^{m-1} u_r \quad (n \geq m)$	$\sum_{r=24}^{50} u_r = \sum_{r=1}^{50} u_r - \sum_{r=1}^{23} u_r,$ $\sum_{r=51}^{100} 2r = \sum_{r=1}^{100} 2r - \sum_{r=1}^{50} 2r$
2	$\sum_{r=1}^n k u_r = k \sum_{r=1}^n u_r,$ where k is a constant.	$\sum_{r=1}^{99} 2r^2 = 2 \sum_{r=1}^{99} r^2$
3	$\sum_{r=1}^n (u_r + v_r) = \sum_{r=1}^n u_r + \sum_{r=1}^n v_r$ $\sum_{r=1}^n (u_r - v_r) = \sum_{r=1}^n u_r - \sum_{r=1}^n v_r$	$\sum_{r=1}^{10} \left(r + \frac{1}{r} \right) = \sum_{r=1}^{10} r + \sum_{r=1}^{10} \frac{1}{r}$ $\sum_{r=1}^{10} \left(r - \frac{1}{r} \right) = \sum_{r=1}^{10} r - \sum_{r=1}^{10} \frac{1}{r}$
4	$\sum_{r=1}^n c = \underbrace{c + c + c + \dots + c}_{n \text{ times}} = nc,$ where c is a constant. In general, $\sum_{r=m}^n c = (n - m + 1)c,$ where c is a constant.	$\sum_{r=1}^{100} 5 = \underbrace{5 + 5 + 5 + \dots + 5}_{100 \text{ times}}$ $= 100(5)$ $= 500$ $\sum_{r=30}^{60} c = (60 - 30 + 1)c = 31c$
5	$\sum_{r=1}^n r = 1 + 2 + 3 + \dots + n = \frac{n}{2}(n+1)$	$\sum_{r=1}^{100} r = 1 + 2 + 3 + \dots + 100$ $= \frac{100}{2}(100+1)$

Example 23

For a constant k , identify whether the following statements are true.

- (a) $\sum_{r=0}^n k = \sum_{r=1}^n k$ T / F (b) $\sum_{r=0}^n r = \sum_{r=1}^n r$ T / F
- (c) $\sum_{r=0}^n k^r = \sum_{r=1}^n k^r$ T / F (d) $\sum_{r=1}^n (kr + a) = k \sum_{r=1}^n r + na$ T / F
- (e) $\sum_{r=1}^n u_r v_r = \sum_{r=1}^n u_r \sum_{r=1}^n v_r$ T / F (f) $\left(\sum_{r=0}^n r \right)^2 = \sum_{r=0}^n r^2$ T / F

■

The following two commonly used results will be given in the question if they are required:

$$\sum_{r=1}^n r^2 = \frac{n}{6}(n+1)(2n+1) \qquad \sum_{r=1}^n r^3 = \left(\frac{n}{2}(n+1) \right)^2$$

Example 24

It is given that $\sum_{r=1}^n r^2 = \frac{n}{6}(n+1)(2n+1)$ and $\sum_{r=1}^n r^3 = \left(\frac{n}{2}(n+1) \right)^2$.

Without using the graphing calculator, evaluate

- (i) $1^2 + 2^2 + \dots + 5^2$ (ii) $1^3 + 2^3 + \dots + 5^3$
- (iii) $\sum_{r=3}^{15} (r+1)^2$ (iv) $\sum_{r=0}^n (4r^3 + 3^r + 2)$
- (v) $\sum_{r=n+1}^{2n} r(r-n)$

Solution

(i) $1^2 + 2^2 + \dots + 5^2 =$

(ii) $1^3 + 2^3 + \dots + 5^3 =$

$$\text{(iii)} \quad \sum_{r=3}^{15} (r+1)^2$$

Method 1 (Expanding the general term):

Method 2 (Rewriting the notation):

$$\text{(iv)} \quad \sum_{r=0}^n (4r^3 + 3^r + 2)$$

Using GP formula
for $\sum_{r=0}^n 3^r$. Note the
number of terms is
 $n+1$.

$$(v) \quad \sum_{r=n+1}^{2n} r(r-n)$$

Method 1 (Expanding the general term):

Using AP formula
for $\sum_{r=n+1}^{2n} r$. Note the
number of terms is
 n .

Method 2 (Writing out the terms in the series):

This method is
situational and
relies on
observation and
experience. If there
is a pattern, it will
yield a simpler and
more elegant
solution.

■

Note that for **Example 24**, the answers for **(i)** to **(iii)** can be checked using the GC as well (see **Example 22**).

Extra Practice:

Annex A: Practice Question 7

6.3 Convergence of a Series

A series $\sum_{r=1}^n u_r$ is said to be **convergent** if there exists a finite number L

such that as $n \rightarrow \infty$, $\sum_{r=1}^n u_r \rightarrow L$. In this case,

$$\sum_{r=1}^{\infty} u_r = \lim_{n \rightarrow \infty} \sum_{r=1}^n u_r = L.$$

Graphing Calculator Activity

Investigate the convergence or divergence of a series using GC.

Activity 1

Enter the equation $Y_1 = \sum_{A=1}^X \left(\frac{1}{2^{A-1}} \right)$.

- 1) Press **[ALPHA]** **[WINDOW]** and select **[2: summation Σ]**.
- 2) To enter **A**, press **[ALPHA]** **[MATH]**.
Investigate the sum by using the **TABLE**.
- 3) Press **[2ND]** **[GRAPH]**.

NORMAL FLOAT DEC REAL DEGREE MP					
Plot1 Plot2 Plot3					
Y1 = $\sum_{A=1}^X \left(\frac{1}{2^{A-1}} \right)$					
Y2 =					
Y3 =					
Y4 =					
Y5 =					
Y6 =					
Y7 =					
Y8 =					
NORMAL FLOAT DEC REAL DEGREE MP					
PRESS + FOR Δ Tb1					
X	Y1				
1	1				
2	1.5				
3	1.75				
4	1.875				
5	1.9375				
6	1.9688				
7	1.9844				
8	1.9922				
9	1.9961				
10	1.998				
11	1.999				
X=1					

What do you observe when X increases?

Values of Y_1 increases.

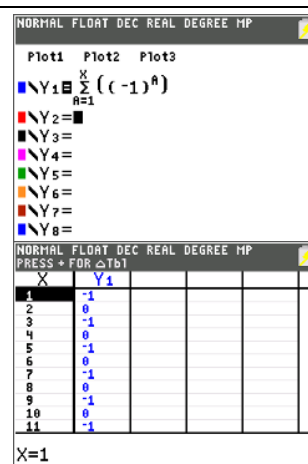
Does the series converge?

Values of Y_1 tends to the value 2. Yes, the series converges to 2.

Activity 2

Enter the equation $Y_1 = \sum_{A=1}^X ((-1)^A)$.

- 1) Press **[ALPHA]** **[WINDOW]** and select **[2: summation Σ]**.
- 2) To enter **A**, press **[ALPHA]** **[MATH]**.
Investigate the sum by using the **TABLE**.
- 3) Press **[2ND]** **[GRAPH]**.



What do you observe when X increases?

Values of Y_1 are either 0 or -1 .

Does the series converge?

The values of Y_1 does not tend to a finite constant number. Therefore, the series is not convergent.

Example 25

Find $\sum_{r=1}^{\infty} e^{2-3r}$ in terms of e .

Explain why $\sum_{r=1}^{\infty} e^{3r-2}$ does not exist.

Solution

■

ANNEX A

Practice Questions on Sequences and Series

Questions on AP

- 1 (a) Find the sum of all the even numbers from 50 to 100 inclusive.
(b) The terms of an arithmetic progression are 72, 69, 66, ..., -24, -27, Find the number of terms whose sum is zero.
(c) An arithmetic progression is such that the tenth term is equal to the sum of the first ten terms. Show that the fifth term is zero. [AM/N90]
(d) An arithmetic progression contains 20 terms. Given that the 8th term is 25 and the sum of the last 8 terms is 404, find the sum of the first 8 terms. [AM/J87]
(e) The sum of the first 6 terms of an arithmetic progression is 55.5 and the sum of the next 6 terms is 145.5. Find the first term and the common difference. [AM/N92]

Questions on GP

- 2 (a) The fourth term of a geometric progression is 6 and the seventh term is -48. Find the sum of the first eleven terms.
(b) The first term of a geometric progression is twelve times the common ratio and the sum to infinity is 4, find the third term. [AM/J91]
- 3 The first term of a geometric progression exceeds the second term by 2, and the sum of the second and third terms is $\frac{4}{3}$. Find the possible values of the first term and the common ratio. Given further that all the terms of the progression are positive, Find the sum to infinity. [AM/J87]
- 4 (a) The sum of an infinite geometric progression is 500. Given that the second term is 80, find the first term and the common ratio. Find also the least number of terms of the progression whose sum exceeds 499. [AM/J90]
(b) A geometric progression has a first term of 18 and a sum to infinity of 30. Each of the terms in the progression is squared to form a new geometric progression. Find the sum to infinity of the new progression. [AM/1997/II/2]
- 5 The first three terms of a geometric progression are $2p+q$, $6p+q$, $14p+q$, where $p \neq 0$. Find
(i) q in terms of p ,
(ii) the common ratio.
Show that the $(n+1)$ th term exceeds the sum of the first n terms by $4p$. [AM/N82]
- 6 An arithmetic progression has first term 8 and common difference d . Given that the 1st, 5th and 21st terms of this progression are three consecutive terms of a geometric progression, find the value of d . [AM/J93]

Questions on Series

7 Evaluate the following without the use of a graphing calculator.

$$(a) \sum_{r=0}^{15} a$$

$$(b) \sum_{r=1}^n (3 - 2r)$$

$$(c) \sum_{r=10}^{22} (150 + r)$$

$$(d) \sum_{r=1}^{100} \frac{1}{10^r}$$

$$(e) \sum_{r=2}^{\infty} (0.8)^{r+1}$$

Answers

1 (a) 1950 (b) 49 (d) 116 (e) 3, 2.5

2 (a) -512.25 (b) $\frac{3}{16}$

3 $r = \frac{1}{3}$, $a = 3$, or $r = -2$, $a = \frac{2}{3}$, 4.5

4 (a) 400, 0.2, 0.8, 4 (b) 386

5 (i) $q = 2p$ (ii) 2

6 $d = 6$

7 (a) $16a$ (b) $2n - n^2$ (c) 2158 (d) $\frac{1}{9}$ (e) 2.56